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EXISTENCE AND UNIQUENESS RESULTS FOR SOME FOURTH ORDER FOUR-POINT AND THREE-POINT BOUNDARY VALUE PROBLEMS

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1. Introduction

In a recent paper [7], Iyase investigated the existence and uniqueness of solutions of fourth order four-point boundary value problem (bvp) of the form

$$x^{(4)} = f(t, x, x', x'', x''') + e(t),$$
$$x(0) = x(\eta_1) = x(\eta_2) = x(1) = 0, \quad \eta_1 \neq \eta_2,$$

where $f : [0,1] \times \mathbf{R}^4 \to \mathbf{R}$ satisfies the Caratheodory conditions, $e \in L^1[0,1]$ and $0 < \eta_i < 1$, $i = 1,2$. Prior to this, similar investigations have been undertaken for fourth order bvps by Aftabzadeh [1], Argwal [3] and Gupta [4], to mention a few; and for third order three-point bvps by Aftabzadeh *et al.* [2] and Gupta and Lakshmikantham [6]. Our present study derives its motivation from these earlier works. Our initial desire was to extend Iyase's result to more general boundary conditions involving the unknown function x as well as its derivatives as was the case in the third order problem [2]; it turned out however that we could do much more than this. We shall in fact show that existence and uniqueness results can be obtained for a wider class of fourth order equations, subject to varied boundary conditions.

It is pertinent to note that fourth order boundary value problems of the type studied here occur in a variety of physical problems. Details of such application can be found in [2, 3, 5].

A function $g : [0,1] \times \mathbf{R}^4 \to \mathbf{R}$ is said to satisfy the Caratheodory condition if

(i) $g(\cdot, y)$ is measurable for every $y \in \mathbf{R}^4$;

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(ii) $g(t,\cdot)$ is continuous for a.e. $t \in [0,1]$;
(iii) for every $r > 0$, there is a function $\gamma_r \in L^1[0,1]$ such that for a.e. $t \in [0,1]$, $|g(t,y)| \leq \gamma_r(t)$ whenever $\|y\| \leq r$.

We shall be concerned with bvps of the forms:

(1.1) $$u^{(iv)} + \phi(u'')u''' = g(t,u,u',u'',u''') + e(t)$$
(1.2) $$u''(0) = u''(1) = u'(\eta_1) = u(\eta_2) = 0, \quad 0 \leq \eta_1, \eta_2 \leq 1;$$
(1.3) $$u^{(iv)} + \psi(u') = g(t,u,u',u'',u''') + e(t)$$
(1.4) $$u''(0) = u''(1) = u'(0) = u(\eta) = 0, \quad 0 \leq \eta \leq 1$$
(1.5) $$u^{(iv)} + \chi(u')u'' = g(t,u,u',u'',u''') + e(t)$$
(1.6) $$u'(0) = u'(1) = u(\eta_1) = u(\eta_2) = 0, \quad \eta_1 \neq \eta_2, 0 \leq \eta_1 < 1, 0 < \eta_2 \leq 1,$$

We shall also consider the unperturbed equation

(1.7) $$u^{(iv)} = g(t,u,u',u'',u''') + e(t)$$

studied by Iyase [7] together with each of the following boundary conditions:

(1.8) $$u(0) = u(1) = u'(0) = u(\eta) = 0, \quad 0 < \eta < 1;$$
(1.9) $$u''(0) = u'''(1) = u'(\eta_1) = u(\eta_2) = 0, \quad 0 \leq \eta_1, \eta_2 \leq 1;$$
(1.10) $$u'(0) = u''(1) = u(\eta_1) = u(\eta_2) = 0, \quad \eta_1 \neq \eta_2, 0 \leq \eta_1 < 1, 0 < \eta_2 \leq 1;$$
(1.11) $$u'(0) = u(1) = u''(\eta_1) = u''(\eta_2) = 0, \quad 0 \leq \eta_1, \eta_2 \leq 1.$$

It will be assumed, in what follows, that $\phi, \psi, \chi : \mathbf{R} \to \mathbf{R}$ are continuous functions, $g : [0,1] \times \mathbf{R}^4 \to \mathbf{R}$ satisfies the Caratheodory conditions and that $e \in L^1[0,1]$.

2. The linear problem

Each of the bvps specified above can be put in the form

(2.1) $$Lu + Nu = e,$$

where $N : X \to Z$ is a nonlinear operator, X and Z are Banach spaces and

(2.2) $$L : D(L) \subset X \to Z, \quad Lu = u^{(iv)}$$

is a linear operator, whose domain, $D(L)$, is defined relative to the boundary condition satisfied. To be more specific, let

$$H^4(0,1) = \begin{cases} u \in [0,1] \to \mathbf{R} \text{ such that } u^{(j)}, j = 0,1,2,3 \\ \qquad \text{is absolutely continuous and} \\ u^{(iv)} \in L^2[[0,1],\mathbf{R}], \quad u^{(j)} \equiv \frac{d^j u}{dt^j} \end{cases}$$

with the usual inner product and corresponding norm $|\cdot|_4$, and let $X = C^3[0,1]$, $Z = L^1[0,1]$. The domain $D(L) \subset C^3[0,1]$ of L is defined by:

$$(2.3)\qquad D(L) = \left\{ \begin{array}{l} u \in H^4(0,1) \text{ such that } u \text{ satisfies one of the boundary} \\ \text{conditions } (1.2), (1.4), (1.6), (1.8), (1.9), (1.10) \text{ or } (1.1) \end{array} \right.$$

For the bvp (1.1), (1.2), for example, the domain of the corresponding operator L is given by

$$(2.4)\qquad D(L) = \{u \in H^4(0,1)\ |u''(0) = u''(1) = u(\eta_1) = u(\eta_2) = 0\}.$$

We now obtain some properties of L.

LEMMA 2.1. *Let L be defined by* (2.2) *and* (2.3). *Then* $\ker L = \{0\}$.

Proof
We verify this for the boundary conditions (1.10) and (1.6), since the proof in the other cases can be obtained in the same way. Assume that $u = \sum_{i=1}^{3} A_i t^i$ is a non-trivial solution of $Lu = 0$ and (1.10). Then from (1.10), $u'(0) = 0 = A_1$, and the other three boundary conditions give:

$$A_0 + A_2\eta_1^2 + A_3\eta_1^3 = 0; \quad A_0 + A_2\eta_2^2 + A_3\eta_2^3 = 0; \quad A_2 + 3A_3 = 0.$$

The discriminant, Δ, for this system (in A_0, A_2, A_3) is given by

$$\Delta = -(\eta_1 - \eta_2)[\eta_1^2 + \eta_1\eta_2 + \eta_2^2 - 3(\eta_1 + \eta_2)].$$

Since $\eta_1 \neq \eta_2, 0 \leq \eta_1 < 1, 0 < \eta_2 \leq 1$, it is clear that

$$\begin{aligned} \eta_1^2 + \eta_1\eta_2 + \eta_2^2 - 3(\eta_1 + \eta_2) &= -\eta_1(1-\eta_1) - \eta_2(1-\eta_2) \\ &\quad - \eta_1(1-\eta_2) - \eta_1 - 2\eta_2 < 0 \end{aligned}$$

and hence that $\Delta \neq 0$. Therefore $A_0 = A_1 = A_2 = A_3 = 0$.

In the case of the boundary condition (1.6), the corresponding discriminant Δ is

$$\Delta = 3(\eta_2 - \eta_1)\left[(\eta_1 + \eta_2) - \frac{2}{3}(\eta_1^2 + \eta_2^2 + \eta_1\eta_2)\right],$$

and it is an easy matter to show that the expression in the square bracket is positive, so that $\Delta \neq 0$.

COROLLARY 2.1. *Subject to the conditions of Lemma 2.1, the equation*

$$(2.5)\qquad Lu = e$$

together with each of the boundary conditions (1.2), (1.4), (1.6), (1.8), (1.9), (1.10), (1.11) *has a unique solution.*

We shall next obtain expressions for the unique solutions in question. Observe first that since $\ker L = \{0\}$, there exists, for each L, a linear map $K : Z \to X$ such that for $z \in Z$, $Kz \in D(L)$ and $LKz = z$; and for $u \in D(L)$, $KLu = u$. Thus $u = Ke$ is the unique solution of (2.5) for each $e \in Z$. Apart from this, the existence of K will play a more crucial role in the proof of the existence theorems.

LEMMA 2.2. *Let L be defined by* (2.2). *Then the corresponding linear map $K : Z \to X$, for each of the boundary conditions* (1.2), (1.4), (1.6), (1.8) *and* (1.10) *is respectively given by:*

$$(2.6)\quad (Kz)(t) = \frac{1}{6}[3\eta_1^2(t-\eta_2) - (t^3 - \eta_2^3)] \int_0^1 \int_0^\tau z(s)\, ds\, d\tau + \int_{\eta_2}^t \int_{\eta_1}^\beta \int_0^\alpha \int_0^\tau z(s)\, ds\, d\tau\, d\alpha\, d\beta;$$

$$(2.7)\quad (Kz)(t) = \frac{(\eta^3 - t^3)}{6} \int_0^1 \int_0^\tau z(s)\, ds\, d\tau + \int_\eta^t \int_0^\beta \int_0^\alpha \int_0^\tau z(s)\, ds\, d\tau\, d\alpha\, d\beta;$$

$$(2.8)\quad (Kz)(t) = \left[\int_0^t + \frac{t^2(t-\eta)}{(\eta-1)} \int_0^t + \frac{t^2(1-t)}{\eta^2(\eta-1)} \int_0^\eta\right] \times \int_0^\beta \int_0^\alpha \int_0^\tau z(s)\, ds\, d\tau\, d\alpha\, d\beta;$$

$$(2.9)\quad (Kz)(t) = \frac{1}{2}\left[(\eta_2^2 - \eta_1^2)\frac{\bar{\psi}(t,\eta_1)}{\bar{\psi}(\eta_2,\eta_1)} - (t^2 - \eta_1^2)\right] \int_0^1 \int_0^\alpha \int_0^\tau z(s)\, ds\, d\tau\, d\alpha + \left[\int_{\eta_1}^t - \frac{\bar{\psi}(t,\eta_1)}{\bar{\psi}(\eta_2,\eta_1)} \int_{\eta_1}^{\eta_2}\right] \int_0^\beta \int_0^\alpha \int_0^\tau z(s)\, ds\, d\tau\, d\alpha\, d\beta;$$

where $\bar{\psi}(t,\eta_1) \equiv 2(t^3 - \eta_1^3) - 3(t^2 - \eta_1^2), \bar{\psi}(\eta_2,\eta_1) \neq 0, \bar{\psi}(\eta_1,\eta_2) = 0;$

$$(2.10)\quad (Kz)(t) = \frac{1}{2}\left[(\eta_2^2 - \eta_1^2)\frac{\bar{\phi}(t,\eta_1)}{\bar{\phi}(\eta_2,\eta_1)} - (t^2 - \eta_1^2)\right] \int_0^1 \int_0^\tau z(s)\, ds\, d\tau + \left[\int_{\eta_1}^t - \frac{\bar{\phi}(t,\eta_1)}{\bar{\phi}(\eta_2,\eta_1)} \int_{\eta_1}^{\eta_2}\right] \int_0^\beta \int_0^\alpha \int_0^\tau z(s)\, ds\, d\tau\, d\alpha\, d\beta;$$

where $\bar{\phi}(t,\eta_1) \equiv (t^3 - \eta_1^3) - 3(t^2 - \eta_1^2),\ \bar{\phi}(\eta_2,\eta_1) \neq 0;\ \bar{\phi}(\eta_1,\eta_1) = 0.$

The expressions (2.6)–(2.10) give the unique solution of $Lu = z$, subject respectively to the boundary conditions (1.2), (1.4), (1.6), (1.8) and (1.10). Each expression can be obtained by repeated integration of $Lu = z$, and using the appropriate boundary condition.

3. The main results

It will be convenient to denote by G_1 and G_2 the following hypotheses:

G_1: there exist functions $\alpha \in C^0[[0,1] \times \mathbf{R}^3, \mathbf{R}]$ and $\beta \in L^1[0,1]$ such that

$$|g(t,u,v,w,z)| \leq \alpha(t,u,v,w)z^2 + \beta(t) \tag{3.1}$$

for every $(u,v,w,z) \in \mathbf{R}^4$ and for a.e. $t \in [0,1]$;

G_2: there exist functions $p,q,r,s \in L^2[0,1]$ and $m \in L^1[0,1]$ such that

$$|g(t,u,v,w,z)| \leq p(t)|u| + q(t)|v| + r(t)|w| + s(t)|z| + m(t) \tag{3.2}$$

Our first set of results — Theorems 3.1, 3.2, 3.3 — have to do with G_1, and is as follows.

THEOREM 3.1. *Let g satisfy hypothesis G_1. Suppose there exist functions $a \in C^1[0,1]$, $b,c,d \in C^0[0,1]$ and $h \in L^1[0,1]$ such that*

$$g(t,u,v,w,z)w \geq a(t)zw + b(t)w^2 + c(t)|vw| + d(t)|uw| + h(t)|w|, \tag{3.3}$$

where

$$a'(t) \leq a_0, \quad b(t) \leq -b_0, \quad c(t) \geq -c_0, \quad d(t) \geq -d_0 \tag{3.4}$$

and a_0, b_0, c_0, d_0 are constants. Then, for all arbitrary continuous functions $\phi : \mathbf{R} \to \mathbf{R}$ and for every $e \in L^1[0,1]$, the bvp (1.1), (1.2) has at least one solution if

$$\frac{a_0 + 2b_0}{2} + \frac{2c_0}{\pi} + \frac{4d_0}{\pi^2} < \pi^2. \tag{3.5}$$

THEOREM 3.2. *Let g satisfy hypothesis G_1 and suppose there exist functions $a,b,c,d, \in C^0[0,1]$ and $h \in L^1[0,1]$ such that*

$$g(t,u,v,w,z)w \geq a(t)|zw| + b(t)w^2 + c(t)|vw| + d(t)|uw| + h(t)|w| \tag{3.6}$$

where

$$a(t) \geq -a_0, \quad b(t) \geq -b_0, \quad c(t) \geq -c_0, \quad d(t) \geq -d_0 \tag{3.7}$$

and a_0, b_0, c_0, d_0 are constants. Then for every $e \in L^1[0,1]$ the bvp (1.7), (1.9) has at least one solution if

$$\pi a_0 + 2b_0 + \frac{4c_0}{\pi} + \frac{8d_0}{\pi^2} < \frac{\pi^2}{2}. \tag{3.8}$$

THEOREM 3.3. *Let g satisfy all the conditions of* Theorem 3.1 *and suppose that*

$$(3.9)\qquad |\Psi(y)| \le \Delta_0|y| + \Delta_1|y|^\gamma; \quad \Psi(y) := \int_0^y \psi(s)\,ds, \quad 1 \le \gamma \le 2;$$

where $\Delta_0 \ge 0, \Delta_1 \ge 0$ are constants. Then for arbitrary $e \in L^1[0,1]$ the bvp (1.3), (1.4) *has at least one solution if*

$$(3.10)\qquad \left(\frac{a_0}{2} + b_0 + \Delta_1\right) + \frac{2c_0}{\pi} + \frac{4d_0}{\pi^2} < \pi^2.$$

The next set of results concern hypothesis G_2 and is as follows.

THEOREM 3.4. *Let g satisfy hypothesis G_2 and suppose that there exist functions $a, b, d \in C^0[0,1]$, $c \in C^1[0,1]$ and $h \in L^1[0,1]$ such that*

$$(3.11)\qquad g(t,u,v,w,z)w \ge a(t)|zw| + b(t)|w^2| + c(t)vw + d(t)|uw| + h(t)|w|,$$

where

$$(3.12)\qquad a(t) \ge -a_0, \quad c'(t) \le c_0, \quad b(t) \ge -b_0, \quad d(t) \ge -d_0$$

and a_0, b_0, c_0 and d_0 are constants. Suppose further that

$$(3.13)\qquad |\chi(y)| \le \Delta_0, \quad \Delta_0 \ge 0 \text{ a constant.}$$

Then, for every $e \in L^1[0,1]$ the bvp (1.5), (1.6) *has at least one solution if*

$$(3.14)\qquad \pi\|s\|_2 + (2\Delta_0 + a_0 + 2\|r\|_2) + \frac{2}{\pi}(\Delta_0 + b_0 + \|q\|_2) + \frac{4}{\pi^2}\|p\|_2 + \frac{1}{\pi^2}(c_0 + 4d_0) < \frac{\pi}{2}$$

THEOREM 3.5. *Let g satisfy hypothesis G_2 and suppose that there exist functions $a, b, c, d \in C^0[0,1]$ and $h \in L^1[0,1]$ such that conditions* (3.6) *and* (3.7) *hold. Then, for arbitrary functions $e \in L^1[0,1]$, each of the bvps* (1.7), (1.8); (1.7), (1.10), *and* (1.7), (1.11) *has at least one solution if*

$$(3.15)\quad \pi\|s\|_2 + (a_0 + 2\|r\|_2) + \frac{2}{\pi}(b_0 + 2\|q\|_2) + \frac{4}{\pi^2}(c_0 + \|p\|_2) + \frac{4}{\pi^3}d_0 < \frac{\pi}{2},$$

$$(3.16)\quad \pi\|s\|_2 + (a_0 + \|r\|_2) + \frac{2}{\pi}(b_0 + 2\|q\|_2) + \frac{4}{\pi^2}(c_0 + \|p\|_2) + \frac{8}{\pi^3}d_0 < \frac{\pi}{2},$$

$$(3.17)\quad \pi\|s\|_2 + (a_0 + 2\|r\|_2) + \frac{4}{\pi}(\|p\|_2 + \|q\|_2 + \frac{b_0}{2}) + \frac{4}{\pi^2}(c_0 + d_0) < \frac{\pi}{2},$$

respectively.

Remarks Theorem 3.1 remains valid subject to the boundary conditions

$$u''(0) = u''(1) = u(\eta_1) = u(\eta_2) = 0,$$

instead of (1.2). Similarly Theorem 3.2 is still valid if the boundary conditions

$$u''(0) = u'''(1) = u(\eta_1) = u(\eta_2) = 0$$

hold, instead of (1.9).

4. Some preliminaries to the proofs

The procedure is essentially the same as in [**2**], the starting point is to recast each of the bvps in the abstract form (2.1), which we then embed in

$$Lu + \lambda Nu = e, \quad 0 \le \lambda \le 1,$$

thus facilitating the application of the Leray-Schauder continuation Theorem of Mawhin [**8**]. We shall illustrate this here for the bvp (1.1), (1.2), since a corresponding setting can then be easily stated for each of the other bvps.

Let L be defined by (2.2) and (2.4), where $X = C^3[0,1]$ and $Z = L^1[0,1]$ are Banach spaces with their usual norms. For each $u \in X, z \in Z$ let $(u,z) = \int_0^1 u(t)z(t)\,dt$ denote the duality pair. Define the nonlinear map $N : X \to Z$ by

$$(Nu)(t) = (u''(t))u'''(t) - g(t, u(t), u'(t), u''(t), u'''(t)),$$

so that N is bounded and continuous. Let $K : Z \to X$ be the linear map (given by (2.6)) such that for $z \in Z, Kz \in D(L)$ and $LKz = z$, and furthermore for $u \in D(L)$, $KLu = u$. Then $KN : X \to X$ is a compact mapping, since (as can be verified by the Arzela-Ascoli's theorem) K maps bounded subsets of Z into relatively compact subset of X.

Now $u \in C^3[0,1]$ is a solution of (1.1), (1.2) if and only if u is a solution of the operator equation

$$Lu + Nu = e \tag{4.1}$$

or, equivalently, if and only if, for $\lambda = 1$, u is a solution of

$$Lu + \lambda Nu = \lambda e, \quad 0 \le \lambda \le 1. \tag{4.2}$$

Observe from Lemma 2.1 that for $\lambda = 0, u \equiv 0$ is the only solution of (4.2). Furthermore, with K defined as above, $u \in C^3[0,1]$ is a solution of (1.1), (1.2) if and only if for $\lambda = 1, u$ is a soltuion of

$$u + \lambda KNu = \lambda Ke. \tag{4.3}$$

Thus the Leray-Schauder continuation theorem can be applied to obtain the existence of a solution of (4.3) for $\lambda = 1$. To achieve this it suffices (see [8]) to show that the set of all possible solutions of the parameter dependent bvp

$$u^{(iv)} + \lambda\phi(u'')u''' = \lambda g(t,u,u',u'',u''') + \lambda e(t), \quad 0 \le \lambda \le 1, \tag{4.4}$$

(1.2) is a-priori bounded in X by a constant, independent of $\lambda \in (0,1)$ and of solutions of (4.4), (1.2). Since the existence of K is assured by Lemma 2.2, the proof of existence of solutions of each of the bvps will follow once the desired a-priori bound is established for an appropriately defined parameter dependent equation and its corresponding boundary conditions.

Now, let

$$||u||_\infty = \sup_{0\le t\le 1} |u(t)|, \quad ||u||_1 = \int_0^1 |u(s)|\,ds, \quad ||u||_2^2 = \int_0^1 |u(s)|^2 ds.$$

We shall have cause to use the following results [2; Lemmas 1–4].

LEMMA 4.1. *Let $u \in C^1[0,1]$. If*

(i) $u(0) = 0$, *then* $||u||_2^2 \le \frac{4}{\pi^2}||u'||_2^2$;

(ii) $u(0) = u(1)$, *then* $||u||_2^2 \le \frac{1}{\pi^2}||u'||_2^2$;

(iii) $u(\eta) = 0, 0 \le \eta \le 1$, *then* $||u||_2^2 \le \frac{4}{\pi^2}M_\eta^2||u'||_2^2, \quad M_\eta = \max\{\eta, 1-\eta\}$;

(iv) $u(0) = u(1) = 0$ *then* $||u||_\infty \le \frac{1}{2}||u'||_1$.

5. Proof of Theorems 3.1, 3.2 and 3.3

We start with Theorem 3.1. Let u be an arbitrary solution of (4.4), (1.2). Then by Lemma 4.1,

$$\begin{gathered} ||u''||_2^2 \le \frac{1}{\pi^2}||u'''||_2^2, \qquad ||u'||_2^2 \le \frac{4}{\pi^2}M_{\eta_1}^2||u''||_2^2 \\ ||u||_2^2 \le \frac{4}{\pi^2}M_{\eta_2}^2||u'||_2^2, \quad ||u''||_\infty = \frac{1}{2}||u'''||_1 \end{gathered} \tag{5.1}$$

so that

$$||u||_2^2 \le \frac{16}{\pi^6}M_{\eta_1}^2 M_{\eta_2}^2||u'''||_2^2 \le \frac{16}{\pi^6}||u'''||_2^2, \tag{5.2}$$

$$||u'||_2^2 \le \frac{4}{\pi^4}M_{\eta_1}^2||u'''||_2^2 \le \frac{4}{\pi^4}||u'''||_2^2, \tag{5.3}$$

since $M_{\eta_1} \le 1, M_{\eta_2} \le 1$. Now multiply (4.4) by u'', integrate from 0 to 1 and use (3.3), (3.4) and the fact that $\int_0^1 u''\phi(u'')u'''\,dt = 0$, to obtain

$$\begin{aligned}
-\int_0^1 (u''')^2 dt &\ge \lambda \int_0^1 [a(t)u''u''' + b(t)u''^2 + c(t)|u'u''| \\
&\quad + d(t)|uu''| + h(t)|u''| + e(t)u''] \, dt \\
&\ge -\lambda \Big[(b_0 + \frac{a_0}{2})\|u''\|_2^2 + c_0\|u'\|_2\|u''\|_2 \\
&\quad + d_0\|u\|_2\|u''\|_2 + (\|h\|_1 + \|e\|_1)\|u''\|_\infty \Big].
\end{aligned}$$

From the estimates (5.1), (5.2), (5.3) it follows that

$$\|u'''\|_2^2 \le \left[\left(b_0 + \frac{a_0}{2} \right) \frac{1}{\pi^2} + \frac{2}{\pi^3} c_0 + \frac{4}{\pi^4} d_0 \right] \|u'''\|_2^2 + \frac{1}{2}(\|h\|_1 + \|e\|_1)\|u'''\|_2,$$

so that

$$\|u'''\|_2 \le \frac{\frac{1}{2}(\|h\|_1 + \|e\|_1)}{1 - \frac{1}{\pi^2}\left[\frac{a_0+2b_0}{2} + \frac{2c_0}{\pi} + \frac{4d_0}{\pi^2}\right]} \equiv \rho \tag{5.4}$$

provided (3.5) holds, as we henceforth assume. Thus

$$\|u''\|_\infty \le \rho, \quad \|u'\|_\infty \le \rho, \quad \|u\|_\infty \le \rho \tag{5.5}$$

by (5.4) and (1.2).

Now let

$$\begin{aligned}
M_\rho &= \max |\alpha(t, u, v, w)| \quad \text{for } t \in [0,1] \quad u, v, w \in [-\rho, \rho] \\
N_\rho &= \max |\phi(w)| \quad \text{for } w \in [-\rho, \rho],
\end{aligned}$$

then from (4.4) and (3.1),

$$\|u^{(iv)}\|_1 \le \rho N_\rho + \rho^2 M_\rho + \|\beta\|_1 + \|e\|_1 \equiv \rho^*$$

Since from (1.2), $u'''(t_1) = 0$ for some $t_1 \in (0,1)$, it readily follows now that

$$\|u'''\|_\infty \le \|u^{(iv)}\|_1 \le \rho^*. \tag{5.6}$$

This together with (5.5) give the desired a-priori bound.

Turning now to Theorem 3.2, observe from the boundary conditions (1.9) and Lemma 4.1 that the following estimates now hold:

$$\|u''\|_2^2 \le \frac{4}{\pi^2}\|u'''\|_2^2, \quad \|u'\|_2^2 \le \frac{4}{\pi^2}M_{\eta_1}^2\|u''\|_2^2, \quad \|u\|_2^2 \le \frac{4}{\pi^2}M_{\eta_2}^2\|u'\|_2^2 \tag{5.7}$$

so that, since $M_{\eta_1} < 1, M_{\eta_2} < 1$,

$$(5.8) \qquad \|u\|_2^2 \le \frac{64}{\pi^6}\|u'''\|_2^2, \quad \|u'\|_2^2 \le \frac{16}{\pi^4}\|u'''\|_2^2.$$

The actual proof of the theorem can be obtained by multiplying

$$(5.9) \qquad u^{(iv)} = \lambda g(t,u,u',u'',u''') + \lambda e(t)$$

by u'' and proceeding as in the proof of Theorem 3.1 using (3.1), (3.6), (3.7) and (3.8). Further details will be omitted here.

We turn lastly to the proof of Theorem 3.3. Again by Lemma 4.1 and the boundary conditions (1.4), we have:

$$\|u''\|_2^2 \le \frac{1}{\pi^2}\|u'''\|_2^2, \quad \|u'\|_2^2 \le \frac{4}{\pi^2}\|u''\|_2^2, \quad \|u\|_2^2 \le \frac{4}{\pi^2}M_\eta^2\|u'\|_2^2,$$
$$\|u''\|_\infty \le \frac{1}{2}\|u'''\|_1, \quad \|u\|_2^2 \le \frac{16}{\pi^6}M_\eta^2\|u'''\|_2^2, \quad \|u'\|_2^2 \le \frac{4}{\pi^2}\|u'''\|_2^2.$$

Now let u be any solution of the bvp

$$(5.10) \qquad u^{(iv)} + \lambda\psi(u') = \lambda g(t,u,u',u'',u''') + \lambda e(t),$$
$$(1.4) \qquad u''(0) = u''(1) = u'(0) = u(\eta) = 0.$$

Then on multiplying by u'', integrating from 0 to 1 and using (3.3), (3.4), and the above estimates, we have readily that

$$(5.11) \qquad \|u'''\|_2^2 \le \frac{1}{\pi^2}\left[\left(\frac{a_0}{2} + b_0\right) + \frac{2c_0}{\pi} + \frac{4d_0}{\pi^2}\right]\|u'''\|^2 + \frac{1}{2}(\|e\|_1 + \|h\|_1)\|u'''\|_1 + |F(u'(1))|$$

But from (1.4),

$$|u'(1)| \le \int_0^1 |u''(s)|\,ds = \|u''\|_1 \le \|u''\|_2,$$

so that by (3.9),

$$|\Psi(u'(1))| \le \Delta_0\|u''\|_2 + \Delta_1\|u''\|_2^2 \le \frac{\Delta_0}{\pi}\|u'''\|_2 + \frac{\Delta_1}{\pi^2}\|u'''\|_2^2.$$

Using this in (5.11), we have that

$$\|u'''\|_2 \le \frac{\frac{1}{2}\left[\|e\|_1 + \|h\|_1 + \frac{2\Delta_0}{\pi}\right]}{1 - \frac{1}{\pi^2}\left[\left(\frac{a_0}{2} + b_0 + \Delta_1\right) + \frac{2c_0}{\pi} + \frac{4d_0}{\pi^2}\right]} \le \rho$$

provided (3.10) holds, as we henceforth assume. Thus, by (1.4)

$$\|u''\|_\infty \le \rho, \quad \|u'\|_\infty \le \rho \quad \text{and} \quad \|u\|_\infty \le \rho$$

and the rest of the proof follows as in the preceding case.

6. Proof of Theorems 3.4 and 3.5

Because of the nature of the boundary conditions, the verification of the a-priori bound is slightly more complicated here. Indeed, starting with Theorem 3.4, observe first from (1.6) that there exist $t_i \in (0,1)$, $i = 1,2,3,4$ such that

$$u''(t_1) = 0, \quad u'(t_2) = 0, \quad u''(t_3) = 0, \quad u'''(t_4) = 0.$$

Thus

$$|u''(0)| \le \left| \int_0^{t_1} u'''(s)\,ds \right| \le \|u'''\|_1, \quad \|u''\|_\infty \le \|u'''\|_1 \le \|u'''\|_2 \tag{6.1}$$

and similarly

$$|u''(1)| \le \|u'''\|_1, \quad |u'''(0)| \le \|u^{(iv)}\|_1, \quad |u'''(1)| \le \|u^{(iv)}\|_2. \tag{6.2}$$

Moreover, by Lemma 4.1,

$$\|u\|_2^2 \le \frac{4}{\pi^2} M_{\eta_1}^2 \|u'\|_2^2, \quad \|u'\|_2^2 \le \frac{1}{\pi^2}\|u''\|_2^2, \quad \|u''\| \le \frac{4}{\pi^2} M_{t_3}^2 \|u'''\|_2^2. \tag{6.3}$$

Now, multiply

$$u^{(iv)} + \lambda\chi(u')u'' = \lambda g(t,u,u',u'',u''') + \lambda e(t), \quad 0 \le \lambda \le 1, \tag{6.4}$$

by u'' and integrate from 0 to 1 to obtain

$$\begin{aligned}
-\int_0^1 (u''')^2 dt &= u'''(0)u''(0) - u'''(1)u''(1) - \lambda \int_0^1 \chi(u')(u'')^2 dt \\
&\quad + \lambda \int_0^1 u'' g(t,u,u',u'',u''') dt + \lambda \int_0^1 u'' e(t)\,dt \\
&\ge \lambda \int_0^1 [a(t)\|u''\|\|u'''\| + b(t)(u'')^2 + c(t)u'u'' + d(t)|uu''|]\,dt \\
&\quad - \lambda \int_0^1 \chi(u')(u'')^2 dt - 2\|u'''\|_1 \|u^{(iv)}\|_1 - (\|e\|_1 + \|h\|_1)\,\|u''\|_\infty,
\end{aligned}$$

by (3.11), (6.1) and (6.2). Therefore, by (3.13), (3.12) and the estimates (6.3),

$$\|u'''\|_2^2 \leq \left[\frac{2}{\pi}\left\{\left(a_0 + \frac{2b_0}{\pi}\right) + \frac{c_0}{\pi^3} + \frac{4d_0}{\pi^3} + \frac{2\Delta_0}{\pi}\right\}\right] \|u'''\|_2^2 + 2\|u'''\|_1\|u^{(iv)}\|_1 + (\|e\|_1 + \|h\|_1)\|u''\|_\infty, \tag{6.5}$$

since $M_{\eta_1} \leq 1, M_{t_3} \leq 1$. But from (6.4), (3.2), (3.13) and (6.3),

$$\|u^{(iv)}\|_1 \leq \left\{\frac{4}{\pi^3}\|p\|_2 + \frac{2}{\pi^2}\|q\|_2 + \frac{2}{\pi}\|r\|_2 + \|s\|_2 + \frac{2}{\pi}\Delta_0\right\} \|u'''\|_2 + (\|m\|_1 + \|e\|_1), \tag{6.6}$$

so that on combining this with the estimate (6.5) we have readily that

$$\|u'''\|_2 \leq \frac{3\|e\|_1 + 2\|m\|_1 + \|h\|_1}{1 - P} = \rho,$$

where

$$P = 2\|s\|_2 + \frac{2}{\pi}(2\Delta_0 + a_0 + 2\|r\|_2) + \frac{4}{\pi^2}(\Delta_0 + b_0 + \|q\|_2) + \frac{8}{\pi^3}\|p\|_2 + \frac{2}{\pi^4}(c_0 + 4d_0),$$

and $P < 1$ by (3.14).

Thus

$$\|u''\|_\infty \leq \rho, \quad \|u'\|_\infty \leq \rho, \quad \|u\|_\infty \leq \rho, \tag{6.7}$$

and by (6.5), $\|u^{(iv)}\|_1 \leq \rho_*$ so that

$$\|u'''\|_\infty \leq \|u^{(iv)}\|_1 \leq \rho_* \tag{6.8}$$

Both (6.7) and (6.8) give the desired a-priori bound.

Next, we give an indication of the proof of Theorem 3.5 for the bvp (1.7), (1.11). From the boundary conditions (1.11) and Lemma 4.1, it will be evident that $u'''(t_1) = 0$ for $t_1 \in (0, 1)$, and that

$$\|u\|_2 \leq \|u'\|_2, \quad \|u'\|_2^2 \leq \frac{4}{\pi^2}\|u''\|_2^2, \quad \|u''\|_2^2 \leq \frac{4}{\pi^2}M_{\eta_1}^2\|u'''\|_2^2.$$

Thus, on multiplying (5.9) by u'' and arguing as in the above, we have readily that

$$\begin{aligned}\|u'''\|_2^2 &\leq 2\|u'''\|_1\|u^{(iv)}\|_1 + \frac{2}{\pi}\left[a_0 + \frac{2}{\pi}b_0 + \frac{4}{\pi^2}(c_0 + d_0)\right] \|u'''\|_2^2 \\ &\quad + (\|e\|_1 + \|h\|_1)\|u'''\|_1 \\ &\leq \frac{2}{\pi}\left[\pi\|s\|_2 + (a_0 + 2\|r\|_2) + \frac{4}{\pi}(\|p\|_2 + \|q\|_2 + \frac{1}{2}b_0) \right. \\ &\quad \left. + \frac{4}{\pi^2}(c_0 + d_0)\right] \|u'''\|_2^2 + (3\|e\|_1 + 2\|m\|_1 + \|h\|_1)\|u'''\|_2,\end{aligned} \tag{6.9}$$

so that by (3.17) $\|u'''\|_2 \le \rho$.
Thus

$$\|u''\|_\infty \le \rho, \quad \|u'\|_\infty \le \rho, \quad \|u\|_\infty \le \rho$$

where $\rho > 0$ is a finite constant. The rest of the proof then follows as before.
The proof, for the other bvps (1.7), (1.8) and (1.7), (1.10) can be similarly obtained; further details are omitted here.

Remarks The existence of at least one solution for each of the bvps in Theorems 3.4 and 3.5 is still assured if instead of the condition (3.6) or (3.11) g is assumed to satisfy

$$(6.10)\quad g(t,u,v,w,z)w \ge -s(t)|z||w| - r(t)w^2 - q(t)|v||w| - p(t)|u||w| - m(t)|w|,$$

a condition that is implied by (3.2), as can be readily verified. Indeed, the following special case of Theorem 3.4 holds:

THEOREM 3.4′. *Let g satisfy hypothesis G_2 and let χ satisfy* (3.13). *Then for arbitrary $e \in L^1[0,1]$, the bvp* (1.5), (1.6) *has at least one solution if*

$$(6.11)\qquad \|s\|_2 + \frac{2}{\pi}\|r\|_2 + \frac{2}{\pi^2}\|q\|_2 + \frac{4}{\pi^3}\|p\|_2 + \frac{2}{\pi}\Delta_0 < \frac{1}{2}\left(\frac{\pi}{\pi+1}\right)$$

A corresponding result can be stated for Theorem 3.5.

7. Uniqueness of solutions

Here we shall be concerned with fourth order differential equations of the forms:

$$(7.1)\qquad u^{(iv)} + Au''' = g(t,u,u',u'',u''') + e(t),$$

$$(7.2)\qquad u^{(iv)} + Au' = g(t,u,u',u'',u''') + e(t),$$

$$(7.3)\qquad u^{(iv)} + Au'' = g(t,u,u',u'',u''') + e(t),$$

with boundary conditions (1.2), (1.4) and (1.6) respectively, where A is a constant. We shall also consider the equation

$$(1.7)\qquad u^{(iv)} = g(t,u,u',u'',u''') + e(t),$$

together with each of the boundary conditions (1.8), (1.9), (1.10) and (1.11).
Our uniqueness theorems will be grouped into two, according as g satisfies hypothesis G_1 or G_2. We start with G_2, and for this we make the following basic assumption:

G_3 : There exist functions $p, q, r, s \in L^2[0,1]$ such that for all $(u_i, v_i, w_i, z_i) \in \mathbf{R}^4$, $i = 1,2$ and for a.e. $t \in [0,1]$

$$
\begin{aligned}
&|g(t,u_2,v_2,w_2,z_2)-g(t,u_1,v_1,w_1,z_1)| \\
&\quad \le p(t)|u_2-u_1|+g(t)|v_2-v_1|+r(t)|w_2-w_1|+s(t)|z_2-z_1|
\end{aligned} \tag{7.4}
$$

THEOREM 7.1. *Let the function g satisfy hypothesis G_3 and let $e \in L^1[0,1]$ be an arbitrary function. Then each of the bvps* (7.3), (1.6); (1.7), (1.8); (1.7), (1.10)*; and* (1.7), (1.11) *has a unique solution if*

$$\|s\|_2+\frac{2}{\pi}\|r\|_2+\frac{2}{\pi^2}\|q\|_2+\frac{4}{\pi^3}\|p\|_2+\frac{2}{\pi}|A|<\frac{1}{2}\left(\frac{\pi}{1+\pi}\right) \tag{7.5}$$

$$\|s\|_2+\frac{2}{\pi}\|r\|_2+\frac{4}{\pi^2}\|q\|_2+\frac{4}{\pi^3}\|p\|_2<\frac{1}{2}\left(\frac{\pi}{1+\pi}\right) \tag{7.6}$$

$$\|s\|_2+\frac{2}{\pi}\|r\|_2+\frac{4}{\pi^2}\|q\|_2+\frac{8}{\pi^3}\|p\|_2<\left(\frac{\pi}{2+\pi}\right) \tag{7.7}$$

$$\|s\|_2+\frac{2}{\pi}\|r\|_2+\frac{4}{\pi^2}(\|q\|_2+\|p\|_2)<\frac{1}{2}\left(\frac{\pi}{1+\pi}\right) \tag{7.8}$$

respectively.

For the case g satisfying G_1, the following assumptions will be made:

G_4 : there exist functions $a \in C^1[0,1]$ and $b,c,d \in C^0[0,1]$ satisfying (3.4) such that for all $(u_i,v_i,w_i,z_i) \in \mathbf{R}^4$, $i=1,2$ and for a.e. $t \in [0,1]$

$$
\begin{aligned}
&(g(t,u_2,v_2,w_2,z_2)-g(t,u_1,v_1,w_1,z_1))(w_2-w_1) \\
&\quad \ge a(t)(z_2-z_1)(w_2-w_1)+b(t)(w_2-w_1)^2 \\
&\qquad +c(t)|v_2-v_1||w_2-w_1|+d(t)|u_2-u_1||w_2-w_1|
\end{aligned} \tag{7.8}
$$

G_5 : there exist functions $a,b,c,d \in C^0[0,1]$ satisfying (3.7) such that for all $(u_i,v_i,w_i,z_i) \in \mathbf{R}^4$, $i=1,2$ and for a.e. $t \in [0,1]$

$$
\begin{aligned}
&(g(t,u_2,v_2,w_2,z_2)-g(t,u_1,v_1,w_1,z_1))(w_2-w_1) \\
&\quad \ge a(t)|z_2-z_1||w_2-w_1|+b(t)(w_2-w_1)^2 \\
&\qquad +c(t)|v_2-v_1||w_2-w_1|+d(t)|u_2-u_1||w_2-w_1|
\end{aligned} \tag{7.9}
$$

THEOREM 7.2. *Let g satisfy hypothesis G_1 and G_4 and let $e \in L^1[0,1]$ be an arbitrary function. Then each of the bvps* (7.1), (1.2); (7.2), (1.4) *has a unique solution if the conditions* (3.4) *and* (3.10)*, with $\Delta_1 = \frac{1}{2}|A|$, hold respectively.*

THEOREM 7.3. *Let g satisfy hypotheses G_1 and G_5. Then for arbitrary function $e \in L^1[0,1]$ the bvp* (1.7), (1.9) *has a unique solution if condition* (3.8) *holds.*

8. Proof of uniqueness results

We start with Theorem 7.1 and observe from condition (7.4) that g satisfies

$$|g(t,u,v,w,z)| \le p(t)|u| + q(t)|v| + r(t)|w| + s(t)|z| + |g(t,0,0,0,0)|, \tag{8.1}$$

for all $(u,v,w,z) \in \mathbf{R}^4$ and for a.e. $t \in [0,1]$. Thus (6.10) holds and the existence of at least one solution for each of bvps in Theorem 7.1 is assured by Theorems 3.4′ and 3.5. It suffices, therefore to verify that solutions are unique. We show this for the bvp (7.3), (1.6).

To this end, let u_1, u_2 be two distinct solutions of (7.3), (1.6) and set $X = u_2 - u_1$. Then

$$\begin{aligned} X^{(iv)} + AX'' &= g(t,u_2,u_2',u_2'',u_2''') - g(t,u_1,u_1',u_1'',u_1'''),\\ X(\eta_1) &= X(\eta_2) = X'(0) = X'(1) = 0 \end{aligned} \tag{8.2}$$

Multiplying (8.2) by X'' and integrating from 0 to 1, we have, on using (7.4) and the estimates (6.3), which now holds for X, that

$$\begin{aligned} \|X'''\|_2^2 \le \frac{2}{\pi}\left[\frac{4}{\pi^3}\|p\|_2 + \frac{2}{\pi^2}\|q\|_2 + \frac{2}{\pi}\|r\|_2 + \|s\|_2 + \frac{2}{\pi}|A|\right]\|X'''\|_2^2 \\ + 2\|X'''\|_1\|X^{(iv)}\|_1. \end{aligned}$$

But, by (8.2) and (7.4),

$$\|X'''\|_1\|X^{(iv)}\|_1 \le \left[\frac{4}{\pi^3}\|p\|_2 + \frac{2}{\pi^2}\|q\|_2 + \frac{2}{\pi}\|r\|_2 + \|s\|_2 + \frac{2}{\pi}|A|\right]\|X'''\|_2^2.$$

Therefore,

$$\|X'''\|_2^2 \le 2\left(1+\frac{1}{\pi}\right)\left[\frac{4}{\pi^3}\|p\|_2 + \frac{2}{\pi^2}\|q\|_2 + \frac{2}{\pi}\|r\|_2 + \|s\|_2 + \frac{2}{\pi}|A|\right]\|X'''\|_2^2, \tag{8.3}$$

so that by (7.5), $\|X'''\|_2^2 \le 0$. Thus $\|X'''\|_2^2 = 0$, and since $\|X\|_2^2 \le \frac{16}{\pi^6}\|X'''\|_2^2$, it is clear that $X(t) = 0$ for every $t \in [0,1]$. Therefore $u_1 = u_2$.

The verification of uniqueness for the other bvps in Theorem 7.1 proceeds as in the above, with obvious modifications in the arguments. Details are omitted here.

We turn now to the proof of Theorems 7.2 and 7.3. Observe first that hypotheses (7.8) and (7.9) respectively imply (3.3) and (3.6), so that under the conditions of Theorem 7.2 (resp 7.3) the existence of solutions is assured by Theorem 3.1 (resp 3.2). It therefore suffices to verify uniqueness of solutions for each of the bvps in Theorems 7.2 and 7.3.

To this end, let u_1, u_2 be two distinct solutions of (7.1), (1.2) and set $X = u_2 - u_1$. Then

$$\begin{aligned} X^{(iv)} + AX''' &= g(t,u_2,u_2',u_2'',u_2''') - g(t,u_1,u_1',u_1'',u_1'''),\\ X''(0) &= X''(1) = X'(\eta_1) = X'(\eta_2) = 0. \end{aligned} \tag{8.4}$$

Multiplying (8.4) by X'' and integrating between 0 and 1, it will follow, in view of (7.8), the estimates (5.2) and (5.3) and the arguments in §5, that

$$\|X'''\|_2^2 \leq \frac{1}{\pi^2}\left[\left(\frac{a_0}{2}+b_0\right)+\frac{2c_0}{\pi}+\frac{4d_0}{\pi^2}\right]\|X'''\|_2^2,$$

so that by (3.4), $\|X'''\|_2^2 = 0$. The fact that $u_1 = u_2$ now follows from the estimate $\|X\|_2^2 \leq \frac{16}{\pi^6}\|X'''\|_2^2$. The case (7.2), (1.4) can be similarly handled.

The proof of Theorem 7.3 can be obtained as in the above, using the appropriate estimates and obvious adaptations of the arguments in the proof of Theorem 3.2. Details of this are omitted here.

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