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Continuous implicit method for the solution of general second order ordinary differential equations

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Abstract

Methods of collocation and interpolation were adopted to generate a continuous implicit scheme for the solution of second order ordinary differential equation. Newton polynomial approximation method was used to generate the unknown parameter in the corrector. This enables us to solve both initial and boundary value problems.

Keywords
Collocation, interpolation, continuous, implicit, Newton's polynomial, corrector.

1.0 Introduction

The second order ordinary differential equation of the form

$$y'' = f(x, y, y') \tag{1.1}$$

subject to $y(a) = \eta_0, y'(a) = \eta_1$ is called initial value problem. When the condition is of the form $y(a) = \eta_0, y(b) = y_2$ for $a \leq x \leq b$ it is called a boundary value problem, where f is a continuous function. Scientific and technological problems often lead to mathematical modeling of real life applications such as motion of projectiles or orbiting bodies, population growth, chemical kinetics and economic growth. Differential equation is often used to model the problems and most times these equations do not have analytic solution, hence an approximate numerical method is required to solve the problems. Equations (1.1) is conventionally solved by first reducing it to the system of first order ordinary differential equation and then one applies the various methods available for solving the system of the first order. This approach is extensively discussed in the literature and we cite few examples among others, [1], [2], [3], [4], [5], [6], [7]. Although this approach has tremendous success yet it has certain drawback. For instance, computer programs associated with the method are often complicated especially when the subroutines to supply the starting values for the methods result in longer computer time and more computational work. In addition Vigo –Aguiar and Ramos [8], stated that these methods do not utilize additional information associated with a specific ordinary differential equation, such as oscillatory nature of the solution.

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Block methods for numerical solution of the first order ordinary differential equations have been proposed by several researchers such as in [18, 20, 21 and 27]. Rosser [19] introduced the 3- point implicit block method based on integration formulae which is basically Newton's cote type. Zanariah et al [26], proposed 3 points implicit block method based on Newton's backward divided difference formula.

Considerable attention has been devoted to development of method to solve special second order ordinary differential equation of the type $y'' = f(x, y)$ directly without reducing it to system of first order. For instance [1],[9], [10], [11] and [12] among others. Hairer and Wanner [13] proposed Nystrom type method and stated other conditions for determining the parameter of the method. Chawla and Sharma [14] proposed a method due to Runge kutta method. Method of linear multistep method have been considered by Awoyemi and Kayode [15], and Kayode [23]. These methods are predictor corrector methods, although the implementation of the method in a pc mode yield good accuracy, the procedure is costly to implement. For instance, pc subroutine are very complicated to write, since they require special techniques for supplying the starting values and for varying the step size, which lead to longer time and more human effort. Jator and Li [16], proposed an order 5 method that was implemented without the need for either predictor or starting values from other methods. Jator [6] proposed an order 6 method based on the same method. Adesanya et al. [17] proposed a two step method for the general solution of second order which is self starting and adopt Newton's polynomial to generate the starting value. Awoyemi et al. [24] recently proposed a self starting Numerov method. This method solves both initial and boundary value solution of ordinary differential equation. Yahaya [12] constructed a Numerov method from a quadratic continuous polynomial solution. This process led to method applied to both initial and boundary value problems.

In this work, we propose a block method for three steps. This method adopt Newton's polynomial approximation to generate the starting value and solves both initial and boundary value problems.

2.0 Methodology

We first state the uniqueness theorem for higher order ordinary differential equation with initial value problem

***Theorem* 2.1**

Let

$$u^{n} = f(x,u,u' \ldots\ldots u^{(n-1)}), u^{k}(x_0) = c_k \tag{2.1}$$

$k = 0, 1 \ldots n-1$, where u are scalars

Let R be the region defined by the inequalities $x_0 \le x \le x_0 + a$, $|s_j - c_j| \le b$, $j = 0,1,\ldots, n-1$, $(a > 0, b > 0)$. Suppose $f(x, s_0, \ldots, s_{n-1})$ is defined in R and in addition

(a) F is non negative and non decreasing in each $x, s_0, \ldots s_{n-1}$ in R

(b) $f(x, c_0, \ldots, c_{n-1}) > 0$ for $x_0 \le x \le x_0 + a$

(c) $c_k \ge 0$, $k = 1,2,\ldots,n-1$

Then the initial value problem (1.1) has a unique solution in R. (see Wend [25]) for details). We consider an approximate solution to (1.1) in power series

$$y(x) = \sum_{j=0}^{k} a_j \phi_j(x) \tag{2.2}$$

$\phi_j = \phi^j, a_j, j = 0(1)2k - 1$ are constants to be determined. Consider a linear multistep method of the form

$$y(x) = \sum_{r=0}^{t-1} \phi_r(x) y_{n+r} + h^2 \sum_{r=0}^{m-1} \varphi_r(x) f_{n+r} \quad (2.3)$$

where $x = [x_n, x_{n+r}]$, k = step length, m = the distinct collocation point, t is the interpolation point, for our method, the step length $k = 3$

$$\phi_r(x) = \sum_{i=0}^{i+m-1} \phi_{i+1,r} p_i(x), r = 0,1,2,...., m-1 \quad (2.4)$$

$$h^2\varphi(x) = \sum_{i=0}^{t+m-1} h^2 \varphi_{i+1,r}(x) p_i(x), r=0,1,2...,m-1 \quad (2.5)$$

$$y(x_{n+r}) = y_{n+r}, r \in [0,1,2,...,t-1] \; y''(x) = f_{n+r}, r = 0,1, m-1 \quad (2.6)$$

To get $\phi_j(x)$ and $\varphi_j(x)$. According to Yahaya [12], Onumanyi arrived at matrix of the form $DC = I$, where I is an identity matrix of dimension $(t+m)\times(t+m)$.

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & . & x_n^{t+i-1} \\ 1 & x_{n+1} & x_{n+1}^2 & . & x_{n+1}^{t+i-1} \\ . & . & . & . & . \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & . & x_{n+t+1}^{t+m-1} \\ 0 & 0 & 2 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 2 & . & (t+m-1)(t+m-2)x_{m-2}^{t+m-2} \end{pmatrix}. \quad (2.7)$$

$$c = \begin{pmatrix} \alpha_{1,0} & \alpha_{1,1} & . & \alpha_{1,t-1} & h^2\varphi_{1,0} & . & h^2\varphi_{1,m-1} \\ \alpha_{2,0} & \alpha_{2,1} & . & \alpha_{2,t-1} & h^2\varphi_{2,0} & . & h^2\varphi_{2,m-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \alpha_{t+m,0} & \alpha_{t+m,1} & . & \alpha_{t+m,t-1} & h^2\varphi_{t+m,0} & . & h^2\varphi_{t+m,m-1} \end{pmatrix} \quad (2.8)$$

3.0 Development value for the unknown

***Theorem* 3.1**

Assuming that $f \in c^{n+1}[a,b]$ and $x_k \in [a,b]$ for k=0, 1, n are distinct values, then $f(x) = y(x) + R_n(x)$, where $y(x)$ is a polynomial that can be used to approximate $f(x)$

For Newton's polynomial

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + ... + a_n(x - x_0)(x - x_1)...(x - x_{n-1}) \quad (3.1)$$

$f(x) \cong y(x)$, $R_n(x)$ is the remainder and has the form

$$R_n(x) = \frac{f^{n+1}}{(x_{n+1})!}(x - x_0)(x - x_1)....(x - x_{n-1})(x - x_n) \quad (3.2)$$

(See Awoyemi et al. [24]) for details.

4.0 Development of three steps method

In developing the method with step length k=3, we consider

$$D=\begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 \end{bmatrix} \quad (4.1)$$

This gives

$$\alpha_1 = -t \ , \ \alpha_2 = t+1$$

$$\varphi_0(t) = \frac{h^2}{360}\left(-3t^5 + 10t^3 - 7t\right)$$

$$\varphi_1 = \frac{h^2}{120}\left(3t^5 + 5t^4 - 20t^3 + 22t\right)$$

$$\varphi_2 = \frac{h^2}{120}\left(-3t^5 + 10t^4 + 10t^3 + 60t^2 + 43t\right)$$

$$\varphi_3 = \frac{h^2}{360}\left(3t^5 + 15t^4 + 20t^3 - 8t\right) \quad (4.2)$$

$$\alpha_1' = -1 \ , \ \alpha_2' = 1$$

$$\varphi_0'(t) = \frac{h^2}{360}\left(-15t^4 + 30t^2 - 7\right)$$

$$\varphi_1'(t) = \frac{h^2}{120}\left(15t^4 + 20t^3 - 60t^2 + 22\right)$$

$$\varphi_2'(t) = \frac{h^2}{120}\left(-15t^4 + 40t^3 + 30t^2 + 120t + 43\right)$$

$$\varphi_3'(t) = \frac{h^2}{360}\left(15t^4 + 60t^3 + 60t^2 - 8\right) \quad (4.3)$$

where $t = \frac{x - x_{n+2}}{h}$. Evaluating (4.2) at x_{n+3} i.e. when $t = 1$ and substituting the result in (2.3) gives

$$12y_{n+3} - 24y_{n+2} + 12y_{n+1} = h^2(f_{n+3} + 10f_{n+2} + f_{n+1}) \quad (4.4)$$

(4.4) has order p=4 and error constant $cp^{+2} = -\frac{1}{240}$

Evaluating (4.2) at x_n i.e. when $t = -2$ and substituting the result in (2.3) gives

$$12y_{n+2} - 24y_{n+1} + 12y_n = h^2(f_{n+2} + 10f_{n+1} + f_n) \quad (4.5)$$

(4.5) has order $p = 4$ and error constant $cp^{+2} = -\frac{1}{240}$.

Evaluating (4.3) at $t = -2$ and substituting the result in (2.3) gives

$$360hy_n - 360y_{n+2} + 360y_{n+1} = h^2(-24f_{n+3} + 9f_{n+2} - 414f_{n+1} - 127f_n) \quad (4.6)$$

Evaluating (4.3) at t =-1 and substituting the result in (2.3) gives

$$360hy'_{n+1} - 360y_{n+2} + 360y_{n+1} = h^2(127f_{n+3} - 546f_{n+2} + 471f_{n+1} - 952f_n) \qquad (4.7)$$

Evaluating (4.3) at t = 0 and substituting the result in (2.3) gives

$$360hy'_{n+2} - 360y_{n+2} + 360y_{n+1} = h^2(7f_{n+3} - 66f_{n+2} - 129f_{n+1} + 8f_n) \qquad (4.8)$$

Evaluating (4.3) at t = 1, and substituting the result in (2.3) gives

$$360hy'_{n+3} - 360y_{n+2} + 360y_{n+1} = h^2(-8f_{n+3} + 129f_{n+2} + 66f_{n+1} - 7f_n) \qquad (4.9)$$

Solving (4.3), (4.4) and (4.5) using matrix inversion method, we obtain the block

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix}+\begin{bmatrix}0&0&-1\\0&0&-1\\0&0&-1\end{bmatrix}\begin{bmatrix}y_{n-1}\\y_{n-2}\\y_n\end{bmatrix}=h^2\begin{bmatrix}\frac{19}{60}&\frac{-13}{120}&\frac{1}{15}\\\frac{22}{15}&\frac{2}{15}&\frac{2}{15}\\\frac{27}{10}&\frac{27}{40}&\frac{17}{60}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix}+\frac{h^2}{360}\begin{bmatrix}0&0&\frac{97}{360}\\0&0&\frac{28}{45}\\0&0&\frac{39}{40}\end{bmatrix}\begin{bmatrix}f_{n-1}\\f_{n-2}\\f_n\end{bmatrix} \qquad (4.10)$$

$$-h\begin{bmatrix}0&0&-1\\0&0&-2\\0&0&-3\end{bmatrix}\begin{bmatrix}y'_{n-1}\\y'_{n-2}\\y'_n\end{bmatrix}$$

Hence, from (4.10)

$$y_{n+1} = y_n + \frac{h^2}{360}(114f_{n+1} - 39f_{n+2} + 24f_{n+3} + 97f_n) + hy'_n \qquad (4.11)$$

$$y_{n+2} = y_n + \frac{h^2}{45}(66f_{n+1} + 6f_{n+2} + 6f_{n+3} + 28f_n) + 2hy'_n \qquad (4.12)$$

$$y_{n+3} = y_n + \frac{h^2}{120}(324f_{n+1} + 81f_{n+2} + 34f_{n+3} + 117f_n) + 3hy'_n \qquad (4.13)$$

4.1 Development of the unknown for $k = 3$

Evaluate the first derivative of (3.1), and neglect a_4 and higher values of a i.e. Newton's polynomial of order 4, we obtain

$$12hy'_n = -25y_n + 48y_{n+1} - 36y_{n+2} + 16y_{n+3} - 3y_{n+4} \qquad (4.14)$$

$$12hy'_{n+1} = -3y_n - 10y_{n+1} + 18y_{n+2} - 6y_{n+3} + y_{n+4} \qquad (4.15)$$

$$12hy'_{n+2} = y_n - 8y_{n+1} + 8y_{n+3} - y_{n+4} \qquad (4.16)$$

$$12hy'_{n+3} = -y_n + 6y_{n+1} - 18y_{n+2} + 10y_{n+3} + 3y_{n+4} \qquad (4.17)$$

Making y_{n+4} the subject in (4.14) and substituting into (4.15)-(4.16) and solving for y_{n+1}, y_{n+2} and y_{n+3} gives

$$y_{n+1} = y_n + \frac{h}{72}(57y'_{n+1} - 10y'_{n+2} + 3y'_{n+3} + 27y'_n) \qquad (4.18)$$

$$y_{n+2} = y_n + \frac{h}{9}(127y'_{n+1} + 2y'_{n+2} + 3y'_n) \qquad (4.19)$$

$$y_{n+3} = y_n + \frac{h}{8}(9y'_{n+1} + 6y'_{n+2} + 3y'_{n+3} + 3y'_{n+n}) \qquad (4.20)$$

Make y_{n+1} the subject in (4.5) and then substitute in (4.6)-(4.8) to solve for f_{n+1}, f_{n+2} and f_{n+3}, give

$$f_{n+1} = -f_n + \frac{1}{h}\left(\frac{-3}{2}y'_{n+1} + \frac{3}{2}y'_{n+2} + \frac{1}{6}y'_{n+3} - \frac{1}{6}y'_n\right) \tag{4.21}$$

$$f_{n+2} = 3f_n + \frac{1}{h}\left(3y'_{n+1} - 6y'_{n+2} + \frac{7}{3}y'_{n+3} + \frac{2}{3}y'_n\right) \tag{4.22}$$

$$f_{n+3} = 25f_n + \frac{1}{h}\left(\frac{39}{2}y'_{n+1} - \frac{69}{2}y'_{n+2} + \frac{17}{2}y'_{n+3} + \frac{13}{2}y'_n\right) \tag{4.23}$$

Comparing (4.20)-(4.23) with the second derivative of (3.1) gives

$$y'_{n+1} = \frac{-486}{14417}y'_n - \frac{504}{14417}hf_n \tag{4.24}$$

$$y'_{n+2} = \frac{16974}{14417}y'_n - \frac{1620}{14417}hf_n \tag{4.25}$$

$$y'_{n+3} = \frac{63485}{14417}y'_n - \frac{45228}{14417}hf_n \tag{4.26}$$

5.0 Numerical example

We test the efficiency of our scheme on linear and non linear second order differential equation.

***Problem* 5.1:**

$$y'' - x(y')^2 = 0$$

$$y(0) = 1, y'(0) = \frac{1}{2}, h = 0.1/40$$

Exact solution $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Grid point	Expected result	Calculated result	Error
0.0025	1.00125000065104	1.00125000014116	5.09882D-10
0.0050	1.00250000520835	1.00250000032393	4.88542D-09
0.0075	1.00375001757828	1.00375001139519	6.18308D-09
0.0100	1.00500004166729	1.00500001779432	2.38729D-08
0.0125	1.00625008138212	1.00625002292868	5.84534D-08
0.0150	1.00750014062974	1.00750005100468	8.96250D-08
0.0175	1.00875022331755	1.00875020629768	1.60840D-07
0.0200	1.01000033335333	1.01000007231814	2.61035D-07
0.0225	1.01125047464542	1.01125041188283	3.55817D-07
0.0250	1.01250065110271	0.012500613537337	3.75653D-07

***Problem* 5.2**

$$y'' = 2y^3$$

$$y(0) = 1, y'(0) = -1, h = 0.1/40$$

Exact solution $y(x) = \frac{1}{x}$

Grid point	Expected result	Calculated result	Error
0.0025	0.997506234413965	0.997506800724006	-4.560D-07
0.0050	0.995024875621891	0.995029339533017	1.941D-06
0.0075	0.992555831265509	0.992564223860174	-8.397D-06
0.0100	0.990099000990099	0.990070873966239	2.814D-05
0.0125	0.98765432098754	0.98759291299131	6.140D-05
0.0150	0.985221674876847	0.985127023087557	9.465D-05
0.0175	0.982800982800983	0.982633524787924	1.675D-04
0.0200	0.980392156862745	0.980155071394184	2.371D-04
0.0225	0.97799511002445	0.977688418640018	3.067D-04
0.0250	0.975609756097561	0.975194774137825	4.149D-04

***Problem* 5.3**

$$y'' = y + e^{3x}$$

$$y(0) = \frac{-3}{32}, y'(0) = \frac{-5}{32}, h = 0.1/40$$

Exact solution $y(x) = \dfrac{4x-3}{32\exp(-3x)}$

Grid point	Expected result	Calculated result	Error
0.0025	-0.094140915761848	-0.094140939393182	2.34D-08
0.0050	-0.094532404142338	-0.094532599228254	1.95D-07
0.0075	-0.094924451608388	-0.094924817272551	3.65D-07
0.0100	-0.095317044390700	-0.095317760663383	7.16D-07
0.0125	-0.095710168480980	-0.095710743379670	5.74D-07
0.0150	-0.096103809629113	-0.096109967248178	6.16D-06
0.0175	-0.09649533403163	-0.096494619870395	7.14D-07
0.0200	-0.096892584872264	-0.096896306302397	3.72D-07
0.0225	-0.097289689232184	-0.097285656289237	4.03D-06
0.0250	-0.097683251173919	-0.097685517015441	2.26D-06

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