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COEFFICIENT BOUNDS FOR CERTAIN CLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS AS RELATED TO SIGMOID FUNCTION

A.M. Gbolagade[1 §], S.O. Olatunji[2], T. Anake[3]
[1]Department of Mathematics
Emmanuel Alayande College of Education
P.M.B. 1010, Oyo, Oyo State, NIGERIA
[2]Department of Pure and Applied Mathematics
Ladoke Akintola University of Technology
Ogbomoso, P.M.B. 4000, Ogbomoso, NIGERIA
[3]Department of Mathematics
Covenant University
Ota, NIGERIA

Abstract: In this work, the authors investigated certain classes of analytic and univalent functions in terms of their coefficient bounds as related to activation sigmoid function with respect to symmetric and conjugate points.

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1. Introduction

Sigmoid function is once the special functions which is a branch of mathematics which is of utmost importance to scientists and engineers who are concerned with actual mathematical calculations such as in physics, engineering, statistics, computer science etc. These special functions do play important role in geometric function

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[§]Correspondence author

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theory. Activation function acts as squashing function such that the output of a neuron in a neural network is between certain values (usually 0 and 1 or -1 and 1). There are three types: threshold function, piecewise-linear function and sigmoid function. Sigmoid function is the most popular of the three with gradient-descendent type algorithms Fadipe et al [2] and Oladipo [7].

Activation function is an information process that is inspired by the way biological nervous system such as brain processes information. It composed a large number of highly interconnected processing element (neurons) working together to solve a specific task. This function works in similar way the brain does, learns by examples and can not be programmed to solve a specific task. Activation function has a wide applications in the real world. It can be used in industries, business, forcasting sales, identifying patterns trend and modelling especially in human cardiovascular system and human population. Sigmoid function can be evaluated using truncated series expansion, look-up tables or piecewise approximation [2].

The logistic sigmoid function $g(z) = \frac{1}{1+e^{-z}}$ is differentiable, which is very useful in weight learning algorithm and serve as a link to the class of univalent functions.

Sigmoid function has the following properties:

(i) It outputs real number between 0 and 1

(ii) It maps a large domain to a small range

(iii) It is one-to-one function hence the information is well preserved

(iv) It increases monotonically

With all the properties mentioned by [2], sigmoid function is perfectly useful in geometric function theory.

Let $H(\omega)$ be the subfamily of $S(\omega) \subset A(\omega) \subset A$ where A denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are regular in the unit disk $U = \{z : |z| < 1\}$ and normalized with $f(0) = 0$ and $f'(0) - 1 = 0$. The above class of functions have been studied by many authors most especially in terms of starlikeness, convexity, close-to- convexity, subordination, convolution, extremal etc the literatures of which littered everywhere.

In 1999, Kanas and Ronning [4] developed and investigated a subclass of A denoted by $A(\omega)$ of the form

$$f(z) = (z - \omega) + \sum_{k=2}^{\infty} a_k (z - \omega)^k \tag{2}$$

which is analytic and regular in the unit disk $U = \{z : |z| < 1\}$ and normalized with $f(\omega) = 0, f'(\omega) - 1 = 0$ and ω is a fixed point in U. They obtained some

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interesting and useful results in terms of ω-starlikeness, ω-convexity and coefficient bounds. Several other authors, the likes of Acu and Owa [1], Selvaraj and Vasanthi [8], Oladipo [5,7], have intensively studied this class of functions and they generated several useful results as contained in the literatures mentioned. For the purpose of clearity, we wish to state the following

$$ST(\omega) = S^*(\omega) = \left\{ f \in S(\omega) : Re\left(\frac{(z-\omega)f'(z)}{f(z)}\right) > 0, z \in U \right\},$$

$$CV(\omega) = S^c(\omega) = \left\{ f \in S(\omega) : 1 + Re\left(\frac{(z-\omega)f''(z)}{f'(z)}\right) > 0, z \in U \right\},$$

and ω is a fixed point in U.

For the purpose of our investigation, the following lemma shall be necessary.

Lemma A. (see [2]) *Let $g(z)$ be a sigmoid function and $G(z) = 2g(z)$, then $G(z) \in P$, $|z| < 1$.*

Lemma B. (see [2]) *Let*

$$G_k(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} z^m \right)^k,$$

then $|G_k(z)| < 2, m = 1, 2....$

Lemma C. (see [2]) *If $G(z) \in P$, then $f(z)$ is a normalized univalent function.*

In this work, our objective is to investigate the class of functions of the form (2) with respects to symmetric and conjugate points as related to modified activated sigmoid function of the form

$$G^{\omega}(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} (z-\omega)^m \right)^k \tag{3}$$

and ω is a fixed point in U.

From Lemma B, we can conviniently say that

$$|G_k^{\omega}| < \frac{A-B}{(1+d)(1-d)^k}, \quad k \geq 1, \ -1 \leq B < A \leq 1, \text{and } |\omega| = d \text{ (see [2], [5])}. \tag{4}$$

Additionally, the following definitions shall be helpful in our investigation.

Definition A. (i) Let $S_s^*(\omega)$ be the subclass of $S(\omega)$ consisting of functions given by (1) satisfying the condition

$$Re\left(\frac{(z-\omega)f'(z)}{f(z) - f(-z)}\right) > 0, z \in U$$

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This class of functions shall be referred to as the class of ω-starlike with respect to symmetric points and ω is a fixed point in U.

(ii) Let $S_c^*(\omega)$ be the subclass of $S(\omega)$ consisting of functions given by (1) satisfying the condition

$$Re\left(\frac{(z-\omega)f'(z)}{f(z)+\overline{f(\overline{z})}}\right) > 0, z \in U$$

and ω is a fixed point in U. This class of functions shall be called ω-starlike with respect to conjugate points.

(iii) Let $S_s^c(\omega)$ be the subclass of $S(\omega)$ consisting of functions given by (1) satisfying the condition

$$Re\left(\frac{((z-\omega)f'(z))'}{(f(z)-f(-z))'}\right) > 0, z \in U$$

This class of functions shall be called ω-convex with respect to symmetric points and ω is a fixed point in U.

In 1982, Goel and Mehrok [3] introduced a subclass of S_s^* denoted by $S_s^*(A,B)$ in terms of subordination. Also, Oladipo [6] in 2012 extended the class of functions studied by [8] and interesting results were obtained. The authors here wish to give the analogue definitions in the same manner by extension as follows.

Definition B. (i) Let $S_s^*(\omega,A,B)$ be the subclass of $S(\omega)$ consisting of functions given by (1) satisfying the condition

$$\frac{2(z-\omega)f'(z)}{f(z)-f(-z)} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)}, -1 \le B < A \le 1, z \in U$$

and ω is a fixed point in U

(ii) Let $S_c^*(\omega,A,B)$ be the subclass of $S(\omega)$ consisting of functions given by (1) satisfying the condition

$$\frac{2(z-\omega)f'(z)}{f(z)+\overline{f(\overline{z})}} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)}, -1 \le B < A \le 1, z \in U$$

and ω is a fixed point in U

(iii) Let $S_s^c(\omega,A,B)$ be the subclass of $S(\omega)$ consisting of functions given by (1) satisfying the condition

$$\frac{2((z-\omega)f'(z))'}{(f(z)-f(-z))'} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)}, -1 \le B < A \le 1, z \in U$$

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and ω is a fixed point in U.

(iv) Let $S_c^c(\omega, A, B)$ be the subclass of $S(\omega)$ consisting of functions given by (1) satisfying the condition

$$\frac{2((z-\omega)f'(z))'}{(f(z)+\overline{f(\overline{z})}} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)}, -1 \le B < A \le 1, z \in U$$

and ω is a fixed point in U

In this work, the authors wish to introduce the class $\sigma_s(\omega, \beta, A, B)$ consisting of analytic functions $f(z)$ of the form (1) and satisfying

$$\frac{2(z-\omega)f'(z)+2\beta(z-\omega)^2f''(z)}{(1-\beta)(f(z)-f(-z))+\beta(z-\omega)(f(z)-f(-z))'} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)},$$
$$-1 \le B < A \le 1, 0 \le \beta \le 1, z \in U$$

and ω is a fixed point in U.

Also, we wish to introduce the class $\sigma_c(\omega, \beta, A, B)$ consisting of analytic functions $f(z)$ of the form (1) and satisfying

$$\frac{2(z-\omega)f'(z)+2\beta(z-\omega)^2f''(z)}{(1-\beta)(f(z)+\overline{f(\overline{z})})+\beta(z-\omega)(f(z)+\overline{f(\overline{z})})'} \prec \frac{1+A(z-\omega)}{1+B(z-\omega)},$$
$$-1 \le B < A \le 1, 0 \le \beta \le 1, z \in U$$

and ω is a fixed point in U

Therefore, by the definition of subordination, it follows that $f \in \sigma_s(\omega, \beta, A, B)$ if and only if

$$\frac{2(z-\omega)f'(z)+2\beta(z-\omega)^2f''(z)}{(1-\beta)(f(z)-f(-z))+\beta(z-\omega)(f(z)-f(-z))'} = \frac{1+Ah(z)}{1+Bh(z)} = G^\omega(z), h \in U \quad (5)$$

and that $f \in \sigma_c(\omega, \beta, A, B)$ if and only if

$$\frac{2(z-\omega)f'(z)+2\beta(z-\omega)^2f''(z)}{(1-\beta)(f(z)+\overline{f(\overline{z})})+\beta(z-\omega)(f(z)+\overline{f(\overline{z})})'} = \frac{1+Ah(z)}{1+Bh(z)} = G^\omega(z), h \in U \quad (6)$$

where $G^\omega(z)$ is as ealier defined.

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2. Main Result

In this section, we state and prove the following

Theorem 2.1. *Let $f \in \sigma_s(\omega, \beta, A, B)$. Then for $k = 2, 3, 4, 5$. $0 \leq \beta \leq 1$*

$$|a_2| < \frac{A-B}{2(1+\beta)(1-d^2)},$$
$$|a_3| < \frac{A-B}{2(1+2\beta)(1-d^2)(1-d)},$$
$$|a_4| < \frac{A-B}{4(1+3\beta)(1-d^2)(1-d)^2},$$
$$|a_5| < \frac{A-B}{4(1+4\beta)(1-d^2)(1-d)^3}.$$

Proof. From (4) and (5), we have

$$\frac{2(z-\omega)+2\sum_{k=2}^{\infty} k(1+\beta(k-1))a_k(z-\omega)^k}{2(z-\omega)+\sum_{k=2}^{\infty}(1-\beta+\beta k)(1-(-1)^k)a_k(z-\omega)^k}$$
$$= 1 - \frac{1}{2}\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}(z-\omega)^m\right) + \frac{1}{4}\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}(z-\omega)^m\right)^2$$
$$-\frac{1}{8}\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}(z-\omega)^m\right)^3 + \frac{1}{16}\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}(z-\omega)^m\right)^4$$
$$-\frac{1}{32}\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}(z-\omega)^m\right)^5$$
$$+ ... + (z-\omega) + 2(1+\beta)a_2(z-\omega)^2 + 3(1+2\beta)a_3(z-\omega)^3 + 4(1+3\beta)a_4(z-\omega)^4$$
$$+5(1+4\beta)a_5(z-\omega)^5 + 6(1+5\beta)a_6(z-\omega)^6 + 7(1+6\beta)a_7(z-\omega)^7 + ...$$
$$= \left[1 + \frac{1}{2}(z-\omega) - \frac{1}{24}(z-\omega)^3 + \frac{1}{240}(z-\omega)^5 - \frac{1}{64}(z-\omega)^6 + \frac{779}{20160}(z-\omega)^7 - ...\right]$$
$$\big[(z-\omega) + (1+2\beta)a_3(z-\omega)^3 + (1+4\beta)a_5(z-\omega)^5 + (1+6\beta)a_7(z-\omega)^7$$
$$+(1+8\beta)a_9(z-\omega)^9 + ...\big],$$
$$(z-\omega) + 2(1+\beta)a_2(z-\omega)^2 + 3(1+2\beta)a_3(z-\omega)^3 + 4(1+3\beta)a_4(z-\omega)^4$$
$$+5(1+4\beta)a_5(z-\omega)^5 + 6(1+5\beta)a_6(z-\omega)^6 + 7(1+6\beta)a_7(z-\omega)^7 + ... =$$
$$(z-\omega) + \frac{1}{2}(z-\omega)^2 + (1+2\beta)a_3(z-\omega)^3 + \left[\frac{1}{2}(1+2\beta)a_3 - \frac{1}{24}\right](z-\omega)^4$$
$$+(1+4\beta)a_5(z-\omega)^5$$
$$+\left[\frac{1}{2}(1+4\beta)a_5 - \frac{1}{24}(1+2\beta)a_3 + \frac{1}{240}\right](z-\omega)^6 + \left[(1+6\beta)a_7 - \frac{1}{64}\right](z-\omega)^7 + ...$$

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Equating the coefficients of the like powers of $(z-\omega)$, we have

$$2(1+\beta)a_2 = G_1$$

$$2(1+2\beta)a_3 = G_2$$

$$4(1+3\beta)a_4 = G_3$$

$$4(1+4\beta)a_5 = G_4$$

By applying lemma B, we have

$$|a_2| < \frac{A-B}{2(1+\beta)(1-d^2)}$$

where $\frac{A-B}{1-d^2} = \frac{1}{2}$

$$|a_3| = \frac{A-B}{2(1+2\beta)(1-d^2)(1-d)}$$

where $\frac{A-B}{(1-d^2)(1-d)} = 0$.

$$|a_4| < \frac{A-B}{4(1+3\beta)(1-d^2)(1-d)^2}$$

where $\frac{A-B}{(1-d^2)(1-d)^2} = -\frac{1}{24}$

$$|a_5| = \frac{A-B}{4(1+4\beta)(1-d^2)(1-d)^3}$$

where $\frac{A-B}{(1-d^2)(1-d)^3} = 0$. which complete the proof.

With various choices of A, B, β, d, many existing and new results could be obtainted. If we set $\omega = 0$ in Theorem 2.1, we have

Corollary A. *Let* $f \in \sigma_s(0,\beta,A,B)$. *Then for* $k = 2,3,4,5$. $0 \le \beta \le 1$

$$|a_2| < \frac{A-B}{2(1+\beta)}$$

$$|a_3| = 0$$

$$|a_4| < \frac{A-B}{4(1+3\beta)}$$

$$|a_5| = 0.$$

If we set $\beta = 1, A = 1, B = -1$ in corollary A, we have

Corollary B. *Let $f \in \sigma_s(\omega, \beta, 1, -1)$. Then for $k = 2, 3, 4, 5.$ $0 \le \beta \le 1$.*

$$|a_2| < \frac{1}{2}$$

$$|a_3| = 0$$

$$|a_4| < \frac{1}{8}$$

$$|a_5| = 0$$

Theorem 2.2. *Let $f \in \sigma_c(\omega, \beta, A, B)$. Then for $k = 2, 3, 4, 5. 0 \le \beta \le 1$.*

$$|a_2| < \frac{A-B}{(1+\beta)(1-d^2)}$$

$$|a_3| < \frac{A-B}{2(1+2\beta)(1-d^2)(1-d)}$$

$$|a_4| < \frac{A-B}{3(1+3\beta)(1-d^2)(1-d)^2}$$

$$|a_5| < \frac{A-B}{4(1+4\beta)(1-d^2)(1-d)^3}.$$

Proof. From (4) and (6), we have

$$\frac{(z-\omega) + \sum_{k=2}^{\infty} k(1+\beta(k-1))a_k(z-\omega)^k}{(z-\omega) + \sum_{k=2}^{\infty}(1-\beta+\beta k)a_k(z-\omega)^k}$$
$$= 1 + \frac{1}{2}(z-\omega) - \frac{1}{24}(z-\omega)^3 + \frac{1}{240}(z-\omega)^5 - \frac{1}{64}(z-\omega)^6 + ...$$

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$$(z-\omega)+2(1+\beta)a_2(z-\omega)^2+3(1+2\beta)a_3(z-\omega)^3$$
$$+4(1+3\beta)a_4(z-\omega)^4+5(1+4\beta)a_5(z-\omega)^5+$$
$$6(1+5\beta)a_6(z-\omega)^6+...=(z-\omega)+\left[(1+\beta)a_2+\frac{1}{2}\right](z-\omega)^2+$$
$$\left[(1+2\beta)a_3+\frac{1}{2}(1+\beta)a_2\right](z-\omega)^3+\left[(1+3\beta)a_4+\frac{1}{2}(1+2\beta)a_3-\frac{1}{24}\right](z-\omega)^4+$$
$$\left[(1+4\beta)a_5+\frac{1}{2}(1+3\beta)a_4-\frac{1}{24}(1+\beta)a_2\right](z-\omega)^5+$$
$$\left[(1+5\beta)a_6+\frac{1}{2}(1+4\beta)a_5-\frac{1}{24}(1+2\beta)a_3\frac{1}{240}\right](z-\omega)^6+...$$

Equating the coefficients of the like powers of $(z-\omega)$, we have

$$\begin{aligned}(1+\beta)a_2&=G_1,\\ 2(1+2\beta)a_3&=G_2,\\ 3(1+3\beta)a_4&=G_3,\\ 4(1+4\beta)a_5&=G_4.\end{aligned}$$

By applying lemma B, we have

$$|a_2|<\frac{A-B}{(1+\beta)(1-d^2)}$$

where $\frac{A-B}{1-d^2}=\frac{1}{2}$

$$|a_3|<\frac{A-B}{2(1+2\beta)(1-d^2)(1-d)}$$

where $\frac{A-B}{(1-d^2)(1-d)}=\frac{1}{4}$

$$|a_4|<\frac{A-B}{3(1+3\beta)(1-d^2)(1-d)^2}$$

where $\frac{A-B}{(1-d^2)(1-d)^2}=\frac{1}{48}$

$$|a_5|<\frac{A-B}{4(1+4\beta)(1-d^2)(1-d)^3}$$

where $\frac{A-B}{(1-d^2)(1-d)^3}=-\frac{5}{288}$. which complete the proof.

If we set $\omega=0$ in Theorem 2.2, we have

Corollary C. *Let* $f \in \sigma_c(0, \beta, A, B)$*. Then for* $k = 2, 3, 4, 5.$ $0 \leq \beta \leq 1$

$$|a_2| < \frac{A - B}{(1 + \beta)},$$

$$|a_3| < \frac{A - B}{2(1 + 2\beta)},$$

$$|a_4| < \frac{A - B}{3(1 + 3\beta)},$$

$$|a_5| < \frac{A - B}{4(1 + 4\beta)}.$$

If we set $\beta = 1, A = 1, B = -1$ in corollary C, we have

Corollary D. *Let* $f \in \sigma_s(\omega, \beta, 1, -1)$*. Then for* $k = 2, 3, 4, 5.$ $0 \leq \beta \leq 1.$

$$|a_2| < 1,$$

$$|a_3| < \frac{1}{3},$$

$$|a_4| < \frac{1}{6},$$

$$|a_5| < \frac{1}{10}.$$

References

[1] Acu Mugur, Shigeyoshi Owa, On some subclasses of univalent functions, *Journal of Inequalities in Pure and Applied Mathematics,* **6**, No. 3, Article 70 (2005), 1-14.

[2] O. A. Fadipe-Joseph, A. T. Oladipo, U. A. Ezeafulukwe, Modified sigmoid function in univalent function theory, *International Journal of Mathematical Sciences and Engineering Application,* **7**, No. 7 (2013), 313-317.

[3] R. M. Goel, B. C. Mehrok, A subclass of starlike functions with respect to symmetric points, *Tamkang J. Math.*, **13**,No. 1 (1982), 11-24.

[4] S. Kanas, F. Ronning, Uniformly starlike and convex functions and other related classes of univalent functions, *Ann. Univ. Mariae Curie-skldowska Section A*, **53**, (1999), 95-105.

[5] A. T. Oladipo, On a subclass of univalent functions, *Advances in Applied Mathematical Analysis,* **4**, No. 2 (2009), 87-93.

[6] A. T. Oladipo, New subclasses of analytic functions with respect to other points, *International Journal of Pure and Applied Mathematics,* **75**, No. 1(2012), 1-12.

[7] A. T. Oladipo, Sigmoid Activation function in the space of univalent functions, (Submitted).

[8] C.Selvaraj, N.Vasanthi, Subclasses of Analytic functions with respect to symmetric and conjugate points, *Tamkang J. Math.*, **42**, No. 1(2011), 87-94.