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MAXIMAL SOLUTION OF NON CLASSICAL DIFFERENTIAL EQUATION ASSOCIATED WITH KURZWEIL EQUATIONS

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ABSTRACT

We prove some new results on existence of solutions to first-order non classical ordinary differential equations associated with Kurzweil equations. Our existence results lean on new definitions of lower and upper solutions introduced in this article. The existence of a maximal solution is guaranteed when the local uniqueness property in the future is established. These results could be of great value in applications to the theory of non classical differential equations in locally convex spaces.

KEYWORDS: Differential Equations, Maximal Solution, Kurzweil Equations

1.0 INTRODUCTION

It is well known that, an important technique in the theory of differential equations is concerned with estimating a function satisfying a differential inequality by means of the extremal solutions of the corresponding differential equation. This comparison principle has been widely employed by many authors in studying the qualitative theory of differential equations [8-10]. Problems of extremal solutions of ordinary differential equations have attracted considerable attention in the literature. Some extremal results at the classical setting can be found in the works of [2, 9-11].

If we desire to develop a similar comparison result in the present noncommutative quantum setting in certain locally convex spaces we must first establish existence of maximal and minimal solutions which can then be used to prove comparison results. These results could be of great value in applications to the theory of non classical differential equations in locally convex spaces. In this paper, employing the properties of the Kurzweil equations we prove existence of maximal solutions.

We strongly rely on the formulations of [1]. The results obtained here are generalizations of similar results in [12] concerning classical ordinary differential equations to this present non commutative quantum setting involving unbounded linear operators on a Hilbert space.

The rest of this paper is organized as follows. In section 2 we present some definitions, preliminary results and establish results concerning the local existence and uniqueness of solution of the non classical ordinary differential equation (ODE). In section 3, we present the major results concerning maximal (minimal) solution of QSDE.

2.0 PRELIMINARY RESULTS AND NOTATIONS

In what follows, as in [1, 6-8], we employ the locally convex topological state space $\tilde{A}$ of noncommutative stochastic processes and adopt the definitions and notations of the spaces $Ad(\tilde{A})$, $Ad(\tilde{A})_{wac}$, $L^p_{loc}(\tilde{A})$, $L^\infty_{loc}(\mathbb{R}_+)$, $BV(\tilde{A})$ and the integrator processes Λ_Π, A^+_f, A_g for f , g ∈ $L^\infty_{\gamma, loc}(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\gamma), loc}(\mathbb{R}_+)$, and E, F, G, H lying in $Loc^2_{loc}(I \times \tilde{A})$. For the definitions of the classes of Kurzweil integrable sesquilinear form-valued maps that belong to the following classes $C(\tilde{A} \times [a, b], W)$ and $\mathcal{F}(\tilde{A} \times [a, b], h_{\eta\xi}, W)$ we refer the reader to the references [4, 6].

We consider the following equivalent form of quantum stochastic differential equation (1.1) introduced in [1] given by

$$\frac{d}{dt}\langle\eta, X(t)\xi\rangle = P(X(t), t)(\eta, \xi)$$

$$X(t_0) = X_0\,, t \epsilon\, [t_0, T]\,, \tag{2.1}$$

Where η, ξ lie in some dense subspaces of some Hilbert spaces which has been defined in [4-6]. For the explicit form of the map $(x, t) \to P(x, t)(\,\eta, \xi\,)$ appearing in equation (2.1), see [4-6]. Equation (1.3) is a first order non-classical ordinary differential equation with a sesquilinear form valued map P as the right hand side.

In [1], the equivalence of the non-classical ordinary differential equation (2.1) with the associated Kurzweil equation

$$\frac{d}{d\tau}\langle\eta, X(\tau)\xi\rangle = DF(X(\tau), t)(\eta, \xi)\;,\; t \,\epsilon\, [t_0, T] \tag{2.2}$$

was established along with some numerical approximations. The map F in (1.2) is given by

$$F(x, t)(\eta, \xi) = \int_0^t P\,(x, s)(\eta, \xi)ds \tag{2.3}$$

Definition 1

- Let $h_{\eta\xi}\,:\,[t_0, T] \to \mathbb{R}$ be a non decreasing function defined and continuous from the left on $[t_0, T]$, and let $W\colon [0, +\infty) \to \mathbb{R}$ be a continuous, increasing function with $W(0) = 0$.
- Let $BV(\hat{\mathcal{A}}) \cap Ad(\hat{\mathcal{A}})_{wac} \coloneqq A$ denote the space of stochastic process that are weakly absolutely continuous and of bounded variation.
- Denote by $\mathcal{F}$ the class $\mathcal{F}\big(\hat{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W\big)$

The following result has been established in [6].

2.1 Lemma: If $x\colon [\alpha, \beta) \to \hat{\mathcal{A}}$ is a solution of (2.2) and the map $F: \hat{\mathcal{A}} \times [t_0, T] \to Sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ satisfies the condition (1.1) in [3] then

$$x(s+) - x(s) = \lim_{\sigma\to s+} x(\sigma) - x(s) = F(x(s), s+)(\eta, \xi) - F(x(s), s)(\eta, \xi)$$

for $s \in [\alpha, \beta)$ and

$x(s) - x(s-) = x(s) - \lim_{\sigma\to s-} x(\sigma) = F(x(s), s)(\eta, \xi) - F(x(s), s-)(\eta, \xi)$ for $s \in (\alpha, \beta]$ where $F(x, s+)(\eta, \xi) = \lim_{\sigma\to s+} F(x, \sigma)(\eta, \xi)$ for $s \in [\alpha, \beta)$ and $F(x, s-)(\eta, \xi) = \lim_{\sigma\to s-} F(x, \sigma)(\eta, \xi)$ for $s \in (\alpha, \beta]$.

Next we consider the case when the solution has to be a function of bounded variation which in this case is continuous from the left and has discontinuity of the first kind given by Lemma 2.1. That is if for some $t_0 \in (a, b)$ the value of the solution x of (2.2) is $x(t_0) = \tilde{x}$ then the right limit at the point t_0 fulfils

$$x(t_0+) = x(t_0) + F(x(t_0), t_0+) - F(x(t_0), t_0) =$$

$$= \tilde{x} + F(\tilde{x}, t_0+) - F(\tilde{x}, t_0) \tag{2.4}$$

Because of the possible discontinuities of a solution it can happen that for some $\tilde{x} \in A$, that is for some

$(\tilde{x}, t_0) \in A \times (a, b)$, the value (2.4) does not belong to $A \times (a, b)$ and this means that the corresponding solution x with $x(t_0) = \tilde{x}$ jumps off the open set $A \times (a, b)$ at the moment t_0 and cannot be continued for $t > t_0$.

Therefore, to prove a local existence theorem for a solution of (1.2) satisfying the initial condition

$$\tilde{x}_{\eta\xi} += \tilde{x}_{\eta\xi} + F(\tilde{x}, t_0 +)(\eta, \xi) - F(\tilde{x}, t_0)(\eta, \xi) \in A(\eta, \xi) \tag{2.5}$$

The next theorem establishes the local existence of solution for equation (2.2).

Theorem 2.2: Assume that the following hold;

- The map $(x, t) \rightarrow F(x, t)(\eta, \xi)$ is of class $\mathcal{F}\left(\hat{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W\right)$
- $(\tilde{x}, t_0) \in A \times [t_0, T]$ is such that (2.4) is satisfied.

Then there exist $\delta^-, \delta^+ > 0$ such that on the interval $[t_0 - \delta^-, t_0 + \delta^+]$ there exists a solution $x: [t_0 - \delta^-, t_0 + \delta^+] \rightarrow \hat{\mathcal{A}}$ of the equation (2.2) for which $x(t_0) = \tilde{x}$.

Proof: The proof is a simple adaptation of arguments employed in Theorem 3.1 in [3] to the present non commutative quantum setting.

Next we establish some conditions that guarantee the existence of a maximal solution.

Since we assumed that the sesquilinear form valued map F is of class $\mathcal{F}$ and that the function $h_{\eta\xi}$ is continuous from the left, a solution of the Kurzweil equation associated with QSDE (2.1) can be in general continued to the right for increasing values of the independent variable.

If the local uniqueness of a solution for increasing values of the independent variable is ensured with the condition that $W: [0, +\infty) \rightarrow \mathbb{R}$ is continuous, nondecreasing, W(r) > 0, for r > 0 and W(0) = 0, then a unique forward maximal solution of the equation (2.2) can be defined when an initial condition $x(t_0) = \tilde{x}$ is prescribed for some $t_0 \in (a, b), (a, b) \subseteq [t_0, T]$ and $\tilde{x} \in A$. It is obvious that a maximal forward solution can be defined only if $(\tilde{x}, t_0) \in A \times [t_0, T]$, i.e. if $\tilde{x} + F(x, t +) - F(x, t) \in A$ because otherwise for possible solution x it can happen that $x(t) \notin A$ for $t > t_0$ and this would contradict the definition of a solution. Assume therefore that (2.4) $\in A$ for every $x \in A$, $t \in (a, b)$, this implies that there are no points in $A \times [t_0, T]$ from which the solution of (2.2) can jump off the space A.

Definition

- Let $x: [t_0, t_0 + \beta] \rightarrow \hat{\mathcal{A}}, \beta > 0$ be a solution of (2.2) on $[t_0, t_0 + \beta]$. The solution $y: \mathrm{I} \rightarrow \hat{\mathcal{A}}$ of (2.2) where $I = [t_0, t_0 + \delta]$ or $I = [t_0, t_0 + \delta), \delta > 0$ is called prolongation of x if $[t_0, t_0 + \beta] \subset \mathrm{I}$ and $x(t) = y(t)$ for $t \in [t_0, t_0 + \beta]$. If $[t_0, t_0 + \beta] \subsetneq \mathrm{I}$, i.e. $\delta > \beta$ then I is called a proper prolongation of x to the right.
- If $(\tilde{x}, t_0) \in A \times [t_0, T]$ then a solution of (2.2) with $x(t_0) = \tilde{x}$ defined for $t \geq t_0$ is maximal if there is a value $b\left(\tilde{x}_{\eta\xi}, t_0\right) > \sigma, \sigma \geq t_0$ such that x exists on $[t_0, b\left(\tilde{x}_{\eta\xi}, t_0\right))$ and cannot be prolonged to a larger interval of the form $[t_0, \beta]$ where $\beta_{\eta\xi} \geq b\left(\tilde{x}_{\eta\xi}, t_0\right)$, or alternatively, there is no proper prolongation to the right of the solution $x: [t_0, b\left(\tilde{x}_{\eta\xi}, t_0\right)) \rightarrow \hat{\mathcal{A}}$ of (2.2).

3.0 MAJOR RESULTS

3.1 Proposition: Assume that the map $(x, t) \rightarrow F(x, t)(\eta, \xi)$ is of class $\mathcal{F}$ and $(\tilde{x}, t_0) \in A \times [t_0, T]$. If equation (2.2) has the local uniqueness property in the future then there exists an interval J with the left end point t_0 and a function $x: \mathrm{J} \rightarrow \hat{\mathcal{A}}$

such that $t_0 \in \mathrm{J}$, $x(t_0) = \tilde{x}$ and $x\colon \mathrm{J} \to \hat{\mathcal{A}}$ is a maximal solution of (2.2). The interval J and the function *x* are uniquely defined by the initial condition $x(t_0) = \tilde{x}$ and the maximality property of the solution.

Proof: Assume that $x_1\colon \mathrm{J}_1 \to \hat{\mathcal{A}}$, $x_2\colon \mathrm{J}_2 \to \hat{\mathcal{A}}$ are two maximal solutions of (2.2) with $x_1(t_0) = x_2(t_0) = \tilde{x}$. The local uniqueness property implies $x_1(t) = x_2(t)$ for every $t \in \mathrm{J}_1 \cap \mathrm{J}_2 \cap [t_0, +\infty)$. Define $x(t) = x_1(t)$ for $t \in \mathrm{J}_1$ and $x(t) = x_2(t)$ for $t \in \mathrm{J}_2$. Then $x\colon \mathrm{J}_1 \cup \mathrm{J}_2 \to \hat{\mathcal{A}}$ is a solution of (2.2) on $\mathrm{J}_1 \cup \mathrm{J}_2$. Since we assumed that the solutions x_1, x_2 are maximal, we have $\mathrm{J}_1 = \mathrm{J}_2 = \mathrm{J}$ and $x_1(t) = x_2(t)$ for $t \in \mathrm{J}$. Hence the maximal solution *x* is unique.

Next we show that a solution $x\colon \mathrm{J} \to \hat{\mathcal{A}}$ exists. Denote by $\mathcal{S}$ the set of all solutions $x\colon \mathrm{J} \to \hat{\mathcal{A}}$ of (2.2) with $x(t_0) = \tilde{x}$ and the interval of definition J for which t_0 is the left endpoint of J_x. The set $\mathcal{S}$ is nonempty by the local existence of a solution given in Theorem (2.2). Define $\mathrm{J} = \bigcap_{x \in \mathcal{S}} \mathrm{J}_x$ If $\mathrm{t} \in \mathrm{J}_z \cap \mathrm{J}_y$ where $\mathrm{y}, \mathrm{z} \in \mathcal{S}$ then *z*(t) = *y*(t) by the assumption of the uniqueness. Hence if we define $x\colon \mathrm{J} \to \hat{\mathcal{A}}$ by the relation *x*(t) = *y*(t) where $\mathrm{y} \in \mathcal{S}$ and $\mathrm{t} \in \mathrm{J}_y$ we obtain a solution of (2.2) defined on J which satisfies the initial condition $x(t_0) = \tilde{x}$. Hence from the definition of J we can see that $x\colon \mathrm{J} \to \hat{\mathcal{A}}$ is a solution of (2.2).

3.2 Proposition: Assume that the map $(x,t) \to F(x,t)(\eta,\xi)$ is of class $\mathcal{F}$ and $(\tilde{x}, t_0) \in A \times [t_0, T]$. Assume that (2.2) has the local uniqueness property in the future. Let $x\colon \mathrm{J} \to \hat{\mathcal{A}}$ be a maximal solution of (2.2) with $x(t_0) = \tilde{x}$ where $t_0 \in \mathrm{J}$ is the left end point of the interval J. Then $[t_0, \beta) \cap (a,b), t_0 < \beta \leq +\infty$.

Proof: It is clear that for the maximal interval J we have $\mathrm{J} \in (a,b)$. Let $t^* \in \mathrm{J}$. Take $y = x(t^*) \in A$. Theorem 2.2 yields the existence of a $\delta > 0$ such that on $[t^*, t^* + \delta]$ there is a solution $v_{\eta\xi}\colon [t^*, t^* + \delta] \to A(\eta,\xi)$ of (2.2) such that $v_{\eta\xi}(t^*) = y_{\eta\xi}$. The point (y, t^*) is a point of local uniqueness in the future and therefore *x* is a prolongation of v and $[t^*, t^* + \delta] \subset \mathrm{J}$. This means that relatively to (a, b) the interval J is open at its right endpoint and the statement holds true.

3.3 Proposition: Assume that the map $(x,t) \to F(x,t)(\eta,\xi)$ is of class $\mathcal{F}$ and $(\tilde{x}, t_0) \in A \times [t_0, T]$. Assume that equation (2.2) has the local uniqueness property in the future. Let $x\colon [t_0, \beta] \to \hat{\mathcal{A}}$ be a maximal solution of (2.2) and let $M \subset A(\eta,\xi) \times [t_0, T] \coloneqq \{\langle \eta, y(t)\xi \rangle \colon y \in A, t \in [t_0, T]\}$ is compact in $\mathbb{C}$. Then there exists $c \in [t_0, \beta)$ such that $(x(t), t) \notin M$ for $t \in (c, \beta)$.

Proof: Assume the contrary that the statement does not hold. Then there is a sequence $t_k \in [t_0, \beta), k \in \mathbb{N}$ such that $\lim_{k\to\infty} x(t_k) = y$ and $(x(t_k), t_k) \in M, k \in \mathbb{N}$. Since M is assumed compact and $b < +\infty$, the sequence $(x(t_k), t_k)_{k\in\mathbb{N}}$ contains convergent subsequence which we denote again by $(x(t_k), t_k)_{k\in\mathbb{N}}$. Then $\lim_{k\to\infty} x_{\eta\xi}(t_k) = y_{\eta\xi}$ and $\left(y_{\eta\xi}, \beta\right) \in M \subset A(\eta,\xi) \times [t_0, T]$. By Theorem 2.2 there exists a δ > 0 such that on $[\beta, \beta + \delta]$ there is a solution v of (2.2) with $\mathrm{v}(\beta) = y$. Define $\mathrm{u}\colon [t_0, \beta + \delta] \to \hat{\mathcal{A}}$ by

$$u(t) = x(t), t \in [t_0, \beta), u(t) = v(t), t \in [\beta, \beta + \delta].$$

Now assume that $s_1 \in [t_0, \beta)$ and $s_2 \in [\beta, \beta + \delta]$. Then for $k \in \mathbb{N}$ sufficiently large we have $t_k \in (s_1, \beta)$ and

$$\int_{s_1}^{s_2} DF(u(\tau),t)(\eta,\xi) = \int_{s_1}^{\beta} DF(u(\tau),t)(\eta,\xi) + \int_{\beta}^{s_2} DF(u(\tau),t)(\eta,\xi) =$$

$$= \int_{s_1}^{t_k} DF(u(\tau),t)(\eta,\xi) + \int_{t_k}^{\beta} DF(u(\tau),t)(\eta,\xi) + \langle \eta, (u(s_2) - u(\beta))\xi \rangle$$

$$= \langle \eta, u(t_k)\xi \rangle - \langle \eta, u(s_1)\xi \rangle + \int_{t_k}^{\beta} DF(u(\tau),t)(\eta,\xi) + \langle \eta, u(s_2)\xi \rangle - \langle \eta, u(\beta)\xi \rangle$$

$$= \langle \eta, x(t_k)\xi \rangle - \langle \eta, u(s_1)\xi \rangle + \langle \eta, u(s_2)\xi \rangle - \langle \eta, y\xi \rangle + \int_{t_k}^{\beta} DF(u(\tau),t)(\eta,\xi) \tag{3.1}$$

By Lemma 2.1 in [3]

$$\left|\int_{t_k}^{\beta} DF(u(\tau),t)(\eta,\xi)\right| \leq h_{\eta\xi}(\beta) - h_{\eta\xi}(t_k)$$

Since h is continuous from the left and $\lim_{k\to\infty} t_k = \beta, t_k \leq \beta,$ we have

$$\lim_{k\to\infty} \int_{t_k}^{\beta} DF(u(\tau),t)(\eta,\xi) = 0$$

Using this and the $\lim_{k\to\infty} x(\,t_k) = y$ we take k→∞ in (3.1) and obtain

$$\int_{s_1}^{s_2} DF(u(\tau),t)(\eta,\xi) = \langle \eta, u(s_2)\xi\rangle + \langle \eta, u(s_1)\xi\rangle.$$

For all other possible positions of $s_1, s_2 \in [t_0, \beta+\delta]$ we obtain the same relation directly from the definition of u. In this way we obtain $u: [t_0, \beta+\delta] \to \hat{\mathcal{A}}$ is a solution of (2.2) on $[t_0, \beta+\delta]$ which is a proper prolongation of the solution *x* which is assumed to be maximal. This is again a contradiction and hence the result is proved.

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