One-Step Implicit Hybrid Block Method for The Direct Solution of General Second Order Ordinary Differential Equations

Timothy A. Anake, David O. Awoyemi and Adetola O. Adesanya

***Abstract* - A one-step implicit hybrid block solution method for initial value problems of general second order ordinary differential equations has been studied in this paper. The one-step method is augmented by the inclusion of off step points to enable the multistep procedure. This guaranteed zero stability as well as consistency of the resulting method. The convergence and weak stability properties of the new method have been established. Results from the new method compared with those obtained from existing methods show that the new method gives better accuracy.**

***Index Term* - one-step, hybrid method, block method, stability, off step.**

I. INTRODUCTION

Conventionally, higher order ordinary differential equations of the form

$$y^{(n)}(x) = f\left(x, y, y', y'', \ldots, y^{(n-1)}\right) \qquad (1)$$

are solved by the reduction of order method, [11], [8] and [14]. An approach widely reported to be cumbersome and prone to accumulation of errors in the course of the integration process, (See [1], [4] [20] and 22]). Direct method is an alternative to the reduction of order method where the problem is approximated directly. This approach has been extensively studied by many researchers in recent years (See [2], [5] [21] and [25]). A commonly adopted method to approximate the solution of (1) directly is the linear multistep method. Implementation of this method has mostly been either in the predictor-corrector mode [17], or the block mode, [6] and [15]. The former is widely reported to be costly in terms of the human effort involved in developing predictors; subroutines to supply starting values for the integration process and the wastage of computer time and memory. In the latter, development of predictors separately is not required; using other methods and writing subroutines to supply starting values is not necessary and the method can be applied as a parallel integrator thereby reducing the usage of computer memory and computing time.

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T. A. Anake is with Department of Mathematics, Covenant University, Otta, Ogun State, Nigeria; (corresponding author phone: +234 808 728 2714; e-mail: timothy.Anake@covenantuniversity.edu.ng.)

D. O. Awoyemi is with Department of Mathematical Sciences, Federal University of Technology, Akure, Ondo State, Nigeria; (e-mail: oniawoyemi@yahoo.com)

O. Adesanya is with Department of Mathematics, Modibo Adama University of Technology, Yola, Adamawa State, Nigeria; (e-mail: torlar10@yahoo.com)

However, the underlying condition for these methods of implementation is the Dahlquist's barrier conditions, [10]. It stipulates that a zero stable linear multistep method is at best of order $p = k+1$ for odd number of steps and $p = k+2$ for even number of steps. Thus, if one desires to reduce the number of steps and hence the number of function evaluations per given order while still achieving a higher order of accuracy and zero stability then, one must circumvent Dahlquist's barrier conditions. To achieve this, the hybrid methods were introduced in [9] and [12].

Generally, these hybrid methods are known to share with Runge-Kutta methods the advantage of easy change in step length and evaluating data at off step points. Although, hybrid methods were considered initially not to be methods on their own, [19], today, many notable authors have used the method independently to derive new numerical schemes of step number $k > 1$, for the solutions of initial value problems of ordinary differential equations. Some examples include hybrid-predictor-corrector method, [7] and [17] and the hybrid block methods, in which different hybrid schemes are combined to form a single block, [15] and [24].

In this paper, we propose a continuous one-step implicit hybrid block method for the direct solution of initial value problems in the class of (1) particularly, when $n=2$ with initial conditions prescribed, that is;

$$y'' = f\left(x, y, y'\right), \quad y(a) = \eta_a,\ y'(a) = \eta_a'. \qquad (2)$$

Where $x \in [a,b]$, $a,b \in \Re$ and f is continuously differentiable in $[a,b]$. Basically, we assume the existence and uniqueness of the solution of (2) according to [23].

II DERIVATION AND SPECIFICATION OF THE METHOD

Suppose the solution of (2) is approximated in the range $x_n \le x \le x_{n+1}$ such that the step length is given by $h = x_{n+1} - x_n$. Suppose also that the approximate solution is of the form of the power series polynomial:

$$P(x) = \sum_{j=0}^{\tau} a_j x^j \qquad (3)$$

Then, imposing $r+s$ conditions on (3), (where r and s represent the number of collocation and interpolation points respectively), polynomials of degree atmost $m = r+s-1$ are obtained as follows:

$$P\left(x_{n+v_1}\right) = y_{n+v_1}$$

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$$P\left(x_{n+\nu_2}\right)=y_{n+\nu_2}$$
$$P''\left(x_{n+\nu_j}\right)=f\left(x_{n+j},y_{n+j},y'_{n+j}\right) \tag{4}$$

Where $j=0,\nu_1,\nu_2,1$ and $\nu_1,\nu_2\in\{x_n,x_{n+1}\}$ are off step points.

This system of equations each with degree atmost m can be written in the matrix form:

$$Ax=b \tag{5}$$

i.e.

$$\begin{bmatrix} 1 & x_{n+\nu_1} & x_{n+\nu_1}^2 & x_{n+\nu_1}^3 & x_{n+\nu_1}^4 & x_{n+\nu_1}^5 \\ 1 & x_{n+\nu_2} & x_{n+\nu_2}^2 & x_{n+\nu_2}^3 & x_{n+\nu_2}^4 & x_{n+\nu_2}^5 \\ 0 & 0 & 0 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 0 & 6x_{n+\nu_1} & 12x_{n+\nu_1}^2 & 20x_{n+\nu_1}^3 \\ 0 & 0 & 0 & 6x_{n+\nu_2} & 12x_{n+\nu_2}^2 & 20x_{n+\nu_2}^3 \\ 0 & 0 & 0 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}=\begin{bmatrix} y_{n+\nu_1} \\ y_{n+\nu_2} \\ f_n \\ f_{n+\nu_1} \\ f_{n+\nu_2} \\ f_{n+1} \end{bmatrix}$$

Using Gaussian elimination method, the unknown coefficients a_j's can be obtained. Putting the a_j's back into (4), gives the continuous one-step implicit hybrid method:

$$y_{n+k}=\alpha_{\nu_1}(t)\,y_{n+\nu_1}+\alpha_{\nu_2}(t)\,y_{n+\nu_2}+h^2\left[\sum_{j=0}^{k}\beta_j(t)\,f_{n+j}+\sum_{i=1}^{2}\beta_{\nu_i}(t)\,f_{n+\nu_i}\right] \tag{6}$$

Where, $x_{n+z}=x_n+zh$; $y_{n+z}=y\left(x_{n+z}\right)$ is the approximation of the exact solution at the grid point x_{n+z} and $f_{n+z}=f\left(x_{n+z},y_{n+z},y''_{n+z}\right)$, for any $z\in\Re$.

The coefficients $\alpha_i(t)$, $\beta_j(t)$; $i=1,2;\ j=0,\nu_i,1$ are continuous coefficients obtained by using the transformation $th=x-x_{n+\nu_2}$ in (5). The method (6) is completely determined by these coefficients for values of some values of $t\in[0,1]$.

The first derivative of the coefficients of (6) with respect to t gives the derivative method:

$$y'_{n+k}=\alpha'_{\nu_1}(t)\,y_{n+\nu_1}+\alpha'_{\nu_2}(t)\,y_{n+\nu_2}+h^2\left[\sum_{j=0}^{k}\beta'_j(t)\,f_{n+j}+\sum_{i=1}^{2}\beta'_{\nu_i}(t)\,f_{n+\nu_i}\right] \tag{7}$$

Hence, using (6) and (7), approximate methods and their derivatives, (needed to set up the block formula), can be derived at the step and off step points respectively.

A careful choice of the off step points is necessary if zero stable methods are to be derived. For the new method, we have carefully chosen off step points to be at equal distances of one third apart, starting from the origin. Thus, $\nu_1=1/3$ and $\nu_2=2/3$.

In particular, the coefficients $\{\alpha_{\nu_i}(t),\beta_j(t)\}$; $i=1,2,\ j=0,\nu_i,1$ which determine (6) are given respectively as follows:

$$\begin{aligned}
\alpha_{1/3}(t)&=-3t\\
\alpha_{2/3}(t)&=3t+1\\
\beta_0(t)&=-\frac{h^2}{1080}\left(243t^5-90t^3+7t\right)\\
\beta_{1/3}(t)&=\frac{h^2}{360}\left(243t^5+135t^4-180t^3+22t\right)\\
\beta_{2/3}(t)&=-\frac{h^2}{360}\left(243t^5+270t^4-90t^3-180t^2-43t\right)\\
\beta_1(t)&=\frac{h^2}{1080}\left(243t^5+405t^4+180t^3-8t\right)
\end{aligned} \tag{8}$$

Evaluating (6) at the step points $x=x_n$ and x_{n+1} so that $t=-2/3$ and $1/3$ respectively, gives the discrete methods

$$y_{n+1}-2y_{n+\frac{2}{3}}+y_{n+\frac{1}{3}}=\frac{h^2}{108}\left[f_{n+1}+10f_{n+\frac{2}{3}}+f_{n+\frac{1}{3}}\right] \tag{9}$$

$$y_{n+\frac{2}{3}}-2y_{n+\frac{1}{3}}+y_n=\frac{h^2}{108}\left[f_{n+\frac{2}{3}}+10f_{n+\frac{1}{3}}+f_n\right] \tag{10}$$

Similarly, the coefficients $\left\{\alpha'_{\nu_i}(t),\beta'_j(t)\right\}$ $i=1,2;\ j=0,\nu_i,1$ in (7) are as follows:

$$\begin{aligned}
\alpha'_{\frac{1}{3}}(t)&=-3\\
\alpha'_{\frac{2}{3}}(t)&=3\\
\beta'_0(t)&=-\frac{h}{1080}\left(1215t^4-270t^2+7\right)\\
\beta'_{\frac{1}{3}}(t)&=\frac{h}{360}\left(1215t^4+540t^3-540t^2+22\right)\\
\beta'_{\frac{2}{3}}(t)&=-\frac{h}{360}\left(1215t^4+1080t^3-270t^2-360t-43\right)\\
\beta'_1(t)&=\frac{h}{1080}\left(1215t^4+1620t^3+540t^2-8\right)
\end{aligned} \tag{11}$$

Evaluating (7) at $x=x_{n+i}$, $i=0\left(\frac{1}{3}\right)1$ implies $t=-\frac{2}{3},-\frac{1}{3},0,$ and $\frac{1}{3}$ respectively. Thus, the following derivative methods are obtained:

$$y'_n-3y_{n+\frac{2}{3}}+3y_{n+\frac{1}{3}}=\frac{h}{1080}\left[-8f_{n+1}+9f_{n+\frac{2}{3}}-414f_{n+\frac{1}{3}}-127f_n\right] \tag{12}$$

$$y'_{n+\frac{1}{3}}-3y_{n+\frac{2}{3}}+3y_{n+\frac{1}{3}}=\frac{h}{1080}\left[7f_{n+1}-66f_{n+\frac{2}{3}}-129f_{n+\frac{1}{3}}+8f_n\right] \tag{13}$$

$$y'_{n+\frac{2}{3}}-3y_{n+\frac{2}{3}}+3y_{n+\frac{1}{3}}=\frac{h}{1080}\left[-8f_{n+1}+129f_{n+\frac{2}{3}}+66f_{n+\frac{1}{3}}-7f_n\right] \tag{14}$$

$$y'_{n+1}-3y_{n+\frac{2}{3}}+3y_{n+\frac{1}{3}}=\frac{h}{1080}\left[127f_{n+1}+414f_{n+\frac{2}{3}}-9f_{n+\frac{1}{3}}+8f_n\right] \tag{15}$$

III IMPLEMENTATION OF THE METHOD

For these methods derived in Section II, block method in the sense of [6] but with modification is adapted for their implementation. The modified version called the one-step implicit hybrid block method is defined in mathematical notations as:

$$h^{\lambda}\,\mathrm{A}Y_m^{(n)}=h^{\lambda}\,\mathrm{B}y_m^{(n)}+h^{\mu-\lambda}\,\mathrm{C}F\left(Y_m\right). \tag{16}$$

Where:

$$Y_m^{(n)} = \left(y_{n+\frac{1}{3}}, y_{n+\frac{2}{3}}, y_{n+1}, y'_{n+\frac{1}{3}}, y'_{n+\frac{2}{3}}, y'_{n+1} \right)^T$$

$$y_m^{(n)} = \left(y_{n-\frac{1}{3}}, y_{n-\frac{2}{3}}, y_n, y'_{n-\frac{1}{3}}, y'_{n-\frac{2}{3}}, y'_n \right)^T$$

$$F(Y_m) = \left(f_n, f_{n+\frac{1}{3}}, f_{n+\frac{2}{3}}, f_{n+1} \right)^T .$$

Also, n represents the order of the derivative of (6), $\lambda \in \Re$ represents the power of h and μ is the order of problem (2). A, B and C are constant coefficient matrices obtained from the combination of methods (9), (10), and (12) – (15) into a block. Furthermore, A is invertible.

Normalizing (16), yields the following new constant coefficient matrices:

$$\mathrm{A}' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{B}' = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathrm{C}' = \begin{bmatrix} \frac{97}{3240} & \frac{19}{540} & -\frac{13}{1080} & \frac{1}{405} \\ \frac{28}{405} & \frac{22}{135} & -\frac{2}{135} & \frac{2}{405} \\ \frac{13}{120} & \frac{3}{10} & \frac{3}{40} & \frac{1}{60} \\ \frac{1}{8} & \frac{19}{72} & \frac{5}{72} & \frac{1}{72} \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}$$

Substituting matrices $\mathrm{A}, \mathrm{B},$ and C in (16) with the new coefficient matrices A', B' and C', the block solutions are obtained for each off step and step point respectively. These solutions are implemented as simultaneous integrators for the approximation of the solutions of problem (2) (without requiring neither special methods to supply starting values nor the development of predictors separately), over the subintervals: $[x_0, x_1], \ldots, [x_{N-1}, x_N]$ of the partition $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$.

IV ANALYSIS OF THE BLOCK METHOD

In this section, fundamental properties of the one-step implicit hybrid block method are discussed.

A Order of the method

Consider the modified block method (16), the associated linear difference operator is defined as follows

Definition 1 *The linear difference operator* L *associated with (16) is defined as*

$$\mathrm{L}\left[y(x); h \right] = h^{\lambda}\, \mathrm{A}\, \mathrm{Y}_m^{(n)} - h^{\lambda}\, \mathrm{B}\, y_m^{(n)} - h^{\mu-\lambda}\, \mathrm{C}\, F(Y_m) \qquad (17)$$

where $y(x)$ *is an arbitrary test function continuously differentiable on* $[a,b]$. *Expanding* $Y_m^{(n)}$ *and* $F(Y_m)$ *component wise respectively in Taylor's series and collecting terms in powers of* h *gives*

$$\begin{aligned} \mathrm{L}\left[y(x); h \right] = {} & \bar{C}_0 y(x) + \bar{C}_1 h y^{(1)}(x) + \bar{C}_2 h^2 y^{(2)}(x) + \cdots \\ & + \bar{C}_p h^p y^{(p)}(x) + \bar{C}_{p+1} h^{p+1} y^{(p+1)}(x) + \cdots \end{aligned} \qquad (18)$$

Where the $\bar{C}_i$*,* $i = 0, 1, \ldots$ *are vectors.*

Definition 2 *The implicit one-step hybrid block method (16) and the associated linear difference operator (17) are said to have order* p *if* $\bar{C}_0 = \bar{C}_1 = \cdots = \bar{C}_p = \bar{C}_{p+1} = 0$ *and* $\bar{C}_{p+2} \neq 0$.

Definition 3 *The term* $\bar{C}_{p+2}$ *called the error constant implies that the one-step implicit hybrid block method (16) has local truncation error given by*

$$t_{n+k} = \bar{C}_{p+2} h^{p+2} y^{(p+2)}(x_n) + \mathrm{O}\left(h^{p+3}\right) \qquad (19)$$

From our computation, the new method has order of accuracy $p = (4, 4, 4, 4, 4, 4)^T$ and has error term given as

$$\bar{C}_{p+2} = \left(\frac{-7}{349920}, \frac{-1}{21870}, \frac{-1}{21960}, \frac{-19}{174960}, \frac{-1}{21870}, \frac{-1}{6480} \right)^T .$$

B Zero stability of the method

Definition 4 *The one-step implicit hybrid block method (16) is said to zero stable as* $h \to 0$ *if its first characteristic polynomial* $\bar{\rho}(z)$ *satisfies*

$$\bar{\rho}(z) = \det\left[z\, \mathrm{A}' - \mathrm{B}' \right] = z^{r-\mu} (z-1)^{\mu} = 0 \qquad (20)$$

where r *is the order of the matrices* A' *and* B'*, and the roots* $z_s, s = 1, \ldots, 6$ *of (20) satisfy the condition that* $|z_s| \leq 1$. *Furthermore, those roots* $|z_s| = 1$ *have multiplicity not exceeding the order of the differential equation.*

The one-step implicit hybrid block satisfies the conditions of Definition 4 since from (16), $r = 6$ and $\mu = 2$. Thus,

$$\det\left[z\, \mathrm{A} - \mathrm{B} \right] = z^4 (z-1)^2 = 0$$

Clearly, the method is zero stable and consistency follows since the order of the new block method is greater than one.

C Convergence and Interval of Absolute Stability

From the foregoing, the convergence of the one-step implicit hybrid block method is established according to [13].

The new method is absolutely stable within an interval of $(-7776, 0)$. Although the method is not A-stable, it is strongly stable with a wide interval of absolute stability which makes it suitable for mildly stiff problems.

V NUMERICAL EXPERIMENT

In this section, the accuracy of the fourth order one-step implicit hybrid block method (16), is experimented on two test problems with a fixed step size $h=1/320$. In each case, the computed result is compared with results obtained from existing methods. The absolute errors are given at some selected points of evaluation in Tables 1 and 2 respectively.

Problem 1

$$y''-x\left(y'\right)^2=0,\quad y\left(0\right)=1,\ y'\left(0\right)=\frac{1}{2}$$

$$\textit{Exact solution: } 1+\frac{1}{2}\ln\left(\frac{2-x}{2+x}\right).$$

For this problem, the new method of order four is compared with the three step, fourth order method proposed in [6] and the one step, one-hybrid, fourth order method proposed in [3]. It is obvious in Table 1 that the new method is more accurate than those in [3] and [6].

Table I for Problem 1

X	Error in [6]	Error [3]	Error in new result
0.1	6.5501E-11	4.9827E-11	2.5056E-12
0.2	5.4803E-10	4.1043E-10	2.0446E-11
0.3	1.9256E-09	1.4286E-09	7.0966E-11
0.4	4.8029E-09	3.5243E-09	1.7482E-10
0.5	1.0006E-08	7.2435E-09	3.5904E-10
0.6	1.8727E-08	1.3336E-08	6.6068E-10
0.7	3.2346E-08	2.2873E-08	1.1328E-09
0.8	5.3969E-08	3.7447E-08	1.8543E-09
0.9	8.8004E-08	5.9504E-08	2.9461E-09
1.0	1.4353E-07	9.2940E-08	4.6013E-09

Problem 2

$$y''+\left(\frac{6}{x}\right)y'+\left(\frac{4}{x^2}\right)y=0,\quad y\left(1\right)=1,\ y'\left(1\right)=1$$

$$\textit{Exact solution: } \frac{5}{3x}-\frac{3}{3x^4}$$

From Table II, it is obvious that in the solution of Problem 2, the new method of order four is more accurate than the 3-step hybrid method proposed in [18] and the 5-step Adam Moulton type method proposed in [16]; both methods are of order six and were implemented in the predictor-corrector mode.

Table II for Problem 2

X	Error in [16]	Error in [18]	Error in new result
1.0094	9.6400E-07	8.5357E-10	2.0169E-10
1.0125	3.6750E-06	1.7846E-09	4.5540E-10
1.0156	3.9320E-06	2.9171E-09	7.9967E-10
1.0188	6.2160E-06	4.2420E-09	1.2305E-09
1.0219	7.4430E-06	5.7509E-09	1.7440E-09
1.0250	7.7370E-06	7.4341E-09	2.3365E-09
1.0281	4.3530E-06	9.2848E-09	3.0043E-09
1.0313	1.1610E-05	1.1295E-08	3.7441E-09

VI CONCLUSION

In this paper, it is shown that continuous implicit one-step hybrid methods can be formulated as implicit hybrid block methods for the direct solution of problems in the class of (2). The new one-step implicit hybrid block method proposed in this paper is of order four and gives very low error terms. The consistency and zero stability of the new method guarantee its convergence in the sense of [13]. Furthermore, the strongly stable method offers a very wide interval of absolute stability suitable for the solution of mildly stiff problems.

Apart from the advantage of utilizing data at off step points, it can also be deduced from Tables I and II respectively, that the method is highly accurate.

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