On The Existence Of Periodic Solutions Of Certain Fourth Order Differential Equations With Delay

Samuel A. Lyase

Department of Mathematics, Computer Science and Information Technology, Lebanon University, Lebanon. P.O. Box 1980, Beirut City, Lebanon. N. Nepal.

Abstract

We derive existence results for the periodic boundary value problem

\[ \begin{align*}
X'''' + aX' + f(t)X + g(t, x(t-r)) &= p(t) \\
x(0) &= x(T), \quad x'(0) = x'(T), \quad x''(0) = x''(T),
\end{align*} \]

(1)

using degree theoretic methods. The uniqueness of periodic solutions is also examined.

Keywords and Phrases: Periodic Solutions, Catastrophe Conditions, Fourth Order Differential Equations with delay. 2000 Mathematics Subject Classification: 34B15, 34C25.

1. Introduction

In this paper we study the periodic boundary value problem

\[ \begin{align*}
X'''' + aX' + f(t)X + g(t, x(t-r)) &= p(t) \\
x(0) &= x(T), \quad x'(0) = x'(T), \quad x''(0) = x''(T),
\end{align*} \]

(1)

and \( f(t), g(t, x(t-r)) \) of continuous k times continuously differentiable or measurable real functions whose kth power of the absolute value is Lebesgue integrable.

We shall also make use of the Sobolev spaces denoted by

\[ \mathbb{H}^k = \{ x : [0, 2\pi] \rightarrow \mathbb{R}, \quad x \text{ absolutely continuous on } [0, 2\pi] \text{ and } \| x \|_{\mathbb{H}^k}^2 = \int_0^{2\pi} |x''''(t)|^2 + \frac{1}{2} \sum_{j=0}^{k-1} |x^{(j)}(t)|^2 \, dt < \infty \} \]

where \( \mathbb{H}^k \) are Sobolev spaces on \([0, 2\pi]\) with norm \( \| \cdot \|_{\mathbb{H}^k} \). When \( f(t) \) is a constant, \( p : [0, 2\pi] \rightarrow \mathbb{R} \) and \( g : [0, 2\pi] \rightarrow \mathbb{R} \) are 2\( \pi \) periodic in \( t \) and \( g \) satisfies certain catastrophe conditions.

The autonomous function \( x(0) \), \( 2\pi \rightarrow \mathbb{R} \) is defined for all \( t \geq 0 \) by \( \xi := t = x(2\pi - t) \).

(1.2)

The differential equation \( x'''' + aX' + f(t)X + g(t, x(t-r)) \)

(1.3)

In which \( b = 0, a \) a constant was the object of a recent study [6].

Results on the existence and uniqueness of periodic solutions were established subject to certain, stringent conditions on \( g \). Fourth order differential equations with delay are used in a variety of physical problems in fields such as Biology, Physics, Engineering and Medicine. In recent year, there have been many publications involving differential equation with delay; see for example, [1,2,4,5,6,8,9]. However, as far as we know, there are few results on the existence and uniqueness of periodic solutions to [1,1].

In what follows we shall use the spaces \( C([0, 2\pi]), C^0([0, 2\pi]) \) and \( L^1([0, 2\pi]) \) of continuous k times continuously differentiable or measurable real functions whose kth power of the absolute value is Lebesgue integrable.

We define \( C^1([0, 2\pi]) \) for all \( n \in \mathbb{N} \).

By substituting \( x(0) = e^{2\pi x(t-1)} \), we can see that the conclusion of the Lemma is true if \( x(t) = x(t-1) \) for all \( n \in \mathbb{N} \).

Proof

By the definition of \( C^1([0, 2\pi]) \) we have

\[ C^1([0, 2\pi]) \]

Therefore \( C^1([0, 2\pi]) \) is the result follows.
If \( x \in L^1([0, 2\pi]) \) we shall write
\[
\delta = \frac{1}{2\pi} \int_0^{2\pi} x(t) dt, \quad \delta(t) = \delta(t) - \overline{\delta}
\]
and
\[
\int_0^{2\pi} \delta(t) dt = 0
\]
We shall consider next the delay equation
\[
x'''' + a(x') + bx' + cx + d(t)x(t-r) = 0
\]
(2.4)
\( m(0) = (2\pi \delta, \delta(0), \delta(2\pi), \delta(0)) = (2\pi \delta, \delta) \)
Where \( a, b, c \) are constants and \( d(t) \in L^1([0, 2\pi]) \)
Here the coefficient \( d \) in (2.4) is not necessarily constant. We have the following results which are also useful in the non-linear case involving (1.11)
Lemma 2.2
Suppose that
\[
\gamma > 0
\]
Then for arbitrary constant \( b \) the equation (2.4) admits in \( H^1_0 \), only the trivial solution for ever
\( r \in [0, 2\pi] \).
We note that \( a \) and \( c \) are not arbitrary.

Proof
If \( x \in H^1_0 \), is a possible solution of (2.4) then, by multiplying (2.4) by \( \overline{x} + \overline{x}(t) \) integrating over \([0, 2\pi]\) noting that
\[
\int_0^{2\pi} \left( \overline{x} + \overline{x}(t) \right) dt = 0
\]
We have that
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \overline{x} + \overline{x}(t) \right) \left[ x'''' + ax' + bx' + dx(t-r) \right] dt = 0
\]
and
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \overline{x} + \overline{x}(t) \right) \left( x'''' + ax' + bx' + d(t)x(t-r) \right) dt = 0
\]
Using (2.5) we get
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \overline{x} + \overline{x}(t) \right) \left[ x'''' + ax' + bx' + d(t)x(t-r) \right] dt = 0
\]
From the periodicity of \( \overline{x} \) we have that
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \overline{x} + \overline{x}(t) \right) \left[ x'''' + ax' + bx' + d(t)x(t-r) \right] dt = 0
\]
Hence
\[
\int_0^{2\pi} \left( \overline{x} + \overline{x}(t) \right) \left[ x'''' + ax' + bx' + d(t)x(t-r) \right] dt = 0
\]
and
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \overline{x} + \overline{x}(t) \right) \left[ x'''' + ax' + bx' + d(t)x(t-r) \right] dt = 0
\]
Using (2.5) we can see that the last expression is non-negative hence
\[
\int_0^{2\pi} \left( \overline{x} + \overline{x}(t) \right) \left[ x'''' + ax' + bx' + d(t)x(t-r) \right] dt = 0
\]
The text is too large to be displayed here. Please refer to the original document for the full text.
Multiplying (3.9) by \(-\frac{dW(t)}{dt}\) and integrating over 
\[0.2\pi \] 
We get 
\[
|\psi| \leq C_1 + C_2 e^{-\frac{1}{2}R} > 0
\]
Thus 
\[
|\psi| \leq C_1 + C_2 e^{-\frac{1}{2}R} > 0 \quad (3.20)
\]
And hence 
\[
|\psi| \leq C_1 + C_2 e^{-\frac{1}{2}R} > 0 \quad (3.21)
\]
Also 
\[
|\psi| \leq C_1 + C_2 e^{-\frac{1}{2}R} > 0 \quad (3.22)
\]
Since \(\psi(0) = \psi(2\pi)\) there exists \(\psi \in [0.2\pi]\)
Such that \(\psi(0) = 0\) hence 
\[
|\psi| \leq C_1 + C_2 e^{-\frac{1}{2}R} > 0 \quad (3.23)
\]
From (3.17) (3.19) (3.22) and (3.23) our result follows.

4. Uniqueness Result
Let \(f(\cdot) = h\) a constant, The following uniqueness result holds:

**Theorem 4.1**
Let \(h, c_1, c_2\) be constants with \(c_1 h = 0\).
Suppose \(g\) a continuous function satisfying
\[
0 < \frac{c_1 g(x) + c_2 g(x - \tau)}{c(x)} < 1 \quad (3.24)
\]
for a.e., \(t \in [0.2\pi]\) and all \(x, \tau \in \mathbb{R}\), \(x \neq \tau\)
where \(c(\cdot) \neq 0\).

Then the boundary value problem 
\[
x'' + c_1 x' + c_2 x + g(t, x(t - \tau)) = p(t)
\]
\[
x(0) = \psi(2\pi), \quad \psi(0) = \psi(2\pi), \quad \psi(0) = \psi(2\pi)
\]
has at most one solution.

**Proof**
Let \(x = x_1 - x_2\) for any two solutions \(x_1, x_2 \in [0.2\pi]\).
Then \(x\) satisfies the boundary value problem
\[
x'' + c_1 x' + c_2 x + g(t, x(t - \tau)) = 0
\]
\[
x(0) = n(2\pi), \quad \psi(0) = n(2\pi), \quad \psi(0) = n(2\pi)
\]
Thus \(x_1\) and \(x_2\) satisfy the boundary value problem
\[
x'' + c_1 x' + c_2 x + g(t, x(t - \tau)) = 0
\]
\[
x(0) = n(2\pi), \quad \psi(0) = n(2\pi), \quad \psi(0) = n(2\pi)
\]
\[
\psi(0) = n(2\pi), \quad \psi(0) = n(2\pi), \quad \psi(0) = n(2\pi)
\]
\[
\psi(0) = n(2\pi), \quad \psi(0) = n(2\pi), \quad \psi(0) = n(2\pi)
\]
\[
\psi(0) = n(2\pi), \quad \psi(0) = n(2\pi), \quad \psi(0) = n(2\pi)
\]
References