EXISTENCE AND UNIQUENESS THEOREMS FOR A CLASS OF THREE POINT FOURTH ORDER BOUNDARY VALUE PROBLEM

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ABSTRACT. Existence and uniqueness results are obtained for the fourth order boundary-value problem of the form:

\[
\begin{cases}
  x' + f(x, x', x'', x''') = g(t, x, x', x'', x''') \\
  x(0) + x'(0) = 0, 0 \leq \eta < 1
\end{cases}
\]

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1. Introduction

In a recent paper [5], Iyase investigated the existence and uniqueness of solutions of four-point boundary value problem (bvp) of the form

\[
\begin{cases}
  x(4) + f(t, x, x', x'', x''') = e(t) \\
  x(0) = x(\eta_1) = x(\eta_2) = x(1) = 0, \eta_1 \neq \eta_2
\end{cases}
\]

where \( f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R} \) satisfies the Caratheodory condition, \( e \in L^1[0, 1] \) and \( 0 \leq \eta_1 < 1, 1 - 1, 2 \).

Following this investigation, Tejumola, Tchenagi and Iyase [6] obtain existence and uniqueness results for a wider class of fourth order equations subject to varied boundary conditions. Prior to these
with norm \( \|x\|_{H^2}^2 = \sum_{j=1}^4 \left( \int x^{(j)}(t)^2 dt + \int x^2 dt \right) \).

\( W^{4,1}(0,1) = \{ x : [0,1] \to \mathbb{R}^4, x, \ldots, x^{(3)} \text{ abs. cont. on } [0,1] \} \),

with norm \( \|x\|_{W^{4,1}[0,1]}^2 = \sum_{j=1}^4 \int |x^{(j)}(t)|^2 dt \)

We now assert

\[ D(L) = \{ x \in W^{4,1}[0,1] : x \text{ satisfies (1.2)} \} \]  \hspace{1cm} (2.3)

We have the following result

**Lemma 2.1:** Let \( L \) be defined by (2.1) and (2.3). Then \( \ker L = \{0\} \).

**Proof:** The proof follows the same procedures as Lemma 2.1 of \( [6] \).

**Corollary 2.1** Subject to the conditions of Lemma 2.1 the equation

\[ Lx = c \]  \hspace{1cm} (2.4)

together with the boundary condition (1.2) has a unique solution

We observe that since \( \ker L = \{0\} \), there exists a linear map \( K : Z \to X \) such that for \( z \in Z \), \( Kz \in D(L) \) and \( LKz = z \), and for \( x \in D(L) \), \( Kx = x \). Thus \( x = Kc \) is the unique solution of (2.4) for each \( c \in Z \).

**Lemma 2.2** Let \( L \) be defined by (2.1). Then the corresponding linear map \( K : Z \to X \) is given by

\[ (Kz)(t) = \frac{1}{6} + \int (t - S)y(s)ds + Bt + Dc^2, \]  \hspace{1cm} (2.5)

where

\[ B = \frac{1}{6} + \int_0^1 \left[ \frac{1}{2} \int_0^1 (1 - S)y(s)ds - \frac{1}{2} \int_0^1 \eta y(s)ds \right] \]

\[ D = -1 + \int_0^1 (1 - S)y(s)ds \]

The expressions above can be obtained by repeated integration of \( Lx - z \) using (1.2).

3. **Existence Theorems**

We have the following results:

**Theorem 3.1:** Let \( g : [0,1] \times \mathbb{R}^4 \to \mathbb{R} \) be a function satisfying the Carathéodory conditions, \( f, h \in (\mathbb{R}, \mathbb{R}) \).
Theorem 3.4: Let \( g : [0, 1] \times \mathbb{R}^4 \to \mathbb{R} \) satisfy the Carathéodory’s conditions, \( f, h : [0, 1] \times \mathbb{R}^4 \to \mathbb{R} \) be continuous functions. Assume that for almost every \( t \in [0, 1] \), the function \( g(t, u, v, w, z) \) is continuously differentiable with respect to \( u, v, w, z \). Suppose that there exists real numbers \( a_0, b_0, c_0 \), and \( c_0 \) such that

\[
\begin{align*}
\frac{\partial}{\partial u}g(t, u, v, w, z) &\geq -a_0, \\
\frac{\partial}{\partial v}g(t, u, v, w, z) &\geq -b_0, \\
\frac{\partial}{\partial w}g(t, u, v, w, z) &\geq -c_0, \\
\frac{\partial}{\partial z}g(t, u, v, w, z) &\leq c_0
\end{align*}
\]

for almost every \( t \in [0, 1] \) and \( u, v, w, z \in \mathbb{R} \). Suppose that \( h(x') \leq k \), \( k \in \mathbb{R} \), and there exists a continuous functions \( \alpha : [0, 1] \times \mathbb{R}^4 \to \mathbb{R} \) and \( \beta(t) \in L^1([0, 1]) \) such that

\[
[g(t, u, v, w, z)] \leq [\alpha(t, u, v, w, z)]|z|^2 + \beta(t) \tag{3.3}
\]

for every \( u, v, w, z \in \mathbb{R} \), \( t \in [0, 1] \). Then for every given \( p(t) \in L^1([0, 1]) \) the boundary value problem (1.1) with (1.2) has at least one solution provided

\[
4a_0 + 2b_0 + c_0 + \pi^2 c_0 + |k| < \pi^3 \tag{3.4}
\]

4. SOME PRELIMINARIES TO THE PROOFS

Following the procedure in [6], we recast the bvp (1.1), (1.2) in the abstract form which we then embed in

\( Lu + \lambda Nu = \psi, \quad 0 \leq \lambda \leq 1. \)

We then apply the Leray-Schauder continuation theorem of Mawhin [7]. Let \( L \) be defined by (2.1) and (2.3) where \( X \) is the Banach Space \( C^2([0, 1]) \) and \( Z \) the Banach Space \( L^1([0, 1]) \) with their usual norms.

For each \( u \in X, z \in Z \), let \( (u, z) - \int u(t)z'(t)dt \) denote the duality pair. We define the nonlinear mapping \( N : X \to Z \) by

\[
(Nu)(t) = f((u^r)u^r' + h(u')u^r'' - g(t, u, u', u''))u^r' \tag{4.1}
\]
is a priori bounded in $C^3[0,1]$, independently of solution $x(t)$ and $\lambda$.

Multiplying (4.5) by $x''$ and integrating over $[0,1]$, we have

\[ \int_0^1 x''' x'' dt + \lambda \int_0^1 f(x') x'''' dt + \lambda \int_0^1 h(x') x'' dt \]

\[ - \lambda \int_0^1 g(x',x'',x^{'\prime\prime}) x'' dt + \lambda \int_0^1 p(t) x''' dt \]

Since $x''(0) = x'(1)$, it follows that $\lambda \int_0^1 f(x') x''' dt = 0$ and from condition (ii) we get

\[ - \int_0^1 \|x''''\|^2 dt \geq \lambda \int_0^1 x(t) x'''' dt + \lambda \int_0^1 \|x'(t)\|^2 dt \]

\[ + \lambda \int_0^1 c(t) |x''''| dt + \lambda \int_0^1 d(t) |x''| dt \]

\[ + \lambda \int_0^1 e(t) |x''''| dt - \lambda \int_0^1 h(x') (x''')^2 dt \]

\[ + \lambda \int_0^1 p(t) x''' dt - \|x''''\|^2 \]

\[ \geq \frac{\lambda}{2} \|x''''\|^2 - \lambda b_2 \|x''''\|^2 - \lambda c_2 \|x''''\| \|x''\| \]

\[ - \lambda \|e\| \|x''''\| \|x''''\| - \lambda \|p\| \|x''''\| \]

\[ - \lambda \|p\| \|x''''\| \]

We observe that $x(0) = x(1)$, there exists $t_1 \in (0,1)$ with $t_1 < \eta < 1$, such that $x''(t_1) = 0$. It follows

\[ \|x''\| \leq \|x''''\|, \|x''\| \leq \frac{1}{\eta} \|x''''\| \]  \hspace{1cm} (5.1)

\[ \|x''''\| \leq \|x''''\|, \|x''''\| \leq \frac{1}{\eta} \|x''''\| \]  \hspace{1cm} (5.2)

Since that $x''(0) = x''(1)$ we have from Lemma 4.1

\[ \|x''\| \leq \frac{1}{\eta} \|x''''\| \]  \hspace{1cm} (5.3)

\[ \|x''''\| \leq \frac{1}{\eta} \|x''\| \]  \hspace{1cm} (5.4)

Using the inequalities (5.1) to (5.4), we obtain

\[ \|x''''\| \leq \left( \frac{\lambda}{2} + \frac{b_2}{\eta} + \frac{c_2}{\eta^2} + \frac{d}{\eta^3} \right) \|x''''\|^2 + \frac{1}{\eta} \|\|x''''\| \|x''''\| + \frac{1}{\eta} \|\|p\| \|x''''\| 

\]
Next we prove Theorem 3.4. Multiplying (4.5) by \( z'' \) and integrating from 0 to 1, we get

\[
\int_0^1 x^n z'' \, dx + \lambda \int_0^1 f(x) x^n z'' \, dx + \lambda \int_0^1 h(x') (z'')^2 \, dx
\]

\[
= \lambda \int_0^1 g(t, x, x', x''') x^n \, dx + \lambda \int_0^1 p(t) z'' \, dx
\]

\[
- \lambda \int_0^1 \int_0^1 \frac{\partial}{\partial t} f(t, x, x', x''', x''') \, dx' \, dx''
\]

\[
+ \lambda \int_0^1 \int_0^1 \frac{\partial}{\partial t} h(t, x, x', x''') x^n \, dx' \, dx''
\]

\[
+ \lambda \int_0^1 \int_0^1 \frac{\partial}{\partial t} p(t, x, x') x^n \, dx' \, dx''
\]

\[
+ \lambda \int_0^1 \int_0^1 \frac{\partial}{\partial t} p(t, x, x') x^n \, dx' \, dx''
\]

\[
+ \lambda \int_0^1 \int_0^1 g(t, x, x', x''') x^n \, dx' \, dx''
\]

\[
+ \lambda \int_0^1 g(t, x, x', x''') x^n \, dx' \, dx''
\]

\[
- \left\| x'' \right\|_2^2 \geq -c_0 \int_0^1 |x''|^2 \, dx - c_0 \int_0^1 |x''|^2 \, dx
\]

\[
- c_0 \int_0^1 |x''|^2 \, dx - \int_0^1 |p(t)| |x''|^2 \, dx
\]

\[
- \int_0^1 |p(t)| |x''|^2 \, dx
\]

\[
\left\| x'' \right\|_2^2 \leq a_0 \left\| x'' \right\|_2^2 + b_0 \left\| x'' \right\|_2^2 + c_0 \left\| x'' \right\|_2^2 + d_0 \left\| x'' \right\|_2^2
\]

\[
+ k \left\| x'' \right\|_2^2 + \left\| \phi \right\| \left\| x'' \right\|_\infty + \left\| \phi \right\| \left\| x'' \right\|_\infty
\]

\[
\leq \left( a_0 + b_0 + c_0 + d_0 \right) \left\| x'' \right\|_2^2 + \left( k + 1 \right) \left\| \phi \right\| \left\| x'' \right\|_2
\]

or

\[
\left\| x'' \right\|_2 \leq \frac{1}{a_0 + b_0 + c_0 + d_0 + k + 1} \left\| x'' \right\|_2
\]

Thus

\[
\left\| x \right\|_\infty \leq \left\| z \right\|_\infty \leq \left\| x'' \right\|_\infty \leq \rho_2
\]
on multiplying (6.4) by \((u_1 - u_2)^2\) and integrating from 0 to 1 we get

\[
- \int_0^1 (u_1 - u_2)^2 \theta \, dt + \int_0^1 \theta' \, dt - \int_0^1 (g(t, u_1, u_2, u_1', u_2') (u_1 - u_2)' \, dt
- \int_0^1 \theta \, dt
\]

(6.6)

Let \(v = u_1 - u_2\), then

\[
\int_0^1 (v')^2 \, dt \geq \int_0^1 a(t) v' \, dt + \int_0^1 b(t) (v')^2 \, dt + \int_0^1 c(t) \theta' \, dt + \int_0^1 d(t) \theta \, dt
- \int_0^1 b(t) v'' \, dt - \int_0^1 c(t) \theta' \, dt + \int_0^1 d(t) \theta \, dt
\]

From condition (6.3) we derive \(\|v''\|_2 \leq 0\) and hence

\[
\|v''\|_2 = 0.\]

Since \(\|v''\|_2 \leq \|v''\|_2 \leq 0\) we have \(v(t) = 0\) and hence \(u_1(t) = u_2(t)\) a.e. \(t \in [0, 1]\) by the continuity of \(u_1(t)\) and \(u_2(t)\).

**Theorem 6.2** Let \(q : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}\) be a Carathéodory function.

Assume that there exists \(a(t), b(t), c(t), d(t) \in L^\infty([0, 1])\) such that

\[
\|g(t, x, x', x''_1, x''_2) - g(t, x, x', x''_1, x''_2)\| \leq a(t) \|x_1 - x_2\| + b(t) \|x_1'' - x_2''\| + c(t) \|x_2 - x_2''\| + d(t) \|x''_1 - x''_2\|
\]

Then for every \(\varphi(t) \in L^1([0, 1])\) the boundary value problem (6.1) with (6.2) has a unique solution provided

\[
\pi^3 \|a\|_\infty + \pi^3 \|b\|_\infty + \pi^3 \|c\|_\infty + \pi^3 \|d\|_\infty + \pi^3 |B| < \pi^5
\]