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ON A SURVEY OF UNIFORM INTEGRABILITY OF SEQUENCES OF RANDOM VARIABLES

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ABSTRACT: *This paper presents explicitly a survey of uniformly integrable sequences of random variables. We also study extensively several cases and conditions required for uniform integrability, with the establishment of some new conditions needed for the generalization of the earlier results obtained by many scholars and researchers, noting the links between uniform integrability and pointwise convergence of a class of polynomial functions on conditional based.*

KEYWORDS: Uniform Integrability, Sequences, Boundedness, Convergence, Monotonicity.

INTRODUCTION

Uniform integrability is an important concept in functional analysis, real analysis, measure theory, probability theory, and plays a central role in the area of limit theorems in probability theory and martingale theory. Conditions of independence and identical distribution of random variables are basic in historic results due to Bernoulli, Borel and A.N. Kolmogorov[1]. Since then, serious attempts have been made to relax these strong conditions; for example, independence has been relaxed to pairwise independence.

In order to relax the identical distribution condition, several other conditions have been considered, such as stochastic domination by an integrable random variable or uniform integrability in the case of weak law of large number. Landers and Rogge [2] prove that the uniform integrability condition is sufficient for a sequence of pairwise independence random variables in verifying the weak law of large numbers.

Chandra [3] obtains the weak law of large numbers under a new condition which is weaker than uniform integrability: the condition of Cesáro uniform integrability. Cabrera [4], by studying the weak convergence for weighted sums of variables introduces the condition of uniform integrability concerning the weights, which is weaker than uniform integrability, and leads to Cesáro uniform integrabilty as a particular case. Under this condition, a weak law of large numbers for weighted sums of pairwise independent random variables is obtained; this condition of pairwise independence can also be dropped at the price of slightly strengthening the conditions of the weights.

Chandra and Goswami [5] introduce the condition of Cesáro α-integrability ($\alpha > 0$), and show that Cesáro α-integrability for any $\alpha > 0$ is weaker than uniform integrability. Under the Cesáro α- integrability condition for some $\alpha > 1/2$, they obtain the weak law of large numbers for sequences of pairwise independence random variables. They also prove that Cesáro α-integrability for appropriate α is also sufficient for the weak law of large numbers to hold for certain special dependent sequences of random variables and h-integrability which is weaker than all these was later introduced by Cabrera [4].

As an application, the notion of uniform integrability plays central role in establishing weak law of large number. The new condition; Cesáro uniform integrability, introduced by Chandra [3] can be used in cases to prove L^p–convergence of sequences of pairwise independent random variables, and in the study of convergence of Martingales. Other areas of application include the approximation of Green's functions of some degenerate elliptic operators as shown by Mohammed [6].

Definition 1 The random variables $X_1, X_2, X_3, \dots, X_n$ are independent if:

$$\mathbb{P}\left(\bigcap_{j=1}^{n}\{X_j \in A_j\}\right) = \prod_{j=1}^{n}\mathbb{P}(X_j \in A_j)$$

$\forall n - tuples$ of measurable sets $(A_1, A, A_3, \dots, A_n)$. A family of random variables $\{X_i\}_{i\in j}$ is independent if for each finite set $J \subset I$, the family $\{X_i\}_{i\in j}$ is independent.

Definition 2 A family of random variables $\{X_i\}$, $i \epsilon J$ is pairwise independent if X_i and X_j are independent whenever $i \neq j$.

Definition 3 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A sequence of random variables $\{X_n\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to converge in probability to a random variable X if for every $\epsilon > 0$, $\mathbb{P}(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$. This is expressed as : $X_n \xrightarrow{p} X$

Definition 4. A sequence of random variables $\{X_n\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to converge in L^p to a random variable X if $X_n \in L^p$ for all n, $X \in L^p$, and $\mathbb{E}|X_n - X|^p \to 0$ as $n \to \infty$. This is expressed as : $X_n \xrightarrow{L^p} X$

Definition 5. Sequence of real valued random variables $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable if and only if, for any $\epsilon > 0$, $\exists\, K > 0$ such that

$$\mathbb{E}\left[|X_n|\, I_{(|X_n| > K)}\right] \leq \epsilon$$

where $I_{(|X_n|>K)}$ is an indicator function of the event $\{|X_n| > K\}$ i.e, the function which is equal one for $\{|X_n| > K\}$ and zero otherwise, and $\mathbb{E}[(\cdot)]$ is an expectation operator. Expectation values are given by integrals for continuous random variables.

NOTION OF USEFUL INEQUALITIES AND RESULTS OF UNIFORM INTEGRABILITY

In this section, we introduce the basic inequalities and results needed for the conditions and applicability of uniformly integrable sequences of random variables.

Markov Inequality

If Y is a random variable such that $\mathbb{E}|Y|^p < \infty$ for some positive real number p which may or may not be a whole number, then for any $\epsilon > 0$,

$\mathbb{P}\{|Y| \geq \epsilon\} \leq \epsilon^{-p}\mathbb{E}|Y|^p.$

Chebyshev's Inequality

Let X be a random variable with finite mean μ and finite variance σ^2. Then

$$\mathbb{P}\{|X-\mu\,| \geq \epsilon\} \leq \frac{\mathbb{E}|X-\mu|^2}{\epsilon^2}$$

This follows immediately by putting $Y = X - \mu$ and $p = 2$ in the Markov's inequality above.

Chebyshev's Sum Inequality

If $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5 \geq \cdots \geq b_n$, then

$$n\sum_{k=1}^{n} a_k b_k \geq \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

Proof: Assume $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq b_3 \geq b_4 \geq \cdots \geq b_n$, then by rearrangement of inequality, we have that:

$$a_1b_1 + \cdots + a_nb_n = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

$$a_1b_1 + \cdots + a_nb_n \geq a_1b_2 + a_2b_3 + \cdots + a_nb_1$$

$$a_1b_1 + \cdots + a_nb_n \geq a_1b_3 + a_2b_4 + \cdots + a_nb_2$$

$$a_1b_1 + \cdots + a_nb_n \geq a_1b_n + a_2b_1 + \cdots + a_nb_{n-1}$$

Now adding these n inequalities gives:

$$n(a_1b_1 + \cdots + a_nb_n) \geq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)$$

Hence, $\quad n\sum_{k=1}^{n} a_k b_k \geq (\sum_{k=1}^{n} a_k\,)(\sum_{k=1}^{n} b_k\,) \quad \boxdot$

RESULTS OF UNIFORM INTEGRABILITY

Lemma 2.1 Uniform integrability implies L^1- boundedness

Let $\{X_n : n \in \mathbb{N}\}$ be uniformly integrable. Then $\{X_n : n \in \mathbb{N}\}$ is L^1- bounded.

Proof: Choose K so large such that $\sup \mathbb{E}[|X_n|; \{|X_n| > K\}] \leq 1$.

Then $\mathbb{E}|X_n| = \mathbb{E}[|X_n| : \{|X_n| \leq K\}] + \mathbb{E}[|X_n| : \{|X_n| > K\}] \leq K + 1$ $\boxdot$

Remark: The converse of Lemma 2.1 is not true, i.e boundedness in L^1 is not enough for uniform integrability. For a counter example, we present the following:

Let U be uniformly distributed on $[o, 1]$ such that:

$$X_n(u) = \begin{cases} n \text{ ,} for\ u \leq \dfrac{1}{n} \\ 0 \text{ ,} otherwise \end{cases}$$

Then $\mathbb{E}[X_n] = n\frac{1}{n} = 1$, so $\{X_n\}$ is L^1 –bounded. But for $K > 0$,

$\mathbb{E}\left[|X_n| I_{(|X_n| > K)}\right] = 1, \forall n \geq K$, so $(|X_n|)_{n \in \mathbb{N}}$ is not uniformly integrable.

Theorem 2.2 Let X be a random variable and $\{X_n : n \in \mathbb{N}\}$ be a sequence of random variables. Then the following are equivalent:

i) $X_n \in L^1 \ \forall n, X \in L^1\ and\ X_n \to X\ in\ L^1$

ii) $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable and $X_n \to X$ in probability.

Proof: Suppose i) holds. By Chebychev's inequality, for $\epsilon > 0$,

$$\mathbb{P}\{|X_n - X\ | > \epsilon\} \leq \epsilon^{-1} \mathbb{E}(|X_n - X|) \to 0$$

So $X_n \to X$ in probability. Moreover, given $\epsilon > 0$, there exists N such that $\mathbb{E}(|X_n - X|) < \frac{\epsilon}{2}$

whenever $n \geq N$. Then we can find $\delta > 0$ so that $\mathbb{P}(A) \leq \delta$ implies $\mathbb{E}(|X| 1_A) < \frac{\epsilon}{2}$,

$\mathbb{E}(|X| 1_A) < \epsilon$, $n = 1,2,3,4, \ldots, N$. Then for $n \geq N$ and $\mathbb{P}(A) \leq \delta$,

$\mathbb{E}(|X|1_A) \leq \mathbb{E}(|X_n - X|) + \mathbb{E}(|X|1_A) \leq \epsilon.$

Hence, $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable. We have shown that i) implies ii).

Suppose on the other hand that ii) holds, then there is a subsequence (n_k) such that $X_{nk} \to X$ almost surely. So, by Fatou's Lemma,

$\mathbb{E}(|X|) \leq \lim_k inf \mathbb{E}(|X_{nk}|) < \infty.$

Now, given $\epsilon > 0$, there exists $K < \infty$ such that, for all n,

$\mathbb{E}\left(|X_n| 1_{(|X_n| \geq K)}\right) < \frac{\epsilon}{3},\ \mathbb{E}\left(|X| 1_{(|X_n| \geq K)}\right) < \frac{\epsilon}{3}.$

Consider the uniformly bounded sequence $X_n^K = (-K) \wedge X_n \vee K$ and set $X^K = (-K) \wedge X \vee K$. Then $X_n^K \to X^K$ in probability, so by bounded convergence, there exists N such that, for all $n \geq N$, $\mathbb{E}(|X_n^K - X^K|) < \frac{\epsilon}{3}$. But then, for $n \geq N$,

$\mathbb{E}(|X_n - X|) \leq \mathbb{E}\left(|X_n| 1_{(|X_n| \geq K)}\right) + \mathbb{E}(|X_n^K - X^K|) + \mathbb{E}\left(|X| 1_{(|X| \geq K)}\right) < \epsilon .$

Therefore, ii) implies i) since $\epsilon > 0$ was arbitrary $\boxdot$

Other authors like [7], [8], [9] have also shown that uniform integrability of functions were related to the sums of random variables.

UNIFORM INTEGRABILITY OF A CLASS OF POLYNOMIALS ON A UNIT INTERVAL

Let $\mathbb{P}_{n,K}(X) : [0,1] \longrightarrow [0,1]$ denote the probability of exactly k successes in an n independent Bernoulli trials with probability x success by any trial. In other words:

$\mathbb{P}_{n,K}(X) = \mathbb{P}_r\{b(n,x) = K\} = \binom{n}{k} x^k (1-k)^{n-k}$

and, for integers $r, s \geq 1$, we define the family of functions $\{S_{n,r,s}\}_{n=1}^{\infty}$ by

$$S_{n,r,s}(x) = \sqrt{n} \sum_{k=0}^{n} \mathbb{P}_{rn,rk}(x) \mathbb{P}_{sn,sk}(x) \qquad (1)$$

The family of polynomials arise in the context of statistical density estimation based on Bernstein polynomials. Specifically, the case $r = s = 1$ has been considered by many authors [10], [11], & [12] while the case $r = 1$ and $s = 2$ was considered by [13]. These same authors have considered issues linked to uniform integrability and pointwise convergence of $\{S_{n,1,1}\}$ and $\{S_{n,1,2}\}$. However, the generalization to any $r, s \geq 1$ has not been considered. In this section, we will establish the following results.

Theorem 3.1 Let r, s be fixed positive integers. Then

i) $0 \leq S_{n,r,s}(x) \leq \sqrt{n}$ for $x \in [0,1]$ and $S_{n,r,s}(0) = S_{n,r,s}(1) = \sqrt{n}$

ii) $\{S_{n,r,s}\}_{n=0}^{\infty}$ is uniformly integrable on $[0,1]$.

iii) $S_{n,r,s}(x) \to \frac{\gcd(r,s)}{\sqrt{rs(r+s)2\pi x(1-x)}}$ for $x \in (0,1)$ as $n \to \infty$.

For the case $r = s = 1$, Babu et al [10, Lemma 3.1] contains the proof of iii). Leblanc et al [13,Lemma 3.1] considered when $r = 1$ and $s = 2$. The proof here generalizes (but follows the same line as) these previous results.

In establishing Theorem 3.1, we first show that for all $0 \leq K \leq n$ and $x \in [0,1]$,

$$\mathbb{P}_{n,r}(x) \geq \mathbb{P}_{2n,2r}(x) \geq \mathbb{P}_{3n,3r}(x) \geq \cdots \qquad (2)$$

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The proof of this inequality is based on a class of completely monotonic functions and hence of general interest [14]. Using completely different methods, Leblanc and Johnson [13] previously showed that $\{\mathbb{P}_{2^j n, 2^j k}(x)\}_{j=0}^{\infty}$ is decreasing in j and hence, (2) is a generalization of the earlier result.

Lemma 3.1 Let $\{a_k\}_{k=1}^{m}$ and $\{b_k\}_{k=1}^{m}$ be real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_m$ and let φ denote the digamma function. Define

$$\emptyset_\delta(x) := \sum_{k=1}^{m} a_k \varphi(b_k x + \delta), \quad x > 0, \delta \geq 0.$$

If $\delta \geq \frac{1}{2}$ and $\sum_{k=1}^{m} a_k \geq 0$, then $\emptyset'_\delta$ is completely monotonic on $(0, \infty)$ and hence $\emptyset_\delta$ is increasing and concave on $(0, \infty)$, see **[14] & [15]** .

Proof: Let $x > 0$, and $\delta \geq \frac{1}{2}$. Then the integral representation of $\varphi^{(n)}(x)$ is:

$$\varphi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1 - e^{-t}} dt, \qquad n = 1,2, \ldots$$

Therefore, for $n = 1,2, \ldots,$

$$(-1)^{n+1} \emptyset^{(n)}{}_\delta(x) = (-1)^{n+1} \sum_{k=1}^{m} a_k b^n{}_k \varphi^{(n)}(b_k x + \delta)$$

$$= \sum_{k=1}^{m} a_k \int_0^\infty \frac{(b_k t)^n e^{-x b_k t}}{e^{-ti\theta}(1 - e^{-t})} dt \tag{3}$$

The assumption $u = b_k t$ yields

$$(-1)^{n+1} \emptyset^{(n)}{}_\delta(x) = \int_0^\infty u^{n-1} e^{-ux} \sum_{k=1}^{m} a_k \eta\left(\frac{u}{b_k}\right) du. \tag{4}$$

where $\eta(x) = xe^{-\delta x}(1-e^{-x})^{-1} > 0$. Calculus shows that, for $\delta \geq \frac{1}{2}$, η is strictly decreasing on $(0,\infty)$ and hence, for every $u > 0$, $\left\{\eta\left(\frac{u}{b_k}\right)\right\}_{k=1}^{m}$ is decreasing [we note that, if $b_k = 0$, there is no difficulty in taking $\eta\left(\frac{u}{b_k}\right) = \eta(\infty) = \lim_{x\to\infty}\eta(x) = 0$, since these terms vanish in (3)]. Since $\{a_k\}_{k=1}^{m}$ is also decreasing, Chebyshev's inequality for sums yields:

$$\sum_{k=1}^{m} a_k\,\eta\left(\frac{u}{b_k}\right) \geq \frac{1}{m}\left(\sum_{k=1}^{m} a_k\right)\left(\sum_{k=1}^{m}\eta\left(\frac{u}{b_k}\right)\right)$$

We see that, if $\sum_{k=1}^{m} a_k \geq 0$, the integrand in (4) is non-negative and hence $(-1)^{n+1}\emptyset^{(n)}{}_{\delta} \geq 0$ on $(0,\infty)$. We conclude that $\emptyset'_{\delta}$ is completely monotonic on $(0,\infty)$ and, in particular, $\emptyset_{\delta}$ is increasing and concave on $(0,\infty)$ whenever $\delta \geq \frac{1}{2}$ and $\sum_{k=1}^{m} a_k \geq 0$. $\boxdot$

Lemma 3.2 [14] Let n, k, j be integers such that $0 \leq k \leq n$ and $j \geq 1$and define

$$Q_{n,k}(j) = \frac{\binom{(j-1)n}{(j-1)k}}{\binom{jn}{jk}} = \frac{\Gamma((j-1)n+1)\Gamma(j(n-k)+1)}{\Gamma(jn+1)\Gamma((j-1)k+1)\Gamma((j-1)(n-k)+1)} \quad .$$

Then $Q_{n,k}(j)$ is decreasing in j and

$$\lim_{j\to\infty} Q_{n,k}(j) = \left(\frac{k}{n}\right)^{k}\left(\frac{n-k}{n}\right)^{n-k}.$$

Proof: The limit is easily verified using Stirling's formula, thus we need only show that $Q_{n,k}(j)$ is decreasing in j. Treating $Q_{n,k}(j)$ as a continuous function in j and differentiating we obtain:

$$Q'_{n,k}(j) = Q_{n,k}(j)\{k(q_j(k) - q_j(n)) + (n-k)(q_j(n-k) - q_j(n))\}$$

where $q_j(x) = \varphi(jx+1) - \varphi(jx - x + 1)$. Now, taking $\delta = 1,\ a_1 = 1,\ a_2 = -1,$ an $b_1 = j$, and $b_2 = j - 1$ in Lemma 3.1, we have that $q_j(x)$ is increasing on $(0, \infty)$ and hence $Q'_{n,k}(j) \leq 0$ for all $j \geq 1$ since $Q_{n,k}(j) > 0$ always $\boxdot$

Corollary 3.1 Let $0 \leq k \leq n$. Then $\{P_{jn,jk}(x)\}_{j=1}^{\infty}$ is decreasing in j for every fixed $x \in [0,1]$.

Proof: $P_{(j-1)n,(j-1)k}(x) \geq P_{jn,jk}(x)$ if and only if, $Q_{n,k}(j) \geq x^k(1-x)^{n-k}$ and we have, by Lemma 3.2.

$$Q_{n,k}(j) \geq \left(\frac{k}{n}\right)^k \left(\frac{1-k}{n}\right)^{n-k} = \sup_{x \in [0,1]} x^k(1-x)^{n-k}$$

which completes the proof.

Proof of Theorem 3.1. First we note that (i) holds since

$$\begin{aligned}
\sum_{k=0}^{n} P_{rn,rk}(x) P_{sn,sk}(x) &\leq \sum_{k=0}^{n} P_{rn,rk}(x) \\
&\leq \sum_{k=0}^{rn} P_{rn,k}(x) \\
&= P_{rn,0}(x) + P_{rn,1}(x) + P_{rn,2}(x) + \cdots + P_{rn,rn}(x) \\
&= \binom{rn}{0} x^0 (1-x)^{rn-0} + \cdots + \binom{rn}{rn} x^{rn} (1-x)^0 \\
&= [x + (1-x)]^{rn} = 1
\end{aligned}$$

with equality if and only if $x = 0,1$. Similarly, (ii) holds since $\{S_{n,1,1}\}_{n=1}^{\infty}$ is uniformly integrable on $[0,1]$ and, by Corollary 3.1, we have $S_{n,r,s}(x) \leq S_{n,1,1}(x)$ for all $x \in [0,1]$.

To prove (iii), let $U_1, U_2, U_3, \ldots, U_n$ and $V_1, V_2, V_3, \ldots, V_n$ be two sequences of independent random variables such that U_i is Binomial (r, x) and V_i is Binomial (s, x). Now, define

$W_i = r^{-1}U_i - s^{-1}V_i$ so that W_i has a lattice distribution with span $\gcd(r, s)/rs$ [17]. We can write $S_{n,r,s}(x)$ in terms of the W_i as:

$$\frac{S_{n,r,s}(x)}{\sqrt{n}} = \sum_{k=0}^{n} P_{rn,rk}(x) P_{sn,sk}(x) = \mathbb{P}\left(\sum_{i=1}^{n} \frac{U_i}{r} = \sum_{i=1}^{n} \frac{V_i}{s}\right) = \mathbb{P}\left(\sum_{i=1}^{n} W_i = 0\right).$$

Now, define the standardized variable $W^*{}_i = W_i\sqrt{rs}/\sqrt{(r+s)x(1-x)}$ so that $\mathrm{Var}(W^*{}_i) = 1$

and note that these also have a lattice distribution, but with span $\gcd(r, s)/\sqrt{rs(r+s)x(1-x)}$. Theorem 3 of Section XV.5 of Feller [16] leads to

$$\lim_{n\to\infty} \frac{S_{n,r,s}(x)}{\sqrt{n}} = \lim_{n\to\infty} \mathbb{P}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{m} W^*{}_i = 0\right) = \frac{\gcd(r,s)\,\emptyset(0)}{\sqrt{nrs(r+s)x(1-x)}} ,$$

where $\emptyset$ corresponds to the standard normal probability density function. The result now follows from the fact that $\emptyset(0) = \frac{1}{\sqrt{2\pi}}$ $\boxdot$

Remark 3.1 Since $Q_{n,k}(j)$ is decreasing, it is obvious that:

$$\binom{(j-1)n}{(j-1)k}\binom{(j+1)n}{(j+1)k} \geq \binom{jn}{jk}^2$$

And since $Q_{n,k}(j-m+1) \geq Q_{n,k}(j+m)$ for $m = 1,2,3,\ldots,j$. we see that the sequence $\{A_m\}_{m=1}^{j}$ define by:

$$A_m = \binom{(j+m)n}{(j+m)k}\binom{(j-m)n}{(j-m)k} \tag{5}$$

is increasing.
Also, Corollary 3.1 trivially leads to a similar family of inequalities for "number of failure"-negative binomial probabilities. As such, let $H_{n,k}$ be the probability of exactly n failures $(n \geq 0)$ before the kth success $(k \geq 1)$ in a sequence of i.i.d. Bernoulli trials with success probability $p \in [0,1]$ so that, for $j = 1,2,3, \ldots,$

$$H_{jn,jk} = \binom{jn+jk-1}{jk-1} p^{jk}(1-p)^{jn} = \frac{k}{n+k} P_{j(n+k),jk} \quad .$$

Hence, as a direct consequence of Corollary 3.1, we have that $\{H_{jn,jk}\}_{j=1}^{\infty}$ is also decreasing.

As a consequence of Theorem 3.1, we have, for any function f bounded on [0,1],

$$\lim_{n\to\infty} \int_0^1 S_{n,r,s}(x)\, f(x)\, dx = \frac{\gcd(r,s)}{\sqrt{rs(r+s)}} \int_0^1 \frac{f(x)}{\sqrt{2\pi(1-x)}} dx \qquad (6)$$

In particular, Kakizawa [12], establish (6) for the case $r = s = 1$

CONCLUSION

Generally, we conclude by pointing out the usefulness of the results to some other interesting areas such as combinatorial and discrete probability inequalities in terms of monotonicity. In addition, the consequence of Theorem 3.1 is a key tool in assessing the performance of nonparametric density estimators based on Bernstein polynomials.

The work complements previous results in the literature with significance in computational analysis and in applied probability. Also, the result is of special interest in the study of uniform integrability of martingales in terms of pointwise boundedness, and equicontinuity of a certain class of functions.

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