PERIODIC BOUNDARY VALUE PROBLEMS FOR
THIRD ORDER ORDINARY DIFFERENTIAL
EQUATIONS WITH DELAY

S. A. IYASE

Department of Mathematics, Statistics and Computer Sciences, University of
Abuja, PMB 117, Abuja, FCT, Nigeria, West Africa

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We study the periodic boundary value problem

\[ \dot{x}(t) + f(t) \dot{x}(t) + g(t, x(t - \tau)) + \phi(t) = \psi(t), \]
\[ x(0) - x(2n) = x(0) - x(2n) = 0, \]

under some resonant conditions on the asymptotic behaviour of \( g(t, x) \) for \( |x| \to \infty \). The uniqueness of periodic solutions is also examined.

1. INTRODUCTION

In this paper we study the periodic boundary value problem

\[ \dot{x} + f(t) \dot{x} + g(t, x(t - \tau)) + \phi(x) = \psi(t), \]
\[ x(0) - x(2n) = x(0) - x(2n) = 0, \]

with fixed delay \( \tau \in [0, 2\pi] \), where \( f : R \to R \) is continuous, \( P : [0, 2\pi] \to R \) and
\( g : [0, 2\pi] \times R \to R \) are \( 2\pi \)-periodic in \( i \) and \( g \) satisfies certain Carathéodory
conditions. The unknown function \( x : [0, 2\pi] \to R \) is defined for \( 0 < t < \tau \) by
\( x(t - \tau) = [2\pi - (t - \tau)] \). We are specifically concerned with the existence of periodic
solutions of eqn. (1.1) under some resonant conditions.

The differential equations

\[ \dot{x} + a \dot{x} + f(x) \dot{x} + g(t, x(t - \tau)) = p(t), \]
\[ x(0) - x(2n) = x(0) - x(2n) = 0, \]

in which \( a = 0 \) is a constant and
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We consider a solution of the form \( x(t) = e^{\lambda t} \) where \( \lambda = \iota n \) with \( \iota^2 = -1 \).

Then Lemma 2.1 will follow if

\[
\psi(n, \iota) = -n^2 + b \cos \pi t = 0
\]

for all \( n \neq 1 \) and \( t \in [0, 2\pi) \).

By (2.2) we get

\[
\psi(n, \iota) = -n^2 + b < 0.
\]

Therefore \( \psi(n, \iota) \neq 0 \) and the result follows. If \( x \in L^1_\iota \) we shall write

\[
\int_0^{2\pi} x(t) dt = 0
\]

so that

\[
\int_0^{2\pi} x(t) dt = 0.
\]

Our next result concerns the delay equation

\[
\dot{x} + \alpha \dot{x} + \beta(t) \dot{x}(t-\tau) + cx = 0 \tag{2.4}
\]

where \( \alpha, c \) are constants and \( b \in L^1_\iota \).

**Theorem 2.1** — Let \( c \neq 0 \). Suppose that \( b(t) \) satisfies

\[
0 < b(t) < 1, \quad t \in [0, 2\pi] \tag{2.5}
\]

Then for arbitrary \( a \) eqn. (2.4) admits in \( W^{1, 2}_{2\pi} \) only the trivial solution.

**Proof**: If \( x \) is a possible solution of (2.4) then since

\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \alpha \dot{x} + cx \right) dt = 0
\]

as can be easily verified, we have from (2.5) that

\[
0 = \frac{1}{2\pi} \int_0^{2\pi} \left( \alpha \dot{x} + \beta(t) \dot{x}(t-\tau) + cx \right) dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\dot{x}^2}{2} - b(t) \ddot{x}(t-\tau) \right) dt.
\]
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**Proof:** Integrating by parts and using the identity

\[-ab = \frac{a^2 - b^2}{2} = \frac{a^2}{2} - \frac{b^2}{2}\]

and noting that

\[\frac{1}{2\pi} \int_0^{2\pi} \xi(t) (\xi(t) + v(t)) \, dt = 0\]

we get

\[\frac{1}{2\pi} \int_0^{2\pi} (\xi(t) + v(t)) \xi(t) \, dt = \frac{1}{2\pi} \int_0^{2\pi} (\xi(t))^2 \, dt\]

Thus,

\[\frac{1}{2\pi} \int_0^{2\pi} (\xi(t))^2 \, dt < \frac{1}{2\pi} \int_0^{2\pi} \xi(t) \, dt\]

\[\frac{1}{2\pi} \int_0^{2\pi} (\xi(t))^2 \, dt < \frac{1}{2\pi} \int_0^{2\pi} \xi(t) \, dt\]

We shall next consider the non-linear delay equation

\[\ddot{x} + f(t, \dot{x}) \dot{x} + g(t, \dot{x}, x(t)) + h(x) = p(t)\]

\[x(0) - x(2\pi) = \dot{x}(0) - \dot{x}(2\pi) = x(\pi) - \dot{x}(\pi) = 0\]

where \(f, h : \mathbb{R} \to \mathbb{R}\) are continuous functions and \(g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}\) is such that \(g(x, \cdot, \cdot)\) is measurable on \([0, 2\pi]\) for each \(x \in \mathbb{R}\) and \(g(t, x, \cdot)\) is continuous on \(\mathbb{R}\) for almost each \(t \in [0, 2\pi]\).

We assume moreover that for each \(r > 0\) there exists \(\gamma_r \in L^2_{2\pi}\) such that \(|g(t, y)| \leq \gamma_r(t)\) for \(s-t \in [0, 2\pi]\) and all \(x \in [-r, r]\) such a \(g\) is said to satisfy Carathéodory's conditions.

**Theorem 3.1** — Let \(g\) be a Carathéodory’s function with respect to the space \(L^2_{2\pi}\) such that

(i) There exists \(s > 0\) such that

\[xg(t, x) = 0\] for \(|x| = s\]
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3.5

Define as in Mawhin and Ward1:

\[ L : \text{dom} L \ni x \rightarrow z, \quad x \rightarrow \gamma^\prime \]
\[ F : x \rightarrow z, \quad x \rightarrow f(x) \]
\[ G : x \rightarrow z, \quad x \rightarrow \gamma(t, x(t-\tau)) \]
\[ H : x \rightarrow z, \quad x \rightarrow h(x) \]
\[ \Lambda : x \rightarrow z, \quad x \rightarrow \Delta(t) \]
\[ G_0 : x \rightarrow z, \quad x \rightarrow \gamma_0(t, x(t-\tau)) \]

The proof of the theorem will follow from Theorem 4.5 of Mawhin7 if we show that the possible solutions of the equation

\[ Lx + \lambda Fx + (1 - \lambda) Ax + \lambda Gx + \lambda \gamma_0 x + \lambda \gamma(t, x(t-\tau)) + \lambda \Delta(t) = \lambda \rho(t) \quad \ldots (3.5) \]

where \( c > 0 \) are a priori bounded independently of \( \lambda \in [0, 1] \).

For \( \lambda = 0 \) we get the equation

\[ \ddot{x} + b(t) \dot{x}(t-\tau) + cx = 0 \]

which by theorem (2.1) has only the trivial solution.

Observe that

\[ 0 \leq (1 - \lambda) b(t) \gamma(t, x(t-\tau)) \leq 0 \gamma(t) + \delta $$

Hence by Lemma 3.1 we get

\[ \frac{1}{2\pi} \int_0^{2\pi} -\hat{\gamma}(t) (\dot{\gamma} + \lambda \hat{\gamma}(t, x(t-\tau))) \dot{x} + [(1 - \lambda) b(t) + \lambda \gamma(t, x(t-\tau))] \]

\[ \ddot{x}(t-\tau) + (1 - \lambda) cx \]

\[ \leq \frac{\delta}{2} \text{ | } \ddot{x} |^2. \]

Thus

\[ 0 \leq \frac{1}{2\pi} \int_0^{2\pi} -\hat{\gamma}(t) (\dot{\gamma} + \lambda \hat{\gamma}(t, x(t-\tau))) \dot{x} + [(1 - \lambda) b(t) \]

\[ + \lambda \gamma(t, x(t-\tau))] \dot{x}(t-\tau) + (1 - \lambda) cx \]

\[ + \lambda \gamma_0(t, x(t-\tau)) + \lambda \Delta(t) - \lambda \rho(t) \] \[ \text{d}t \]
From (3.8) we obtain
\[ x(t) = x(t^*) + \frac{1}{2\pi} \int_0^{2\pi} x(s) \, ds. \]
Hence
\[ |x| \leq \beta_0 + |\dot{x}| = \beta_0 + \beta_1 \]
for some \( \beta_1 > 0 \).

From eqn. (3.5) and by continuity of \( h \) we obtain
\[ |\dot{x}| \leq \beta_0 \quad \text{for some} \quad \beta_0 > 0. \]

Now since \( \dot{x}(0) = x(2\pi) \), there exists \( t_0 \in (0, 2\pi) \) such that \( \dot{x}(t_0) = 0 \). Hence
\[ \dot{x}(t) = \dot{x}(t_0) + \frac{1}{2\pi} \int_0^{2\pi} \dot{x}(s) \, ds. \]

Therefore
\[ |\dot{x}| \leq \beta_0 \text{ for some } \beta_0 > 0. \]

Hence
\[ |\dot{x}| = |x| + |\dot{x}| = |x| \leq \beta_0 + \beta_1 + \beta_0 = \beta_0. \]

Choosing \( \gamma > \beta_0 > 0 \) we obtain the required \textit{a priori} bound in \( C^1 [0, 2\pi] \) independently of \( x \) and \( A \).

**4. Uniqueness Result**

If in (1.1) \( f(x) = a \), \( h(x) = d \) where \( a \) and \( d \) are constants, then we have the following uniqueness result.

**Theorem 4.1** — Let \( a \) and \( d \) be constants with \( d > 0 \). Suppose \( g \) is a Carathéodory function satisfying
\[ 0 < g(t, x_1) - g(t, x_2) < b(t) \]
for all \( x_1, x_2 \in \mathbb{R} \), \( t = t_0 \), where \( b(t) \in L^1_{[0, 2\pi]} \) is such that \( 0 < b(t) < 1 \). Then for all arbitrary constant \( a \) and every \( \tau \in [0, 2\pi] \) the boundary value problem
\[ \dot{x} + ax^2 + g(t, \dot{x} (t - \tau)) + dx = p(t) \]
\[ x(0) - x(2\pi) - \dot{x}(0) - \dot{x}(2\pi) = \dot{x}(\tau) = 0 \]
has at most one solution.