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ON THE EXISTENCE OF PERIODIC SOLUTIONS OF CERTAIN THIRD ORDER NON-LINEAR DIFFERENTIAL EQUATIONS WITH DELAY

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Introduction

This paper is devoted to the study of the periodic boundary value problem

$$\dddot{x} + f(\dot{x})\ddot{x} + b\dot{x} + g(t, x(t-\tau)) = P(t) \tag{1.1}$$
$$x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0$$

with fixed delay $\tau \in (0, 2\pi)$, where $b < 0$ is a constant, $f: \mathbf{R} \to \mathbf{R}$ is continuous, $P: [0, 2\pi] \to \mathbf{R}$ and $g: [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ are 2π periodic in t and g satisfies certain Caratheodory conditions. The unknown function $x: [0, 2\pi] \to \mathbf{R}$ is defined for $0 < t \le \tau$ by $x(t-\tau) = x[2\pi - (t-\tau)]$.

The differential equation

$$\dddot{x} + a\ddot{x} + f(x)\dot{x} + g(t, x(t-\tau)) = P(t)$$

in which $a \neq 0$ is a constant was the object of a recent study [6]. Results on the existence and uniqueness of 2π periodic solutions were established subject to certain resonant conditions on g.

In the sequel we shall use the spaces $C([0, 2\pi])$, $C^k([0, 2\pi])$ and $L^k([0, 2\pi])$ of continuous, k terms continuously differentiable or measurable real functions whose k^{th} power of the absolute value is Lebesgue integrable. We shall make use of the Sobolev spaces $W^{3,1}_{2\pi}$ and $H^1_{2\pi}$ respectively defined by

$$W^{3,1}_{2\pi} = \{x: [0, 2\pi] \to \mathbf{R} \mid x, \dot{x}, \ddot{x} \text{ are absolutely continuous on } [0, 2\pi]\}$$

with norm

$$|x|_{W^{3,1}_{2\pi}} = \int_0^{2\pi} |x(t)|\,dt + \int_0^{2\pi} |\dot{x}(t)|\,dt + \int_0^{2\pi} |\ddot{x}(t)|\,dt + \int_0^{2\pi} |\dddot{x}(t)|\,dt$$

and

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where a, b are constants and $c \in L^1_{2\pi}$.

THEOREM 2.1

Let $b < 0$ *and set* $b^{-1}c(t) = \Gamma(t) \in L^1_{2\pi}$

Suppose that

$$0 < \Gamma(t) < 1, t \in [0, 2\pi] \tag{2.5}$$

Then for arbitrary a, *the equation (2.4) admits in* $W^{3,1}_{2\pi}$ *only the trivial solution.*

PROOF

We shall rewrite (2.4) in the form

$$\begin{aligned} &b^{-1}\dddot{x} + b^{-1}a\ddot{x} + \dot{x} + \Gamma(t)x(t-\tau) = 0 \\ &x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0 \end{aligned} \tag{2.6}$$

If x is a possible solution of (2.4) then since

$$\frac{1}{2\pi}\int_0^{2\pi} b^{-1}a(\bar{x} + \dot{\tilde{x}}(t))\ddot{x}(t)dt = 0$$

as can be easily verified, we have from (2.5) that

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_0^{2\pi}(\bar{x} + \dot{\tilde{x}}(t))[b^{-1}\dddot{x} + b^{-1}a\ddot{x} + \dot{x} + \Gamma(t)x(t-\tau)]dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}(\bar{x} + \dot{\tilde{x}}(t))[b^{-1}\dddot{x} + \dot{x} + \Gamma(t)x(t-\tau)]dt \\ &= \frac{-b^{-1}}{2\pi}\int_0^{2\pi}\ddot{\tilde{x}}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi}(\bar{x} + \dot{\tilde{x}}(t))\Gamma(t)x(t-\tau)dt \\ &\geq \frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t) + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\bar{x}^2dt + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\bar{x}\tilde{x}(t-\tau)dt \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\dot{\tilde{x}}(t)x(t-\tau)dt. \end{aligned}$$

Using the identity

$$\begin{aligned} ab &= \frac{[a+b]^2}{2} - \frac{a^2}{2} - \frac{b^2}{2} \\ &= \frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\bar{x}^2dt + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\bar{x}\tilde{x}(t-\tau)dt \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\{\frac{[x(t-\tau) + \dot{\tilde{x}}(t)]^2}{2} - \frac{\dot{\tilde{x}}^2(t)}{2} - \frac{x^2(t-\tau)}{2}\}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\bar{x}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\bar{x}\tilde{x}(t-\tau)dt \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\left\{\frac{[x(t-\tau) + \dot{\tilde{x}}(t)]^2}{2} - \frac{\dot{\tilde{x}}^2(t)}{2} - \frac{\tilde{x}^2(t-\tau)}{2} - \bar{x}\tilde{x}(t-\tau) - \frac{\bar{x}^2}{2}\right\}. \\ &= \frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt - \frac{1}{2\pi}\int_0^{2\pi}\frac{\Gamma(t)}{2}[\dot{\tilde{x}}^2(t) + \tilde{x}^2(t-\tau)]dt \end{aligned}$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi}[\bar{x}+\dot{\tilde{x}}(t)][b^{-1}\dddot{x}+\dot{x}+V(t)x(t-\tau)]dt$$

$$=\frac{1}{2\pi}[-b^{-1}\int_0^{2\pi}\ddot{\tilde{x}}^2(t)dt+\frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt\ +\frac{1}{2\pi}\int_0^{2\pi}[\bar{x}+\dot{\tilde{x}}(t)]V(t)x(t-\tau)dt$$

$$\geq\frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt+\frac{1}{2\pi}\int_0^{2\pi}[\bar{x}+\dot{\tilde{x}}]V(t)x(t-\tau)dt$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt-\frac{1}{2\pi}\int_0^{2\pi}\frac{V(t)}{2}[\dot{\tilde{x}}^2(t)+\tilde{x}^2(t-\tau)]dt$$

$$+\frac{1}{2\pi}\int_0^{2\pi}(\frac{V(t)}{2}\left\{[x(t-\tau)+\dot{\tilde{x}}(t)]\right\}^2+\bar{x}^2)dt$$

$$\geq\frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt-\frac{1}{2\pi}\int_0^{2\pi}\frac{V(t)}{2}[\dot{\tilde{x}}^2(t)+\tilde{x}^2(t-\tau)]dt$$

$$\geq\frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt-\frac{1}{2\pi}\int_0^{2\pi}\frac{(\Gamma(t)+\varepsilon)}{2}[\dot{\tilde{x}}^2+\tilde{x}^2(t-\tau)]dt$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2(t)dt-\frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)\left[\frac{\dot{\tilde{x}}^2}{2}+\frac{\tilde{x}^2(t-\tau)}{2}\right]dt$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\varepsilon\left[\frac{\dot{\tilde{x}}^2}{2}+\frac{\tilde{x}^2(t-\tau)}{2}\right]dt$$

Arguing like in theorem 2.1

$$\geq\delta|\tilde{x}|^2_{H^1_{2\pi}}-\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left[\frac{\dot{\tilde{x}}^2}{2}+\frac{\tilde{x}^2(t-\tau)}{2}\right]dt$$

Using Wirtingers Inequality

$$\geq\delta|\tilde{x}|^2_{H^1_{2\pi}}-\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left[\frac{\dot{\tilde{x}}^2}{2}+\frac{\dot{\tilde{x}}^2(t-\tau)}{2}\right]dt$$

$$=\delta|\tilde{x}|^2_{H^1_{2\pi}}-\frac{\varepsilon}{2\pi}\int_0^{2\pi}\dot{\tilde{x}}^2dt$$

$$\geq\delta|\tilde{x}|^2_{H^1_{2\pi}}-\varepsilon|\tilde{x}|^2_{H^1_{2\pi}}$$

$$=(\delta-\varepsilon)|\tilde{x}|^2_{H^1_{2\pi}}.$$

We shall consider the non-linear delay equation

$$\dddot{x}+f(\dot{x})\ddot{x}+b\dot{x}+g(t,x(t-\tau)) = P(t) \tag{3.2}$$
$$x(0)-x(2\pi) = \dot{x}(0)-\dot{x}(2\pi) = \ddot{x}(0)-\ddot{x}(2\pi)=0$$

where $f:\mathbf{R}\to\mathbf{R}$ is a continuous function and $g:[0,2\pi]\times\mathbf{R}\to\mathbf{R}$ is such that $g(\cdot,x)$ is measurable on $[0, 2\pi]$ for each $x\in\mathbf{R}$ and $g(t,\cdot)$ is continuous on $\mathbf{R}$ for almost each $t\in[0,2\pi]$.

We assume moreover that for each $r>0$ there exists $\gamma_r\in L^1_{2\pi}$ such that

$$|g(t,x)|\leq\gamma_r(t) \tag{3.3}$$

Let

$$X = W_{2\pi}^{4,1}, \ Z = L_{2\pi}^{1}$$

dom $L = \{x \in X : x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = \ddot{x}(0) - \ddot{x}(2\pi) = 0$ and $\dot{x}, \ddot{x}$ are absolutely continuous on $[0, 2\pi]\}$

Define as in [2]

$$L: \quad \text{dom } L \subset X \to Z, \ x \to \dddot{x} + b\dot{x}$$
$$F: \quad X \to Z, \ x \to f(\dot{x})\ddot{x}(t)$$
$$G: \quad X \to Z, \ x \to \tilde{Y}(t, x(t-\tau))x(t-\tau)$$
$$H: \quad X \to Z, \ x \to h(t, x(t-\tau))$$
$$A: \quad X \to Z, \ x \to \Gamma(t)x(t-\tau)$$

Thus (3.8) is equivalent to

$$Lx + Fx + Gx + Hx = P(t) \tag{3.9}$$

The existence of a solution will follow from theorem 4.5 in [3] if we show that the possible solutions of the equation

$$Lx + \lambda Fx + (1-\lambda)Ax + \lambda Gx + \lambda Hx = \lambda P(t) \tag{4.0}$$

In dom L are *a priori* bounded independently of $\lambda \in [0,1]$.

By our construction we have that

$$0 < (1-\lambda)\Gamma(t) + \lambda\tilde{Y}(t, x(t-\tau)) \le \Gamma(t) + \frac{\delta}{2} \tag{4.1}$$

Hence by Lemma 3.1 we have

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\bar{x} + \dot{\tilde{x}}(t)\right]\left\{b^{-1}\dddot{x} + \lambda b^{-1}f(\dot{x})\ddot{x} + \dot{x} + [(1-\lambda)\Gamma(t) + \lambda\tilde{Y}(t, x(t-\tau))]x(t-\tau)\right\}dt$$
$$\ge \frac{\delta}{2}|\tilde{x}|^2_{H^1_{2\pi}}$$

Thus $0 = \frac{1}{2\pi}\int_0^{2\pi}\left[\bar{x} + \dot{\tilde{x}}(t)\right]\left\{b^{-1}\dddot{x} + \lambda b^{-1}f(\dot{x})\ddot{x} + \dot{x} + [(1-\lambda)\Gamma(t)\right.$

$$\left. + \lambda\tilde{Y}(t, x(t-\tau))]x(t-\tau) + \lambda h(t, x(t-\tau) - \lambda P(t)\right\}dt$$
$$\ge \frac{\delta}{2}|\tilde{x}|^2_{H^1_{2\pi}} - 2\pi\left(|Y_0|_{L^1} + |P|_{L^1}\right)\left(|\bar{x}| + |\dot{\tilde{x}}|_c\right)$$
$$\ge \frac{\delta}{2}|\tilde{x}|^2_{H^1_{2\pi}} - \beta\left(|\bar{x}| + |\tilde{x}|^2_{H^1_{2\pi}}\right) \tag{4.2}$$

Hence

$$|\tilde{x}|^2_{H^1_{2\pi}} \le \beta_1\left(|\bar{x}| + |\tilde{x}|_{H^1_{2\pi}}\right)$$

for some constant $\beta_1 > 0$. On integrating (4.0) and using the arguments in [5] we obtain

$$|x|_{H^1_{2\pi}} \le \beta_2 \tag{4.3}$$

where $\beta_2 > 0$ is a constant. The inequality (4.3) implies that

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