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On The Existence Of Periodic Solutions Of Certain Fourth Order Differential Equations With Dealy

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Abstract

We derive existence results for the periodic boundary value problem

$$x^{(iv)} + a\dddot{x} + f(\dot{x})\ddot{x} + c\dot{x} + g(t, x(x-\tau)) = p(t)$$

$x(0) = x(2\pi)$, $\dot{x}(0) = \dot{x}(2\pi)$, $\ddot{x}(0) = \ddot{x}(2\pi)$, $\dddot{x}(0) = \dddot{x}(2\pi)$

using degree theoretic methods. The uniqueness of periodic solutions is also examined.

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1. Introduction

In this paper we study the periodic boundary value problem

$$x^{(iv)} + a\dddot{x} + f(\dot{x})\ddot{x} + c\dot{x} + g(t, x(t-\tau)) = p(t) \quad (1.1)$$

$x(0) = x(2\pi)$, $\dot{x}(0) = \dot{x}(2\pi)$, $\ddot{x}(0) = \ddot{x}(2\pi)$, $\dddot{x}(0) = \dddot{x}(2\pi)$

with fixed delay $\tau \in [0, 2\pi)$ Where $c \neq 0$ is a constant, $p: [0, 2\pi] \to R$ and $g: [0, 2\pi] \times \Re \to \Re$ are 2π periodic in t and g satisfies certain Caratheodory conditions.

The unknown function $x: [0, 2\pi] \to R$ is defined for $0 < t < \tau$ by $x(t-\tau) = x(2\pi-(t-\tau))$

The differential equation $x^{(iv)} + a\dddot{x} + b\ddot{x} + h(x)\dot{x} + g(t, x(t-\tau)) = p(t)$ (1.2)

In which $b < 0$ is a constant was the object of a recent study [6].

Results on the existence and uniqueness of 2π periodic solutions were established subject to certain resonant conditions on g. Fourth order differential equations with delay occur in a variety of physical problems in fields such as Biology, Physics, Engineering and Medicine. In recent year, there have been many publications involving differential equation with delay; see for example [1,2,4,5,6,8,9]. However, as far as we know, there are few results on the existence and uniqueness of periodic solution to [1.1].

In what follows we shall use the spaces $C([0, 2\pi])$, $C^k([0, 2\pi])$ and $L^k([0, 2\pi])$ of continuous, k times continuously differentiable or measurable real functions whose kth power of the absolute value is Labesgue integrable.

We shall also make use of the sobolev space defined by

$H^k_{2\pi} = \{x : [0, 2\pi] \to R \mid x, \dot{x}$ are absolutely continuous on $[0, 2\pi]$ and $\dddot{x} \in L^2[0, 2\pi]$ with norm $|x|^2_H =$

$$\left(\frac{1}{2\pi}\int_0^{2\pi} x^2(t)dt\right)^2 + \frac{1}{2\pi}\sum_{i=1}^{2}\int_0^{2\pi} |x^i(t)|^2 dt,$$

$$x^i = \frac{d^i x}{dt^i}$$

2. The Linear cases

In this section we shall first consider the equation:

$$x^{(iv)}(t) + a\dddot{x}(t) + b\ddot{x}(t) + c\dot{x}(t) + dx(t-\tau) = 0 \quad (2.1)$$

$x(0) = x(2\pi)$, $\dot{x}(0) = \dot{x}(2\pi)$, $\ddot{x}(0) = \ddot{x}(2\pi)$, $\dddot{x}(0) = \dddot{x}(2\pi)$

Where a, b, c, d, are constants

Lemma 2.1 Let $c \neq 0$ and Let $a/c < 0$

Suppose that:

$$0 < d/c < n, \; n \geq 1 \quad (2.2)$$

Then (2.1) has no non-trivial 2π periodic solution for any fixed $\tau \in [0, 2\pi)$.

Proof

By substituting $x(t) = e^{\lambda t}$ with $\lambda = in$, $i^2 = -1$. We can see that the conclusion of the Lemma is true if $\Phi(n, \tau) = an^3 - cn + d \sin n\tau \neq 0$ for all $n \geq 1$ and $\tau \in [0, 2\pi)$ (2.3)

By (2.2) we have

$$c^{-1}\phi(n, \tau) = \frac{a}{c}n^3 - n + \frac{d}{c}\sin n\tau$$

$$\frac{a}{c}n^3 - n + \frac{d}{c} \leq \frac{a}{c}n^3 < 0$$

Therefore $\Phi(n, \tau) \neq 0$ and the result follows

If $x \in L^1\,[0, 2\pi]$ we shall write

$$\bar{x} = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt, \tilde{x}(t) = x(t) - \bar{x}$$

So that

$$\int_0^{2\pi} \tilde{x}(t)dt = 0$$

We shall consider next the delay equation

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + c\dot{x} + d(t)x(t-\tau) = 0 \qquad (2.4)$$

$x(0) = x(2\pi), \dot{x}(0) = \dot{x}(2\pi), \ddot{x}(0) = \ddot{x}(2\pi),$
$\dddot{x}(0) = \dddot{x}(2\pi)$

Where a, b, c are constants and $d \in L^1_{2\pi}$
Here the coefficient d in (2.4) is not necessarily constant. We have he following results which apart from being of independent interest are also useful in the non-linear case involving (1.1)

Lemma 2.2 Let $c \neq 0$ and let $a/c < 0$ Set $\Gamma(t) = c^{-1}d(t) \in L^2_{2\pi}$ Suppose that

$$0 < \Gamma(t) < 1 \qquad (2.5)$$

Then for arbitrary constant b the equation (2.4) admits in $H^1_{2\pi}$ only the trivial solution for every $\tau \in [0, 2\pi)$.
We note that a and c are not arbitrary.

Proof

If $x \in H^1_{2\pi}$ is a possible solution of (2.4) then on multiplying (2.4) by $\bar{x} + \tilde{x}$ (t) and integrating over $[0, 2\pi]$ noting that

$$\frac{1}{2\pi}\int_0^{2\pi} (\bar{x} + \dot{\tilde{x}}(t))$$

$$c^{-1}\left[x^{(iv)} + a\dddot{x} + b\ddot{x}\right] = -\frac{1}{2\pi}\frac{a}{c}\int_0^{2\pi} \ddot{x}^2(t)dt$$

We have that

$$0 = \frac{1}{2\pi}\int_0^{2\pi} (\bar{x} + \tilde{x}(t))\left\{c^{-1}[x^{(iv)} + a\dddot{x} + b\ddot{x}\ ] + \dot{x} + \Gamma(t)x(t-\tau)\ \right\}dt$$

$$= \frac{1}{2\pi}\frac{a}{c}\int_0^{2\pi} \ddot{\tilde{x}}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi} (\bar{x} + \tilde{x}(t))[\dot{x} + \Gamma(t)x(t-\tau)dt]$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} (\bar{x} + \tilde{x}(t))\{\dot{x}(t) + \Gamma(t)x(t-\tau)\}dt$$

$$\frac{1}{2\pi}\int_0^{2\pi} \dot{\tilde{x}}^2(t)dt + \int_0^{2\pi} \Gamma(t)\bar{x}\tilde{x}(t-\tau)dt + \frac{1}{2\pi}\int_0^{2\pi} \Gamma(t)\bar{x}^2 dt$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \Gamma(t)\bar{x}\tilde{x}(t-\tau)dt$$

Using the identity

$$ab = \frac{(a+b)^2}{2} - \frac{a^2}{2} - \frac{b^2}{2}$$

We get

$$= \frac{1}{2\pi}\int_0^{2\pi} \dot{\tilde{x}}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi} \Gamma(t)\bar{x}^2 dt + \frac{1}{2\pi}\int_0^{2\pi} \Gamma(t)\bar{x}\tilde{x}(t-\tau)dt$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \Gamma(t)\left\{\frac{[x(t-\tau) + \tilde{x}(t)]^2}{2} - \frac{\tilde{x}^2}{2} - \frac{\tilde{x}^2(t-\tau)}{2} - \bar{x}\tilde{x}(t-\tau) - \frac{\bar{x}^2}{2}\right\}dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \dot{\tilde{x}}^2(t)dt - \frac{1}{2\pi}\int_0^{2\pi} \frac{\Gamma(t)}{2}\left[\tilde{x}^2 + \tilde{x}^2(t-\tau)\right]dt +$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\Gamma(t)}{2}\ \left\{\left[x(t-\tau) + \tilde{x}(t)\right]^2 + \bar{x}^2\ \right\}dt$$

Using (2.5) we get

$$0 \geq \frac{1}{2\pi}\int_0^{2\pi} \dot{\tilde{x}}^2(t)dt - \frac{1}{2\pi}\int_0^{2\pi} \frac{\Gamma(t)}{2}\left[\tilde{x}^2 + \tilde{x}^2(t-\tau)\right]dt$$

From the periodicity of $\tilde{x}$ we have that

$$\int_0^{2\pi} \dot{\tilde{x}}^2(t)dt = \int_0^{2\pi} \dot{\tilde{x}}^2(t-\tau)dt$$

Hence

$$0 \geq \tfrac{1}{2}\left[\tfrac{1}{2\pi}\int_0^{2\pi} \dot{\tilde{x}}^2(t-\tau) - \Gamma(t)\tilde{x}^2(t-\tau)\right]dt$$

$$\tfrac{1}{2}\left[\tfrac{1}{2\pi}\int_0^{2\pi} (\dot{\tilde{x}}^2(t) - \Gamma(t)\tilde{x}^2(t-\tau)\right]dt \qquad (2.6)$$

Using (2.5) we can see that the last expression is non-negative hence

$$\widetilde{y}(t,x)=\begin{cases}(cx)^{-1}g(t,x) & |x|\geq R_1\\(cR_1)^{-1}g(t,R_1) & 0<x<R_1\\-(cR_1)^{-1}g(t,-R_1) & -R_1<x<0\\\Gamma(t) & x=0\end{cases}\quad (3.6)$$

Then

$$0<\widetilde{y}(t,x)\leq\Gamma(t)+\delta/2 \quad (3.7)$$

for a.e $t\in[0,2\pi]$ and all $x\in\Re$. Moreover the function $\widetilde{y}(t,x)$ satisfy Caratheodory's conditions and $\tilde{g}$: $[0,2\pi]\times\Re\rightarrow\Re$ defined by

$$\widetilde{g}(t,x(t-\tau))=g(t,x(t-\tau))-cx(t-\tau)\widetilde{y}(t,x(t-\tau)) \quad (3.8)$$

is such that for a.e $t\in[0,2\pi]$ and all $x\in\Re$.

$|\widetilde{g}(t,x(t-\tau))|\leq\alpha(t)$ for some $\alpha(t)\in L^2_{2\pi}$.

Let $\lambda\in[0,1]$ be such that

$$c^{-1}[x^{(iv)}+a\dddot{x}+\lambda f(\dot{x})\ddot{x}]+\dot{x}+(1-\lambda)\Gamma(t)x(t-\tau)+\lambda\widetilde{y}(t,x(t-\tau))x(t-\tau)$$
$$+c^{-1}(1-\lambda)b\ddot{x}+\lambda c^{-1}\widetilde{g}(t,x(t-\tau))-c^{-1}\lambda p(t)=0 \quad (3.9)$$

For $\lambda=0$ we obtain (2.1) which by Lemma 2.2 admits only the trivial solution

For $\lambda=1$ we get the original equation (1.1). To prove that equation (3.1) has at least one solution, we show according to the Leray-Shauder Method that the possible solution of the family of equations (3.9) are apriori bounded in $C^3[0,2\pi]$ independently of $\lambda\in[0,1]$.

Notice that by (3.5) one has

$$0\leq(1-\lambda)\Gamma(t)+\lambda\widetilde{y}(t,x(t-\tau))\leq\Gamma(t)+\delta/2 \quad (3.10)$$

Then using Theorem 2.1 with $V(t)=(1-\lambda)\Gamma(t)+\lambda\widetilde{y}(t,x(t-\tau))$ and Cauchy Schwarz inequality we get

$$0=\frac{1}{2\pi}\int_0^{2\pi}(\bar{x}+\tilde{x}(t))\{c^{-1}[x^{(iv)}+a\dddot{x}+\lambda f(\dot{x})\ddot{x}]+\dot{x}+(1-\lambda)\Gamma(t)x(t-\tau)+\lambda\widetilde{y}(t,x(t-\tau))+\lambda\widetilde{g}(t,x(t-\tau))+(1-\lambda)b\ddot{x}-\lambda p(t)\}dt.$$

$$\geq\delta/2|\tilde{x}|^2_{H^1_{2\pi}}-(|\alpha|_2+|p|_2)(|\bar{x}|+|\dot{\tilde{x}}|_2)$$
$$\geq\delta/2|\tilde{x}|^2_{H^1_{2\pi}}-\beta(|\bar{x}|+|\tilde{x}|_{H^1_{2\pi}})$$

Thus

$$|\tilde{x}|^2_{H^1_{2\pi}}\leq\frac{2\beta}{\delta}(|\bar{x}|+|\tilde{x}|^2_{H^1_{2\pi}}) \quad (3.11)$$

With $\beta>0$ independent of x and λ. Integrating (3.9) over $[0,2\pi]$ We obtain

$$(1-\lambda)\int_0^{2\pi}\Gamma(t)x(t-\tau)dt=-c^{-1}\lambda\int_0^{2\pi}g(t,x(t-\tau)) \quad (3.12)$$

Since $\Gamma(t)>0$ we derive that

$$\frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)dt=\overline{\Gamma}>0 \quad (3.13)$$

Hence if $x(t)\geq r$ for all $t\in[0,2\pi]$, (3.3) and (3.12) implies that $(1-\lambda)\overline{\Gamma}<0$ contradicting $\overline{\Gamma}>0$. Similarly if $x(t)\leq -r$ for all $t\in[0,2\pi]$ we reach a contradiction.

Thus there exists a. $t_1\in[0,2\pi]$ such that $|x(t_1)|<r$. Le t_2 be such that

$$\bar{x}=x(t_2)=x(t_1)+\int_{t_1}^{t_2}\dot{x}(s)ds.$$

This implies that $|\bar{x}|\leq r+2\pi|\tilde{x}|_{H^1_{2\pi}}$

Substituting this in (3.11) we get

$$|\tilde{x}|^2_{H^1_{2\pi}}\leq c_1|\tilde{x}|_{H^1_{2\pi}}$$

$$\text{or } |\tilde{x}|_{H^1_{2\pi}}\leq c_1, c_1>0 \quad (3.14)$$

Now

$$|x|_{H^1_{2\pi}}\leq|\bar{x}|+|\tilde{x}|_{H^1_{2\pi}}\leq r+(2\pi+1)c_1=c_2 \quad (3.15)$$

Thus

$$|\dot{x}|_2\leq c_3, c_3>0 \quad (3.16)$$

From (3.16)we have

$$|x|_\infty\leq c_4, c_4>0 \quad (3.17)$$

Multiplying (3.9) by $-\dot{x}(t)$ and integrating over $[0,2\pi]$ we have

$$|\ddot{x}|_2^2\leq|a|^{-1}(|\dot{x}|_2^2+|1+\tfrac{\delta}{2}||\dot{x}|_2|x|_\infty+|\alpha|_2|\dot{x}|_2+|p|_2|\dot{x}|_2)$$

Hence

$$|\ddot{x}|_2\leq c_5, c_5>0 \quad (3.18)$$

And thus

$$|\dot{x}|_\infty\leq c_6, c_6>0 \quad (3.19)$$

Multiplying (3.9) by $-\dddot{x}(t)$ and integrating over $[0,2\pi]$
We get

$$|\dddot{x}|_2^2 \leq |f(\ddot{x})|_\infty |\dddot{x}|_2^2 + |1+\tfrac{a}{2}||\dddot{x}|_2 |x|_\infty + |c|^{-1}|\alpha|_2 |\dddot{x}|_2 + |p|_2 |\dddot{x}|_2 + |b||\dddot{x}|_2$$

Thus

$$|\dddot{x}|_2 \leq c_7, c_7 > 0 \qquad (3.20)$$

And hence

$$|\ddot{x}|_\infty \leq c_8, c_8 > 0 \qquad (3.21)$$

Also

$$|x^{(iv)}|_1 \leq c_9, c_9 > 0 \qquad (3.22)$$

Since $\dddot{x}(0) = \dddot{x}(2\pi)$ there exists $t_o \in [0,2\pi]$ Such that $\dddot{x}(t_o) = 0$ Hence

$$|\dddot{x}|_\infty \leq c_{10}, c_{10} > 0 \qquad (3.23)$$

From (3.17), (3.19), (3.22) and (3.23) our result follows.

4. Uniqueness Result

If in (1.1), $f(\dot{x}) = b$ a constant. The following uniqueness results holds.

Theorem 4.1

Let a, b, c, be constants with $c \neq 0$ $a/c < 0$.
Suppose g is a caratheodony function satisfying

$$0 < \frac{\dot{g}(t, x_1(t-\tau)) - g(t, x_2(t-\tau))}{c(x_1 - x_2)} < \Gamma(t)$$

For a.e., $t \in [0,2\pi]$ and all $x_1, x_2 \in R$ $x_1 \neq x_2$ where $\Gamma \in L^2_{2\pi}$

Then the boundary value problem

$$x^{iv} + a\dddot{x} + b\ddot{x} + c\dot{x} + g(t, x(t-\tau)) = p(t)$$
$$x(0) = x(2\pi), \dot{x}(0) = \dot{x}(2\pi), \ddot{x}(0) = \ddot{x}(2\pi), \dddot{x}(0) = \dddot{x}(2\pi) \qquad (4.1)$$

has art most one solution.

Proof

Let $u = x_1 - x_2$ for any two solutions x_1, x_2 of (4.1).
Then u satisfies the boundary value problem

$$c^{-1}[u^{(iv)} + a\dddot{u} + b\ddot{u}] + \dot{u} + \beta(t)u(t-\tau) = 0$$
$$u(0) = u(2\pi), \dot{u}(0) = \dot{u}(2\pi), \ddot{u}(0) = \ddot{u}(2\pi), \dddot{u}(0) = \dddot{u}(2\pi)$$

Where $\beta(t) \in L^2_{2\pi}$ is defined by

$$\beta(t) = \frac{g(t, x_1(t-\tau)) - g(t, x_2(t-\tau))}{c(x_1 - x_2)}$$

If $u = x_1 - x_2 \neq 0$ and since $0 < \beta(t) \leq \Gamma(t)$ for a.e $t \in [0,2\pi]$ then using the arguments of theorem 2.1 we have that $u = 0$ and thus $x_1 = x_2$ a. e.

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