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EXISTENCE AND UNIQUENESS THEOREMS FOR A CLASS OF THREE POINT FOURTH ORDER BOUNDARY VALUE PROBLEM

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ABSTRACT. Existence and uniqueness results are obtained for the fourth order boundary-value problem of the form:

$$\begin{cases} x'' + f(x'')x''' + h(x')x'' = g(t,x,x',x'',x''') \\ x(0) = x''(0) = x''(1) = x(\eta) = 0, \ 0 < \eta < 1 \end{cases}$$

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1. INTRODUCTION

In a recent paper [5] Iyase investigated the Existence and Uniqueness of solutions of four-point boundary value problem (bvp) of the form

$$x^{(4)} = f(t,x,x',x'',x''') + e(t)$$
$$x(0) = x(\eta_1) = x(\eta_2) = x(1) = 0, \ \eta_1 \neq \eta_2,$$

where $f : [0,1] \times \Re^4 \to \Re$ satisfies the Caratheodory condition, $e \in L^1[0,1]$ and $0 \leq \eta_i \leq 1, \ i = 1,2$

Following this investigation, Tejumola, Tchenagi and Iyase [6] obtain Existence and Uniqueness results for a wider class of fourth order equations subject to varied boundary conditions. Prior to these

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with norm $\|x\|^2_{H^3} = \sum_{j=1}^{3} \int (x^{(i)}(t)^2 dt + \int x dt)^2$.
$W^{4,1}(0,1) = \{x : [0,1] \to \Re|^{x}_{x_1}, x'', x''' \text{ abs. cont. on } [0,1]\}$,
with norm $\|x\|^2_{W[0,1]^{4,13}} = \sum_{j=1}^{4} \int \|x^{(j)}(t)\| dt$
We now set

$$D(L) = \{x \in W^{4,1}(0,1) : x \text{ satisfies } (1.2)\} \tag{2.3}$$

We have the following result

Lemma 2.1: Let L be defined by (2.1) and (2.3). Then $\ker L = \{0\}$.

Proof. The proof follows same procedures as Lemma 2.1 of [6]

Corollary 2.1 Subject to the conditions of Lemma 2.1 the equation

$$Lx = e \tag{2.4}$$

together with the boundary condition (1.2) has a unique solution

We observe that since $\ker L = \{0\}$, there exists a linear map $K : Z \to X$ such that for $z \in Z$, $Kz \in D(L)$ and $LKz = z$; and for $x \in D(L)$, $KLx = x$. Thus $x = Ke$ is the unique solution of (2.4) for each $e \in Z$.

Lemma 2.2 Let L be defined by (2.1). Then the corresponding linear map $K : Z \to X$ is given by

$$(Kz)(t) = \frac{1}{6} + \int (t - S)^3 y(s) ds + Bt + Dt^3, \tag{2.5}$$

where

$B = \frac{1}{6} + \left[\eta^2 \int_0^1 (1 - S) y(s) ds - \frac{1}{\eta} \int_0^{\eta} (\eta - S)^3 y(s) ds\right]$

$D = -\frac{1}{6} + \int_0^1 (1 - S) y(s) ds$

The expressions above can be obtain by repeated integration of $Lx - z$ using (1.2)

3. EXISTENCE THEOREMS

We have the following results:

Theorem 3.1: Let $g : [0,1] \times \Re^4 \to \Re$ be a function satisfying the Caratheodory conditions, $f, h \in (\Re, \Re)$.

$x, x', x'', x''' \in \Re$ a.e $t \in [0,1]$. Then for every $p(t) \in L^1([0,1])$ problem (1.1) with (1.2) has at least one solution provided $\pi^4\|a\|_\infty + \pi^3\|b\|_\infty + 2\pi^2\|c\|_\infty + 8\|e\|_\infty + \pi^3|k| < \pi^5$

Theorem 3.4: Let $g : [0,1] \times \Re^4 \to \Re$ satisfy the Caratheodory's conditions, $f, h : [0,1] \times \Re^4 \to \Re$ be continuous functions. Assume that for almost every $t \in [0,1]$, the function $g(t,u,v,w,z)$ is continuously differentiable with respect to u,v,w,z. Suppose that there exists real numbers a_0, b_0, c_0, and e_0 such that
$\frac{\partial}{\partial u}g(t,u,v,w,z) \geq -a_0$, $\frac{\partial}{\partial v}g(t,0,v,w,z) \geq -b_0$,
$\frac{\partial}{\partial w}g(t,0,0,w,z) \geq -c_0$, $\frac{\partial}{\partial z}|g(t,0,0,0,z)| \leq e_0$
for a.e $t \in [0,1]$ and $u,v,w,z \in \Re$. Suppose that $h(x') \leq k$, $k \in \Re$, and there exists a continuous functions $\alpha : [0,1] \times \Re^4 \to \Re$ and $\beta(t) \in L^1([0,1])$ such that

$$|g(t,u,v,w,z)| \leq |\alpha(t,u,v,w,z)||z|^2 + \beta(t) \tag{3.3}$$

for every $u,v,w,z \in \Re$ a.e $t \in [0,1]$. Then for every given $p(t) \in L^1([0,1])$ the boundary value problem (1.1) with (1.2) has at least one solution provided

$$4a_0 + 2b_0 + \pi c_0 + \pi^2 e_0 + |k| < \pi^3 \tag{3.4}$$

4. SOME PRELIMINARIES TO THE PROOFS

Following the procedure in [6], we recast the bvp (1.1),(1.2) in the abstract form which we then embed in
$Lu + \lambda Nu = e$, $0 \leq \lambda \leq 1$.
We then apply the Leray-Schauder continuation theorem of Mawhin [7]. Let L be defined by (2.1) and (2.3) where X is the Banach Space $C^3([0,1])$ and Z the Banach Space $L^1([0,1])$ with their usual norms. For each $u \in X, z \in Z$, let $(u,z) = \int u(t)z(t)dt$ denote the duality pair. We define the nonlinear mapping $N : X \to Z$ by

$$(Nu)(t) = f((u'')u''' + h(u')u'' - g(t,u,u',u'',u'''))u''' \tag{4.1}$$

is apriori bounded in $C^3[0,1]$ independently of solution $x(t)$ and λ. Multiplying (4.5) by x'' and integrating over $[0,1]$, we have

$$\int_0^1 x^{(iv)}x''dt + \lambda\int_0^1 f(x'')x'''x''dt + \lambda\int_0^1 h(x')(x'')^2dt$$
$$-\lambda\int_0^1 g(t,x,x',x'',x''')x''dt + \lambda\int_0^1 p(t)x''dt$$

Since $x''(0) = x'(1)$, it follows that $\lambda\int_0^1 f(x'')x'''x''dt = 0$ and from condition (ii) we get

$$\begin{aligned}
-\int_0^1 \|x'''\|dt \;\geq\; & \lambda\int_0^1 a(t)x''x'''dt + \lambda\int_0^1 b(t)(x'')^2dt \\
& +\lambda\int_0^1 c(t)|x'x''|dt + \lambda\int_0^1 d(t)|x''|dt \\
& +\lambda\int_0^1 e(t)|xx'|dt - \lambda\int_0^1 h(x')(x'')^2dt \\
& +\lambda\int_0^1 p(t)x''dt - \|x'''\|_2^2 \\
\geq\; & \frac{\lambda}{2}a_o^o\|x''\|_2^2 - \lambda b_o\|x''\|_2^2 - \lambda c_o\|x''\|_2\|x''\|_2 \\
& -\lambda\|d\|_1\|x''\|_\infty - e_o\|x\|_2\|x''\|_2 - \lambda k\|x''\|_2^2 \\
& -\lambda\|p\|_1\|x''\|_\infty.
\end{aligned}$$

We observe that $x(0) = x(\eta) = 0$, there exists $t_1 \in (0,1)$ with $t_1 < \eta < 1$, such that $x_1'(t_1) = 0$. It follows

$$\|x\|_\infty \leq \|x'\|_2,\; \|x\|_2^2 \leq \frac{4}{\pi^2}\|x\|_2^2 \tag{5.1}$$

$$\|x'\|_\infty \leq \|x''\|_2,\; \|x\|_2^2 \leq \frac{4}{\pi^2}\|x''\|_2^2. \tag{5.2}$$

Since that $x''(0) = x''(1)$ we have from Lemma 4.1

$$\|x''\|_2^2 \leq \frac{1}{\pi^2}\|x'''\|_2^2 \tag{5.3}$$

$$\|x''\|_\infty \leq \frac{1}{2}\|x'\|_2 \tag{5.4}$$

Using the inequalities (5.1) to (5.4), we obtain
$\|x'''\|_2^2 \leq \left(\frac{a_0}{2\pi^2} + \frac{b_0}{\pi^2} + \frac{c_0}{\pi^3} + \frac{8e_0}{\pi^5} + \frac{|k|}{\pi^2}\right)\|x'''\|_2^2 + \frac{1}{2}\|d\|_1\|x'''\|_2 + \frac{1}{2}\|p\|_1\|x'''\|_2$

Next we prove Theorem 3.4. Multiplying (4.5) by x'' and integrating from 0 to 1, we get

$$\begin{aligned}
&\int_0^1 x''x^{iv}dt + \lambda\int_0^1 f(x'')x'''x''dt + \lambda\int_0^1 h(x')(x'')^2dt \\
= &\ \lambda\int_0^1 g(t,x,x'x'',x''')x''dt + \lambda\int_0^1 p(t)x''dt \\
- &\ \lambda\int_0^1\int_0^1 \frac{\partial y}{\partial u}(t,sx,x',x'',x''')x''xdsdt \\
&+\lambda\int_0^1\int_0^1 \frac{\partial y}{\partial v}(t,0,sx',x'',x''')x''x'dsdt \\
&+\lambda\int_0^1\int_0^1 \frac{\partial y}{\partial w}(t,0,0,sx'',x''')(x'')^2dsdt \\
&+\lambda\int_0^1\int_0^1 \frac{\partial y}{\partial z}(t,0,0,0,sx''')x'''x''dsdt \\
&+\lambda\int_0^1 g(t,0,0,0,0)x''dt + \lambda\int_0^1 p(t)x''dt
\end{aligned}$$

$$\begin{aligned}
-\|x'''\|_2^2 \ \geq & \ -a_0\int_0^1 |x''||x|dt - b_0\int_0^1 |x''||x'|dt - c_0\int_0^1 |x''|^2dt \\
& -c_0\int_0^1 |x'''||x''|dt - \int_0^1 \beta(t)|x''|dt - k\int_0^1 |x''|^2dt \\
& -\int_0^1 |p(t)||x''|dt
\end{aligned}$$

$$\begin{aligned}
\|x'''\|_2^2 \ \leq & \ a_0\|x''\|_2^2\|x\|_2 + b_0\|x''\|_2\|x'\|_2 + c_0\|x''\|_2^2 + c_0\|x'''\|_2\|x''\|_2 \\
& +k\|x''\|_2^2 + \|\beta\|_1\|x''\|_\infty + \|p\|\|x''\|_\infty \\
\leq & \left(\frac{4a_0}{\pi^3} + \frac{2b_0}{\pi^3} + \frac{c_0}{\pi^2} + \frac{c_0}{\pi} + \frac{|k|}{\pi^2}\right)\|x'''\|_2^2 + \frac{1}{2}\|\beta\|_1\|x'''\|_2
\end{aligned}$$

or

$$\|x'''\|_2 \leq \frac{\frac{1}{2}\pi^3(\|\beta\|_1 + \|p\|_1)}{\pi^3 - (4a_0 + 2b_0 + \pi c_0 + \pi^2 c_0 + |k|)} = \rho_2$$

Thus

$$\|x\|_\infty \leq \|x'\|_\infty \leq \|x''\|_\infty \leq \rho_2$$

on multiplying (6.4) by $(u_1 - u_2)''$ and integrating from 0 to 1 we get

$$-\int_0^1 [(u_1 - u_2)''']^2 dt + B\int_0^1 [u_1'' - u_2'']^2 dt -$$
$$\int_0^1 (g(t, u_1, u_1', u_1'', u_1''')(u_1 - u_2)'')dt$$
$$-\int_0^1 (g(t, u_2, u_2', u_2'', u_2''')(u_1 - u_2)'')dt \tag{6.6}$$

Let $v = u_1 - u_2$, then

$$\begin{aligned}
\int_0^1 (v''')^2 dt &\geq \int_0^1 a(t)v''v'''dt + \int_0^1 b(t)(v'')^2 dt + \int_0^1 c(t)|v'v''|dt \\
&\quad + \int_0^1 e(t)|vv'|dt - B\int_0^1 (v'')^2 dt \\
&\geq -\frac{1}{2}a_0\int_0^1 |v''|^2 dt - b_0\int_0^1 b|v''|^2 dt - c_0\int_0^1 |v'||v''|dt \\
&\quad -e_0\int_0^1 |v|v'|dt - B\int_0^1 |v''|^2 dt
\end{aligned}$$

$$\begin{aligned}
\|v'''\|_2^2 &\leq \frac{a_0}{2}\|v'''\| + b_0\|v'''\|_2^2 + c_0|v'|_2|v''|_2 + e_0|v|_2|v'|_2 + |B|\|v''\|_2^2 \\
&\leq \left(\frac{a_0}{2\pi^2} + \frac{b_0}{\pi^2} + \frac{2c_0}{\pi^3} + \frac{8e_0}{\pi^5} + \frac{B}{\pi^2}\right)\|v'''\|_2^2
\end{aligned}$$

or

$$\left[2\pi^5 - (\pi^3 a_0 + 2\pi^3 b_0 + 4\pi^2 c_0 + 16e_0 + 2\pi^3|B|)\right]\|v'''\|_2^2 \leq 0$$

From condition (6.3) we derive $\|v'''\|_2^2 \leq 0$ and hence $\|v'''\|_2^2 = 0$. Since $\|v'\|_\infty \leq \|v'\|_2 \leq 0$ we have $v(t) = 0$ and hence $u_1(t) = u_2(t)$ a.e $t \in [0,1]$ by the continuity of $u_1(t)$ and $u_2(t)$.

Theorem 6.2 Let $g : [0,1] \times \Re^4 \to \Re$ be a Caratheodory function. Assume that there exists $a(t), b(t), c(t), d(t) \in L^\infty([0,1])$ such that

$$|g(t, x_1, x_1', x_1'', x_1''') - g(t, x_2, x_2', x_2'', x_2''')| \leq$$
$$a(t)|x_1 - x_2| + b(t)|x_2' - x_2''| + c(t)|x_1'' - x_2''| + d(t)|x_1''' - x_2'''|$$

Then for every $p(t) \in L^1([0,1])$ the boundary value problem (6.1) with (6.2) has a unique solution provided

$$\pi^4\|a\|_\infty + \pi^3\|b\|_\infty + \pi^2\|c\|_\infty + 8\|e\|_\infty + \pi^3|B| < \pi^5$$