# NON-RESONANT OSCILLATIONS FOR SOME FOURTH-ORDER DIFFERENTIAL EQUATIONS WITH DELAY

By S. A. IYASE
Department of Mathematics, Statistics and Computer Science, University of Abuja, Nigeria

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ABSTRACT

We use coincidence degree arguments in order to derive the existence and uniqueness of periodic solution of equation (1.1).

## 1. Introduction

In this paper we prove the existence and uniqueness of $2\pi$-periodic solution for the fourth order differential equation with time delay of the form

$$x^{iv}(t) + a\dddot{x}(t) + b\ddot{x}(t) + c\dot{x}(t) + g(t, x(t-\tau)) = p(t)$$
$$x^{(i)}(0) = x^{(i)}(2\pi), i = 0, 1, 2, 3, \qquad (1.1)$$

where $a, b, c$ are constants, $g$ is a Carathéodory's function, $p \in L^1_{2\pi}$ and $\tau \in [0, 2\pi]$ is a fixed time delay. The unknown function $x: [0, 2\pi] \to \mathbb{R}$ is defined for $0 \leq t \leq \tau$ by $x(t-\tau) = x(2\pi - (t-\tau))$. Various fourth order boundary value problems are used to model deformations of elastic beams that have found applications in structures such as aircraft, buildings, ships and bridges. Some of these equations have been extensively studied in recent years (see [1], [4], and references therein). Similarly some problems in biological or physiological systems can be modelled by fourth order differential equations with time delay, for instance the oscillatory movements of muscles that occur from the interaction of a muscle with its load (see [7]).

In section 2 of this paper we shall consider the problem of non-existence of non-trivial $2\pi$-periodic solutions of some linear analogues of (1.1). In section 3 we shall prove that under suitable conditions on the constants $a, b, c$ and on the asymptotic behaviour of the ratio $\frac{g(t,y)}{by}$ the equation (1.1) possesses at least one $2\pi$-periodic solution for each $p \in L^1_{2\pi}$. The technique of proof uses coincidence degree theory [6] and the *a priori* estimates are obtained by adapting the methods established in [3]. Finally, in section 4 we shall obtain uniqueness results.

We use the following notations and definitions. Let $\mathbb{R}$ denote the real line and $I$ the interval $[0, 2\pi]$. The following spaces will be used: $L^p_{2\pi} = L^p(I, \mathbb{R})$ are the usual Lebesgue spaces, $1 \leq p < \infty$, with $x \in L^p_{2\pi}$, $2\pi$-periodic

$$H^k_{2\pi} = H^k(I, \mathbb{R}) = \left[\begin{array}{l} x: I \to \mathbb{R}. x, \dot{x} \dots x^{k-1} \text{ are absolutely} \\ \text{continuous } x^k \varepsilon L^2_{2\pi} \text{ and} \\ x^{(i)}(0) = x^i(2\pi), i = 0, 1, 2, 3 \dots k-1 \end{array}\right.$$

 

with norm $\|x\|^2_{H^k_{2\pi}} = \left(\frac{1}{2\pi}\int_0^{2\pi} x(t)dt\right)^2 + \frac{1}{2\pi}\sum_{i=1}^k \int_0^{2\pi} |x^i(t)|^2 dt$, and $W^{k,1}_{2\pi} = \{x : I \to \mathbb{R}, x, \dot{x} \ldots x^{k-1}$ are absolutely continuous, $x^k \in L^2_{2\pi}$ and $x^i(0) = x^i(2\pi), i = 0, 1, 2, 3 \ldots k-1\}$ with norm

$$\|x\|^2_{W^{k,1}_{2\pi}} = \frac{1}{2\pi}\sum_{i=0}^{k}\int_0^{2\pi} |x^{(i)}(t)| dt.$$

A function $x \in W^{4,1}_{2\pi}$ is a solution of (1.1) if it satisfies (1.1) almost everywhere on $\mathbb{R}$. For such a solution, we set $x = \overline{x} + \tilde{x}$ where

$$\overline{x}(t) = a_0 + \sum_{k=1}^{n}(a_k \cos kt + b_k \sin kt) \tag{1.2}$$

$$\tilde{x}(t) = \sum_{k=n+1}^{\infty}(a_k \cos kt + b_k \sin kt). \tag{1.3}$$

## 2. The linear case

To motivate our study, we consider in this section the problem of non-existence of non-trivial periodic solutions for some linear analogue of (1.1). Specifically, we consider the equation

$$\begin{gathered} x^{iv} + a\dddot{x} + b\ddot{x} + c\dot{x} + d(t)x(t-\tau) = 0 \\ x^i(0) = x^i(2\pi), i = 0, 1, 2, 3 \end{gathered} \tag{2.1}$$

where $a, b, c$ are constants and $d(t) \in L^1_{2\pi}$.

Our result is as follows.

**Theorem 2.1.** *Let $n \geq 1$ be an integer and let the following conditions be satisfied:*

(i) $a > 0$,

(ii) $\frac{c}{a} > n^2$,

(iii) $b < \frac{-n^2}{a}$,

(iv) $n^2 \leq b^{-1}d(t) \leq (n+1)^2$ *holds uniformly a.e in* $t \in [0, 2\pi]$, *with strict inequalities* $n^2 < b^{-1}d(t)$, $b^{-1}d(t) < (n+1)^2$ *holding on subsets of* $[0, 2\pi]$ *of positive measure,*

*and suppose that there exist constants* $\delta > 0$, $\eta > 0$ *with*

(v) $0 < \frac{\delta - a}{a^2} < \eta$,

*then the boundary value problem* (2.1) *has no non-trivial periodic solution in* $W^{4,1}_{2\pi}$.

Proof. First, we rewrite (2.1) in the form

$$b^{-1}[x^{iv} + a\dddot{x} + c\dot{x}] + [\ddot{x} + \Gamma(t)x(t-\tau)] = 0 \tag{2.2}$$

where $\Gamma(t) = b^{-1}d(t) \in L^1_{2\pi}$. Let $x = \overline{x} + \tilde{x} \in H^3_{2\pi}$ be any solution of (2.2). Then, on multiplying (2.2) by $\overline{x}(t-\tau) - \tilde{x}(t)$ and integrating over $I$, we obtain $I_1 + I_2 = 0$

where

$$I_1 = \frac{b^{-1}}{2\pi} \int_0^{2\pi} (\overline{x}(t-\tau) - \tilde{x}(t))[x^{iv} + a\dddot{x} + c\dot{x}]dt \tag{2.3}$$

and

$$I_2 = \frac{1}{2\pi} \int_0^{2\pi} (\overline{x}(t-\tau) - \tilde{x}(t)[\ddot{x} + \Gamma(t)x(t-\tau)]dt. \tag{2.4}$$

■

To estimate $I_1$, we obtain first from definitions (1.2), (1.3) and orthogonality of $\overline{x}$ and $\tilde{x}$ that

$$\frac{1}{2\pi}\int_0^{2\pi} (\overline{x}(t-\tau) - \tilde{x}(t))[\overline{x}^{iv} + \tilde{x}^{iv}]dt = \frac{1}{2\pi}\int_0^{2\pi} \overline{x}(t-\tau)\overline{x}^{iv} - \frac{1}{2\pi}\int_0^{2\pi} \tilde{x}\tilde{x}^{iv} dt$$

$$= \sum_{k=1}^{n} k^4(a_k^2 + b_k^2)\cos k\tau - \sum_{k=n+1}^{\infty} k^4(a_k^2 + b_k^2),$$

$$\frac{1}{2\pi}\int_0^{2\pi} (\overline{x}(t-\tau) - \tilde{x}(t))[\dddot{x} + \dddot{\tilde{x}}]dt = \frac{1}{2\pi}\int_0^{2\pi} \overline{x}(t-\tau)\dddot{x}\, dt = \sum_{k=1}^{n} k^3(a_k^2 + b_k^2)\sin k\tau,$$

$$\frac{1}{2\pi}\int_0^{2\pi} (\overline{x}(t-\tau) - \tilde{x}(t))[\dot{\overline{x}} + \dot{\tilde{x}}]dt = \frac{1}{2\pi}\int_0^{2\pi} \overline{x}(t-\tau)\dot{\overline{x}}dt = \sum_{k=1}^{n} k(a_k^2 + b_k^2)\sin k\tau.$$

Thus

$$I_1 = b^{-1}\sum_{k=1}^{n}\{[k^4\cos k\tau + (ak^3 - ck)\sin k\tau](a_k^2 + b_k^2)\} - b^{-1}\sum_{k=n+1}^{\infty} k^4(a_k^2 + b_k^2),$$

$$|I_1| \leq |b|^{-1}\sum_{k=1}^{n}\{[k^4|\cos k\tau| + |ak^3 - ck||\sin k\tau|](a_k^2 + b_k^2)\} + |b|^{-1}\sum_{k=n+1}^{\infty} k^4(a_k^2 + b_k^2),$$

$$\leq |b|^{-1}\sum_{k=1}^{\infty} k^4(a_k^2 + b_k^2) + a|b|^{-1}\max_{1\leq k\leq n}\left|\frac{c}{a} - k^2\right|\sum_{k=1}^{\infty} k^4\,(a_k^2 + b_k^2)\,.$$

From condition (ii) we have

$$|I_1| \leq |b|^{-1}\sum_{k=1}^{\infty} k^4(a_k^2 + b_k^2) + |b|^{-1}a\eta\sum_{k=1}^{\infty} k^4(a_k^2 + b_k^2)$$

where

$$\eta = \max_{1\leq k\leq n}|\frac{c}{a} - k^2| = |\frac{c}{a} - 1|,$$

and from condition (iii) we get

$$\begin{aligned}
|I_1| &\le a\sum_{k=1}^{\infty} k^2(a_k^2+b_k^2) + a^2\eta\sum_{k=1}^{\infty} k^2(a_k^2+b_k^2)\\
&= a|\dot{x}|^2_{L^2_{2\pi}} + a^2\eta|\dot{x}|^2_{L^2_{2\pi}}\\
&\le (a+a^2\eta)|x|^2_{H^1_{2\pi}}.
\end{aligned}$$

The term $I_2$ can be estimated as in [5] to obtain

$$\begin{aligned}
I_2 &\ge \frac{1}{2\pi}\int_0^{2\pi}[\dot{\ddot{x}}^2(t) - \Gamma(t)\ddot{x}^2(t)]dt\\
&+\frac{1}{2\pi}\int_0^{2\pi}[\dot{\ddot{x}}^2(t-\tau) - \Gamma(t)\ddot{x}^2(t-\tau)]dt\\
&+\frac{1}{2\pi}\int_0^{2\pi}[\Gamma(t)\ddot{x}(t-\tau) - \dot{\ddot{x}}^2(t-\tau)]dt\\
&\ge \delta|x|^2_{H^1_{2\pi}} \text{ for some } \delta > 0.
\end{aligned}$$

Therefore

$$0 = I_1 + I_2 \ge (\delta - (a+a^2\eta)|x|^2_{H^1_{2\pi}}.$$

Using condition (v) we conclude that $x = 0$.

## 3. The non-linear case

We shall be concerned here with the non-linear boundary valve problem of the form

$$\begin{gathered}
x^{iv} + a\dddot{x} + b\ddot{x} + c\dot{x} + g(t, x(t-\tau)) = p(t)\\
x^{(i)}(0) = x^{(i)}(2\pi) \quad i = 0,1,2,3
\end{gathered} \tag{3.1}$$

where $a, b, c$ are constants and $p \in L^1_{2\pi}, g: I\times\mathbb{R}\to\mathbb{R}$ is such that $g(t+2\pi, x) = g(t, x)$ and is a Carathéodory's function with respect to $L^1_{2\pi}$, that is

(i) $g(., x)$ is measurable on $I$ for each $x \in \mathbb{R}$ ,
(ii) $g(t, .)$ is continuous on $\mathbb{R}$ for a.e $t \in I$,
(iii) for each $r > 0$ there exists $Y_r \in L^1_{2\pi}$ such that

$$|g(t, x)| \le Y_r(t)3.2) \tag{3.2}$$

for almost every $t \in I$ and all $x \in \mathbb{R}$ such that $|x| \le r$.

First, we shall prove the following lemma.

**Lemma 3.1.** *Let all the conditions of Theorem* 2.1 *be satisfied. Assume that* $\alpha, \beta \in L^1_{2\pi}$ *satisfy the following conditions:*

$$n^2 \le b^{-1}\alpha(t) \le b^{-1}\beta(t) \le (n+1)^2 \tag{3.3}$$

*for a.e* $t \in [0, 2\pi]$ *where* $n \ge 1$ *is an integer, and* $n^2 < b^{-1}\alpha(t)$, $b^{-1}\beta(t) < (n+1)^2$ *on subsets of* $[0, 2\pi]$ *of positive measure. Suppose that there exist constants* $\varepsilon > 0$ *and* $\delta_o > 0$ *with*

$$b^{-1}\alpha(t) - \varepsilon \le b^{-1}d(t) \le b^{-1}\beta(t) + \varepsilon,$$

*then*

$$\int_0^{2\pi} |x^{iv} + a\dddot{x} + b\ddot{x} + c\dot{x} + d(t)x(t-\tau)|dt \ge \delta_o|x|_{W^{4,1}_{2\pi}}. \tag{3.4}$$

PROOF. Assume that the conclusion of the lemma does not hold. Then there exists a sequence

$$\{x_n\} \in W^{4,1}_{2\pi} \text{ with } |x_n|_{W^{4,1}_{2\pi}} = 1$$

and a sequence $\{d_n(t)\} \in L^1_{2\pi}$ with

$$\frac{\alpha(t)}{b} - \frac{1}{n} \le \frac{d_n(t)}{b} \le \frac{\beta(t)}{b} + \frac{1}{n} \tag{3.5}$$

for a.e, $t \in [0, 2\pi]$ such that

$$|x_n^{iv} + a\dddot{x}_n + b\ddot{x}_n + c\dot{x}_n + d_n(t)x_n(t-\tau)|_{L^1_{2\pi}} < \frac{1}{n}. \tag{3.6}$$

(3.5) implies

$$|b^{-1}d_n(t)| \le \mu(t) \tag{3.7}$$

for some $\mu \in L^1_{2\pi}$ where $\mu(t)$ can be taken as

$$\left|\frac{\beta(t)}{b} + 1\right| + \left|\frac{\alpha(t)}{b} - 1\right|$$

using the compact embeddings

$$W^{4,1}_{2\pi} \subset W^{3,1}_{2\pi} \subset W^{2,1}_{2\pi}$$

and the continuous embedding of $W^{2,1}_{2\pi}$ into $C^1_{2\pi}$, and, going if necessary to subsequences, we can assume that

$$x_n \to x \text{ in } C^1[0, 2\pi]$$
$$\ddot{x}_n \to \ddot{x} \text{ in } L^1_{2\pi}$$
$$\dddot{x}_n \to \dddot{x} \text{ in } L^1_{2\pi}.$$

By (3.5) we have

$$\frac{\alpha(t)}{b} \le b^{-1}d(t) \le \frac{\beta(t)}{b}. \tag{3.8}$$

Moreover by corollary 4.8.11 of [2] and (3.7) we have that

$$d_n(t) \to d(t) \text{ in } L^1_{2\pi}.$$

Now for all $\psi \in L^\infty_{2\pi}$ we have

$$|b|^{-1}|\int_0^{2\pi} \psi(t)[d_n(t)x_n(t-\tau) - d(t)x(t-\tau)]dt|$$

$$\leq |b|^{-1}\{|\int_0^{2\pi} d_n(t)[x_n(t-\tau) - x(t-\tau)]\psi(t)dt| + |\int_0^{2\pi} (d_n(t) - d(t))\psi(t)x(t-\tau)dt|\}$$

$$\leq k|x_n(t-\tau) - x(t-\tau)| + |b|^{-1}|\int_0^{2\pi} (d_n(t) - d(t))x(t-\tau)\psi(t)dt|$$

where

$$k = |\psi|_{L^\infty}.|\mu(t)|_{L^1}.$$

So that

$$b^{-1}d_n(t)x_n(t-\tau) \to b^{-1}d(t)x(t-\tau) \text{ in } L^1_{2\pi}.$$

By (3.6) we deduce that

$$x_n^{iv} \to a\dddot{x} - b\ddot{x} - c\dot{x} - d(t)x(t-\tau).$$

The weak closedness of the graph of the linear operator $\frac{d^4}{dx^4}$ implies that $x \in W^{4,1}_{2\pi}$ and

$$x^{iv} + a\dddot{x} + b\ddot{x} + c\dot{x} + d(t)x(t-\tau) = 0.$$

By (3.8) and Theorem 2.1 we deduce that $x = 0$. Thus $x_n \to 0$. This contradicts

$$|x_n|_{W^{4,1}_{2\pi}} = 1 \text{ for all } n. \quad \blacksquare$$

We shall now prove the following existence result for equation (3.1).

**Theorem 3.2.** *Let $a, b, c$ be constants such that*
(i) $a > 0$,
(ii) $\frac{c}{a} > n^2$,
(iii) $b < -n^2$,
*and let $g$ be a Carathéodory function such that the inequalities*

$$n^2 \leq \frac{\alpha(t)}{b} \leq \liminf_{|x|\to\infty} \frac{g(t,x)}{bx} \leq \limsup_{|x|\to\infty} \frac{g(t,x)}{bx} \leq \frac{\beta(t)}{b} \leq (n+1)^2 \qquad (3.9)$$

*hold uniformly for a.e $t \in I$, where $n \geq 1$ is an integer, $\alpha, \beta \in L^1_{2\pi}$, and the strict inequalities*

$$n^2 < b^{-1}d(t), b^{-1}d(t) < (n+1)^2$$

*hold on subsets of $I$ of positive measure. Suppose that there exist constants $\delta > 0, \eta = |\frac{c}{a} - 1|$ with*

$$0 < \frac{\delta - a}{a^2} \leq \eta,$$

*then the boundary value problem* (3.1) *has at least one solution in* $W^{4,1}_{2\pi}$.

PROOF. Let $\varepsilon > 0$ be associated to $\alpha, \beta$ in Lemma 3.1. Then by (3.9) there exists a constant $R = R(\varepsilon)$ such that

$$\frac{\alpha(t)}{b} - \varepsilon \leq \frac{g(t,x)}{bx} \leq \frac{\beta(t)}{b} + \varepsilon$$

for a.e $t \in I$ and all $x \in \mathbb{R}$ with $|x| \geq R$.

Define a function $\tilde{Y}: \mathbb{R} \to \mathbb{R}$ by

$$\tilde{Y}(t,x) = \begin{cases} x^{-1}g(t,x) \text{ if } |x| \geq R \\ xR^{-2}g(t,R) + (1 - xR^{-1}\beta(t), 0 \leq x < R \\ xR^{-2}g(t,R) + (1 + xR^{-1})\beta(t), -R < x \leq 0. \end{cases} \tag{3.10}$$

Hence

$$\frac{\alpha(t)}{b} + \varepsilon \leq \frac{\tilde{Y}(t,x)}{b} \leq \frac{\beta(t)}{b} + \varepsilon \tag{3.11}$$

for a.e $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$.

Define $\tilde{g}$ and $\phi$ by $\tilde{g}(t,x) = \tilde{Y}(t,x)x.\phi(t,x) = g(t,x) - \tilde{g}(t,x)$ and observe that both $\tilde{g}$ and $\phi$ are Carathéodory's functions.

Thus there exists $Y_R \in L^1_{2\pi}$ such that

$$|\phi(t,x)| \leq Y_R(t) \tag{3.12}$$

for a.e $t \in I$ and all $x \in \mathbb{R}$ where

$$Y_R = Y_R(\alpha, \beta).$$

The equation (3.1) is thus equivalent to

$$\begin{gathered} x^{iv} + a\dddot{x} + b\ddot{x} + c\dot{x} + \tilde{Y}(t, x(t-\tau))x(t-\tau) + \phi(t,x) = p(t) \\ x^{(i)}(0) = x^{(i)}(2\pi), i = 0, 1, 2, 3. \end{gathered} \tag{3.13}$$

To apply coincidence degree theory [6] to (3.1) written in the form (3.13) we set

$$X = W^{4,1}_{2\pi}, \quad Z = L^1_{2\pi}$$

$$domL = \{x \in X: x^{(i)}(0) = x^{(i)}(2\pi) \text{ and } \dddot{x} \text{ is absolutely continuous on } [0, 2\pi]\}.$$

$$\begin{aligned} L: domL \quad &\subset X \to Z, \quad x \to x^{iv} + a\dddot{x} + b\ddot{x} + c\dot{x} \\ H: domL \quad &\subset X \to Z, \quad x \to \tilde{g}(t, x(t-\tau)) \end{aligned}$$

$$\begin{aligned} A: domL &\subset X \to Z, \quad x \to \beta(t)x(t-\tau) \\ G: domL &\subset X \to Z, \quad x \to \phi(t, x(t-\tau)) \\ T: domL &\subset X \to Z, \quad x \to -p(t). \end{aligned}$$

It is easily checked that $H$ and $G$ are well defined and $L$-compact on bounded subsets of $X$, and that $L$ is a linear Fredholm mapping of index zero.

Thus problem (3.1) is equivalent to solving the equation

$$Lx + Gx + Hx + Tx = 0 \tag{3.14}$$

in *domL*, where $\lambda \in [0, 1]$.

By theorem 4.5 of [6], equation (3.14) will have a solution if we can show that for each $\lambda \in [0, 1]$ and each $x \in domL$ such that

$$Lx + (1-\lambda)Ax + \lambda Gx + \lambda Hx + \lambda Tx = 0 \tag{3.15}$$

we have

$$|x|_{W^{4,1}_{2\pi}} < \rho \text{ for some } \rho > 0.$$

Let $x \in domL$ satisfy (3.15) for some $\lambda \in [0, 1]$. Then

$$\begin{aligned} &x^{iv} + a\dddot{x} + b\ddot{x} + c\dot{x} + (1-\lambda)\beta(t) + \lambda\tilde{Y}(t, x(t-\tau)x(t-\tau) \\ &\quad + \lambda\phi(t, x(t-\tau)) - \lambda p(t) = 0 \end{aligned}$$

and by (3.11) we have

$$\frac{\alpha(t)}{b} - \varepsilon \le \frac{(1-\lambda)\beta(t)}{b} + \lambda\frac{\tilde{Y}(t, x(t-\tau))}{b} \le \frac{\beta(t)}{b} + \varepsilon.$$

Therefore using Lemma 3.1 and (3.12) we get

$$\begin{aligned} 0 = |x^{iv} &+ a\dddot{x} + b\ddot{x} + c\dot{x} + (1-\lambda)\beta(t) + \lambda\tilde{Y}(t, x(t-\tau))x(t-\tau) \\ &+ \lambda\phi(t, x(t-\tau)) - \lambda P(t)|_{L^1_{2\pi}} \ge \delta_o|x|_{W^{4,1}_{2\pi}} - |Y_R|_{L^1_{2\pi}} - |P|_{L^1_{2\pi}} \end{aligned}$$

and hence

$$|x|_{W^{4,1}_{2\pi}} \le \delta_o^{-1}(|Y_R|_{L^1_{2\pi}} + |P|_{L^1_{2\pi}}).$$

To complete the proof we take any $\rho > \delta_o^{-1}(|Y_R|_{L^1_{2\pi}} + |P|_{L^1_{2\pi}})$. ■

## 4. Uniqueness result

In this section we shall prove a uniqueness result for equation (3.1).

**Theorem 4.1.** *Let all the conditions of Theorem 3.1 hold with g satisfying*

$$n^2 \le \frac{\alpha(t)}{b} \le \frac{g(t,x) - g(t,y)}{b(x-y)} \le \frac{\beta(t)}{b} \le (n+1)^2 \tag{4.1}$$

*for a.e $t \in I$ and $x \neq y \in \mathbb{R}$. Then the boundary value problem* (3.1) *has a unique solution for each $p \in L^1_{2\pi}$.*

PROOF. Since condition (4.1) implies (3.9), Theorem 3.1 ensures the existence of at least one solution.

Now let $x$ and $y$ be solutions of (3.1) and set $v = x - y$. Then $v$ is a solution of the boundary value problem

$$\begin{gathered} v^{iv} + a\dddot{x} + b\ddot{v} + c\dot{v} + g(t, v + y) - g(t, y) = 0 \\ v^{(i)}(0) = v^{(i)}(2\pi), i = 0, 1, 2, 3. \end{gathered} \tag{4.2}$$

Define $f: I \times R \to R$ by

$$f(t) = \begin{cases} v^{-1}[g(t, v + y) - g(t, y)] \text{ if } v \neq 0 \\ \alpha(t) \quad \text{if} \quad v = 0 \end{cases}.$$

Then (4.2) can be written in the form

$$v^{iv} + a\dddot{v} + b\ddot{v} + c\dot{v} + f(t)v = 0$$

with

$$\frac{\alpha(t)}{b} \leq \frac{f(t)}{b} \leq \frac{\beta(t)}{b}$$

for a.e $t \in I$ and all $v \in \mathbb{R}$.

By Theorem 2.1 we deduce that $v = 0$, i.e. $x = y$ a.e. ■

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