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OSCILLATING PERIODIC SOLUTIONS OF SOME NON-LINEAR DEFERENTIAL EQUATIONS

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1 INTRODUCTION

We consider here ordinary differential equations of the form

$$u^{(m)} = \lambda u + p(t)g(u) - \epsilon f(t) \tag{1.1}$$

where λ and ϵ are real parameters, $P, f \in C \ ([o, +\infty], R)$ $g \in C \ (R, R)$. Our purpose is to obtain existence of ω - periodic solutions of (1.1). We shall first consider a periodic existence boundary value problem (PBVP) associated with (1.1) and obtain existence of solutions by employing the alternative method for non-linear problems at resonance [1]. We shall then relate these solutions to the ω - periodic solutions of (1.1).

Our results generalize the corresponding results of [4] in several ways. For example we obtain our results for any $\lambda \in R$ and for arbitrary m. In the case $\lambda = -1$, $f(t) = \cos\omega t$ and $p(t) = -\mu$, in equation (1.1) Ezeilo [2] has obtained existence results using the Leray - Schauder technique.

2 PRELIMINARIES

Let X be a real Hilbert Space with inner product $(,.)$ and norm $\|.\|$. Let $L : D(L) \subset X \to X$ be a linear operator with finite dimentional kernel X_o and

$$u^{(m)} = \lambda u + p(t)g(u) - \varepsilon f(t) \qquad 2.5$$
$$u^{(i)}(o) = u^{(i)}(\omega), \quad (i = 0, 1, 2...m - 1)$$

in which P, f are continuous functions on $[o, \omega]$. Our procedure is to shift all considerations away from (2.5) to the modified PBVP

$$u^{(m)} = \lambda\phi_\alpha(u) + p(t)g(\phi_\alpha(u)) - \varepsilon f(t) \qquad 2.6$$
$$u^{(i)}(o) = u^{(i)}(\omega), \quad (i = 0, 1, 2...m - 1)$$

and to show that the equation (2.6) has an ω-periodic solution for all arbitrary λ and ε. The existence of periodic solution of (2.5) will then follow in view of (2.4) after it is verified that every ω- periodic solution $u(t)$ of (2.6) satisfies $|u(t)| \leq \alpha$.

3 EXISTENCE OF PERIODIC SOLUTIONS OF 2.6

We shall prove the following

Lemma 3.1

Let $m \geq 1$. Then the PBVP

$$\begin{aligned} u^{(m)} &= 0 \\ u(o) &= u^{(i)}(\omega), \quad (i = 0, 1, 2, ...m - 1) \end{aligned}$$

has non-trivial periodic solutions.

Proof

$$u^{(m)} = 0$$

implies that

$$u(t) = \sum_{j=1}^{m-1} b_j t^{m-j} + b_m$$

for some b, $t \in R$.

Now

$$u^{(i)}(t) = \sum_{j=1}^{m-i} i!(m-j)b_i t^{m-j-i}$$

as follows

$$Px = \frac{1}{\omega}\int_o^\omega x(s)ds$$

and $Hx_1 = y_1$ if and only if $y_1^{(m)} = x_1, y_1^{(i)}(o) = y_1^{(i)}(\omega), \quad i = 0, 1, 2, ...m - 1$

Clearly $dom(N_\alpha) = X$ since ϕ_α is bounded. Thus the PBVP (2.6) may be translated into the operator equation

$$Lu = N_\alpha u \tag{3.2}$$

For $u \in X$, set $u = u_o + u_1$ where $u_o = Pu$ and $u_1 = (I - P)u$. Thus (2.6) is equivalent to the coupled system of equations

$$u_1 = H(I - P)N(u_o + u_1) \tag{3.3}$$

and

$$PN_\alpha(u_o + u_1) = 0$$

Since ϕ_α is bounded there exists a constant $J > 0$ such that $\|N_\alpha u\| \leq J$ for all $u \in X$.
Given $u_o \in X_o$ we can find a constant $c > o$ independent of $u_o \in X_o$ such that

$$\max_{t\in[o,\omega]} |u_1(t)| \leq c \text{ for any solution } u_1 \text{ of (3.3).}$$

Now choose $R > \alpha > 0$ large enough so that

$$|u_1(t)| \leq R - \alpha \tag{3.4}$$

Since $u_o = span\{1\}$ we can see that if $u_o \in X_o$ and $\|u_o\| = R$ then either $u_o = R$ or $u_o = -R$.
If $u_o = R$ then (2.4) and hypothesis (i) implies

$$\begin{aligned}(N_\alpha(u_o + u_1), u_o) &= R\textstyle\int_o^\omega[\lambda\phi_\alpha(u) + p(t)g(\phi_\alpha(u)) - \varepsilon f(t)]dt \\ &= R\textstyle\int_o^\omega[\lambda\alpha + p(t)g(\alpha) - \varepsilon f(t)]dt \\ &= R\omega[\lambda\alpha + M_p g(\alpha) - \varepsilon M_f] \geq 0\end{aligned}$$

Similarly if $u_o = -R$ we again obtain that $(N_\alpha(u_o + u_1), u_o \geq 0$. Hence the hypothesis of theorem is satisfied and the proof is complete.
Lemma 3.3

Assume that

$$M_f = \frac{1}{\omega}\int_o^\omega f(t)dt = 0 \text{ and } g(0) = 0$$

Then for any $\alpha > 0$ and for any of the following choices of λ, p and g given by

where $g_\alpha = \max_{-\alpha \le u \le \alpha} |g(u)|$
then the equation (1.1) has a solution which is both oscillatory and ω-periodic.
Proof

From Lemma 3.3, the modified PBVP (2.6) has a solution u on $[o, \omega]$ and $u(\eta) = 0$ for some $\eta \in [o, \omega]$. Let $u(t)$ be any ω-periodic solution of (2.6) then

$$|u^{(m)}(t)| \le |\lambda|\alpha + |P|g_\alpha + |\epsilon||f| \qquad 4.2$$

and thus, since

$$|u^{(m-k)}(t)| = 0 \text{ at some } t(k = 1, 2, ...m - 1)$$

we will derive successively from (4.2) that

$$|u^{(m-k)}(t)| \le \omega^k[|\lambda|\alpha + g_\alpha M_{|P|} + |\epsilon|M_{|f|}].$$

In particular

$$|u'(t)| \le \omega^{m-1}[|\lambda|\alpha + g_\alpha M_{|P|} + |\epsilon|M_{|f|}]$$

Hence,

$$u(t) = u(\eta) + \int_\eta^\omega u'(s)ds$$

and

$$|u(t)| \le \omega^m[|\lambda|\alpha + g_a M_{|P|} + |\epsilon|M_{|f|}] \le \alpha$$

Since the existence of a positive constant α such that (4.1) is satisfied is given, then the solution obtained above for the PBVP (2.6) is a solution of the PBVP (2.5). This solution may then be extended periodically on $[o, \infty]$ having at least one zero on $[o, \omega]$, one zero on $[\omega, 2\omega]$ and so on. Thus there exists arbitrarily large zeros for u. Also from the remark, u changes sign on $[o, \omega]$, $[\omega, 2\omega]$ and so on. This completes the proof.

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