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Block Algorithm for General Third Order Ordinary Differential Equation

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ABSTRACT

We present a block algorithm for the general solution of $y''' = f(x, y, y', y'')$. The numerical algorithm is developed by the methods of interpolation of the power series approximate solution and collocation of the differential system of the approximant at selected grid points to generate a continuous method. Block method is adopted to simultaneously generate all the parameters needed to implement the method. The method was tested on numerical examples and to investigate the efficiency of the method

KEYWORDS: parallel block method, Collocation, Interpolation, approximant, continuous method.

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INTRODUCTION

The general third order ordinary differential equation which is of the form

$$y''' = f\left(x, y, y', y''\right), \quad y\left(a\right) = \eta_1, y'\left(a\right) = \eta_2, y''\left(a\right) = \eta_3 \tag{1}$$

Recent researches in the direct solution of higher order ordinary differential equation include Awoyemi {[2], [3], [4]}, Adesanya et al [1], Kayode {[10], [12]}. These methods mentioned, there implementation are predictor- corrector mode, like other linear multistep methods and other standard methods, are usually applied to initial value problems

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as a single formula but the drawback of this method are (i) they are not self starting (ii) they advance the numerical integration of the ordinary differential equation in one step at a time, which leads to overlapping of the piecewise polynomial solution model. See Olabode [7] for details. The advantages of the continuous method are widely reported by Awoyemi [5].

In order to correct the setback of the method of predictor – corrector method, Fatunla [10], Olabode et al [8], Yahaya [6] and Badmus et al [9] proposed block method for the solution of higher order ordinary differential equations with limitation to special type of the form $y^{(n)} = f(x, y)$. Thus these methods are self starting and eliminate the use of predictors.

Jator [17], Yahaya et al [18], proposed block method for solving second order initial value problems. This method sequentially generates the parameter to implement the method. In this paper, we present a method for solving third order ordinary differential equation using parallel block method. The method was found to be efficient and give better accuracy.

1. Method of Solution

We consider an approximate solution to Equation (1) in power series

$$y(x) = \sum_{j=0}^{k} a_j \, \Psi^j(x) \tag{2}$$

The third derivative of Equation (2) is given as

$$y''' = \sum_{j=0}^{k} j(j-1)(j-2)a_j \, \Psi^{j-3}(x) \tag{3}$$

Considering Equations (2) and (3) generates the differential system

$$\sum_{j=0}^{k} j(j-1)\,(j-2)a_j \Psi^{j-3}(x) = f(x, y, y', y'') \tag{4}$$

Collocating Equation (4) at $x = x_{n+j}, j = 1(1)k$ and interpolating Equation (2) at $x = x_{n+j}, j = 0(1)k-1$ yield a system which can be expressed in matrix form as

$$AX = B \tag{5}$$

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 \\ 1 & x_{n+3} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 \\ 0 & 0 & 0 & 6 & 24x_{n+5} & 60x_{n+5}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+2} \\ y_{n+4} \\ f_{n+1} \\ f_{n+3} \\ f_{n+5} \end{bmatrix} \tag{6}$$

Solving Equation (6) for $a_j, j = 0(1)k$, substituting back into Equation (2) and simplifying gives an equation of the form

$$y(x) = \sum_{r=0}^{4} \Psi_r y_{n+r}(x) + \sum_{r=1}^{5} \beta_r f_{n+r}(x) \tag{7}$$

Where the coefficient of $y_{n+r}(x)$ and $f_{n+r}(x)$ are found to be

$$\Psi_0(t) = \frac{1}{8}\left(t^2 + 2t\right)$$

$$\Psi_2(t) = -\frac{1}{4}\left(t^2 + 4t\right)$$

$$\Psi_4(t) = \frac{1}{8}\left(t^2 + 6t + 8\right)$$

$$\beta_1(t) = \frac{h^3}{480}(t^5 - 10t^3 + 60t^2 + 144t)$$

$$\beta_3(t) = -\frac{h^3}{240}(t^5 + 5t^4 - 30\,\mathrm{t}^3 - 200t^2 - 256t)$$

$$\beta_5(t) = \frac{h^3}{480}(t^5 + 10t^4 + 30t^3 + 20t^2 - 16t) \tag{8}$$

Where $t = \dfrac{x - x_{n+4}}{h}$

Evaluating Equation (8) at $t = -3$, -1, and 1 respectively gives

$$32y_{n+5} - 60y_{n+4} + 40y_{n+2} - 12y_n = h^3(3f_{n+5} + 64f_{n+3} + 13f_{n+1}) \quad (9)$$

$$12y_{n+4} - 32y_{n+3} + 24y_{n+2} - 4y_n = h^3(-f_{n+5} + 12f_{n+3} + 5f_{n+1}) \quad (10)$$

$$4y_{n+4} - 24y_{n+2} + 32y_{n+1} - 12y_n = h^3(-f_{n+5} + 8f_{n+3} + 9f_{n+1}) \quad (11)$$

Evaluating the first derivative and second derivative of Equation (8) at $t = -3(1)1$ gives the values of y_{n+i}^{j}, $\text{i} = 0(1)5,\ j = 0(1)2$

2. Block method

We propose a block method in the form

$$A^0 h^\gamma Y_m^{(n)} = h^\gamma \sum_{i=0}^{k} A^{(i)}\, Y_{m-i}^{(n)} + h^\mu \sum_{i=0}^{k} B^{(i)}\, F_{m-i}, \quad (12)$$

where the power of the derivative is n, μ is the order of the differential equation, γ is the power relative to the derivative of the differential equation. $A^{(i)}$ and A^0 are $\mathfrak{R}\times\mathfrak{R}$ matrices. The modification in is Equation (12) such that

$$h^\gamma Y_m^{(n)} = [y_{n+1}, y_{n+2}, \dots, hy'_{n+1}, hy'_{n+2}, \dots, h^2y'_{n+1}, h^2y'_{n+2}, \dots h^n y_{n+m}]^T \quad (13)$$

$$h^\gamma Y_{m-i}^{n} = \left[y_{n-1}, y_{n-2}, \dots, y_n, hy'_{n-1}, hy'_{n-2}, \dots, hy'_n, h^2y''_{n-1}, h^2y''_{n-2}, \dots h^2y_n'', \dots, h^n y_m^n\right]^T \quad (14)$$

$$F_{m-1}[f_{n-1}, f_{n-2}, \cdots f_n, f_{n+1}, f_{n+2}, \cdots f_{n+m}]^T \quad (15)$$

Hence, solving for $y_{n+i}^{j}, i = 0(1)5, j = 1{,}2$ gives

$$A^0 = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix} \quad (16)$$

$$A^{i} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & \frac{25}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 00 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

$$B^{(i)} = \begin{bmatrix} \frac{131}{480} & \frac{57}{30} & \frac{891}{160} & \frac{172}{15} & \frac{1875}{96} & \frac{75}{96} & \frac{31}{12} & \frac{153}{32} & \frac{21}{3} & \frac{875}{96} & \frac{17}{12} & \frac{25}{12} & \frac{9}{4} & \frac{13}{6} & \frac{25}{12} \\ \frac{-72}{480} & \frac{-24}{30} & \frac{-252}{160} & \frac{-24}{15} & 0 & \frac{-38}{96} & \frac{-10}{12} & \frac{-18}{32} & \frac{2}{3} & \frac{250}{96} & \frac{-7}{12} & \frac{-2}{12} & \frac{3}{4} & \frac{10}{6} & \frac{25}{12} \\ \frac{21}{480} & \frac{7}{30} & \frac{81}{160} & \frac{12}{15} & \frac{125}{96} & \frac{11}{96} & \frac{3}{12} & \frac{9}{32} & \frac{1}{3} & \frac{75}{96} & \frac{2}{12} & \frac{1}{12} & 0 & \frac{1}{6} & \frac{10}{12} \end{bmatrix}^{T} \tag{18}$$

3. Analysis of the Method

According to Lambert [13], the necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable.

3.1 Order of the method

We have used the method proposed in Lambert[13] and Fatunla [14] to obtain the order of our methods as follows; expanding Equation (9) in Taylor series yields $c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0$ And $c_6 = 4$ it was found that all the derived equation give same order with variable error constant.

3.2 Zero Stability of the modified block method

We propose a theorem below for the zero stability of our method.

Theorem 1

Let the order of the differential Equation (1) be μ and the order of the matrices A^0 and A^i be r. If

(1) As $h^{\mu} \to 0, \left|\lambda A^{0} - A^{i}\right| = \lambda^{r-\mu}\left(\lambda - 1\right)^{\mu}$, and

(2) Those roots satisfying $|\lambda| = 1$ have multiplicity not exceeding μ, (the order of the differential equation).

Then the method (2) is zero stable

Proof.

Let A^0 *and* A^i be defined as above. Suppose $h^{\mu} \to 0$, then for

$$\lambda A^{0} - A^{i} = \begin{bmatrix} \lambda & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \lambda & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & \lambda & 0 & -1 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & -\frac{9}{2} \\ 0 & 0 & 0 & \lambda & -1 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 & \lambda-1 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & -\frac{25}{2} \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & -1 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & -1 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda-1 & 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda-1 \end{bmatrix}$$

$$\begin{aligned} \left|\lambda A^{0} - A^{i}\right| &= \lambda^{15} - 3\lambda^{14} + 3\lambda^{13} - \lambda^{12} \\ &= \lambda^{12}\left(\lambda^{3} - 3\lambda^{2} + 3\lambda - 1\right) \\ &= \lambda^{12}\left(\lambda - 1\right)^{3}. \end{aligned}$$

Clearly, those roots satisfying $|\lambda| = 1$ have multiplicity equals 3 and this does not exceed the order of Equation (1).

4. Numerical examples

Problem 1

$$y''' + 4y' = x, \; y(0) = y'(0) = 0, y''(0) = 1, h = 0.1, \; 0 \le x \le 1$$

Theoretical solution: $y(x) = \frac{3}{16}(1 - \cos 2x) + \frac{1}{8}x^2$

Problem 2

$$y''' + y' = 0, \; y(0) = 0, y'(0) = 1, y''(0) = 2, h = 0.1$$

Theoretical solution: $y(x) = 2(1 - \cos x) + \sin x$

x	Exact result	New result Order 6	Error in [14] Order 6	Error in [20] Order 8	Error in New result
0.1	0.0049875166547	0.00498751875	-	1.6654D -08	2.0952D-09
0.2	0.0198010636244	0.01980108	4.4881D-06	3.8095D -07	1.6375D-08
0.3	0.0439995722044	0.04399968375	-	1.5664D -06	1.1154D-07
0.4	0.0768674919974	0.07686848	4.0676D-06	3.9865D -06	9.8800D-07
0.5	0.1174433176497	0.1174402769	-	7.9597D -06	3.0406D-06
0.6	0.1645579210356	0.1645489084	1.3761D-05	1.3680D -05	9.0126D-06
0.7	0.2168811070620	0.21686419474	-	2.1195D -05	1.6965D-05
0.8	0.2729749143149	0.2729481379	3.0465D-05	3.0396D -05	2.6772D-05
0.9	0.3313503927549	0.331312257	-	4.1008D -05	3.8135D-05
1.0	0.3905275318525	0.3904769353	5.2651D-05	5.2605D -05	5.0596D-05

Table 1 for Problem 1

x	Exact result	New result	Error in Awoyemi [15]	Error in Olabode [16]	Error in New result
0.1	0.1098250860907	0.109825087699	-	1.6654D-08	1.6088D-09
0.2	0.2385361751125	0.2385361885	8.8507D-07	3.8095D-07	10387D-08
0.3	0.3847228410128	0.38484725778	-	1.5664D-07	2.9572D-08
0.4	0.5472963543028	0.54729658572	6.6921D-06	3.9865D-06	2.3147D-07
0.5	0.7242604148234	0.724259960	-	7.9597D-06	4.5420D-07
0.6	0.9139712435767	0.9139697789	2.3718D-05	1.3680D-05	1.4746D-06
0.7	1.1145333126687	1.114530439	-	2.1195D-05	2.8734D-06
0.8	1.3239426722051	1.323937959	5.5181D-05	3.0396D-05	4.6826D-06
0.9	1.5401069730861	1.540100051	-	4.1008D-05	6.9217D-06
1.0	1.7608663707162	1.760856775	1.0338D-05	5.2605D-05	9.5974D-06

Table II for problem II

Conclusion

We have proposed a direct method for the solution of general third order ordinary differential equation. This method is self starting and does not need developing separate predictors as proposed in predictor and corrector method. It was found to be more efficient and cost effective than existing method

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