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PERIODIC BOUNDARY VALUE PROBLEMS FOR FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS

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1 Introduction

The main motivation for this study is a recent paper by A.R. Aftabizadeh, Jian-Mingxu and Chaitan P. Gupta [1]. In that paper the authors study the third order periodic boundary value problem

$$\dddot{u} + f(\dot{u})\ddot{u} + h(u)\dot{u} = g(t,u,\dot{u},\ddot{u}) - e(t)$$
$$u(0) = u(1),\ \dot{u}(0) = \dot{u}(1),\ \ddot{u}(0) = \ddot{u}(1)$$

A uniqueness result was established for the equation

$$\dddot{u} + A\ddot{u} + k\dot{u} = g(t,u) - e(t)$$
$$u(0) = u(1),\ \dot{u}(0) = \dot{u}(1),\ \ddot{u}(0) = \ddot{u}(1)$$

where in each case g satisfies Caratheodory's conditions and A, k are constants.

Recently a number of papers have appeared dealing with the existence of solutions and in some cases, with the uniqueness of solutions of fourth order boundary value problems. For example see Aftabizadeh [2], Yang [7], Gupta [5], Agarwal [4]. All these papers considered the Non-resonance problems near the first eigenvalue.

In this paper we shall extend the study in [1] to periodic boundary value problems of the fourth order. We shall use Mawhin's version of Leray-Schauder continuation theorem and Wirtinger's type inequalities to obtain the necessary a-priori estimates. We shall also establish a uniqueness result for the equation

$$u^{iv} + A\dddot{u} + B\ddot{u} + C\dot{u} = g(t,u) - e(t)$$
$$u(0) = u(1),\ \dot{u}(0) = \dot{u}(1),\ \ddot{u}(0) = \ddot{u}(1), \dddot{u}(0) = \dddot{u}(1).$$

2 Existence Results

Let R denote the reals, $C^k[0,1]$ the space of all k-times continuously differentiable functions defined on [0,1] with norm

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$$\|f\|_k = \sup\|f^{(i)}(t)\|, \quad t \in [0,1].$$

Let $L^p[0,1], 1 \le p \le \infty$ denote the usual Lebesque Spaces with norm $\|\cdot\|_p$.

Also, let X and Z be real normed vector spaces.

$$L : \text{dom } L \subset X \to Z$$

a Fredholm mapping of Index zero, and D an open bounded subset of X. Then $C_L(D)$ will denote the class of **mappings**

$$F : \text{dom } L \cap \overline{D} \to Z$$

which are of the form $F = L + G$ with $G : \overline{D} \to Z$, L-compact on $\overline{D}$ and which satisfy $0 \notin F(\text{dom } L \cap \partial D)$ where ∂D denotes the boundary of D.

THEOREM 2.1 (J. **Mawhin [6]** Theorem IV.4)

Let $H \in C_L(D)$ **be one-to-one** on $\overline{D}$ and such that for some $z \in H(\text{dom } L \cap D)$ and for every $(x, \lambda) \in (\text{dom } L \cap \partial D) \times (0,1)$

$$\lambda Fx + (1-\lambda)(Hx - z) \ne 0$$

Then the equation

$$Fx = 0$$

has at least one solution in dom $L \cap \overline{D}$.

We shall apply a version of theorem 2.1 to obtain existence results for the periodic boundary value problem

$$\begin{aligned} &x^{iv} + f(\ddot{x})\dddot{x} + h(\dot{x})\ddot{x} = g(t, x, \dot{x}, \ddot{x}, \dddot{x}) - p(t) \\ &x(0) = x(1),\ \dot{x}(0) = \dot{x}(1),\ \ddot{x}(0) = \ddot{x}(1),\ \dddot{x}(0) = \dddot{x}(1) \end{aligned} \tag{2.1}$$

where $f, h \in C(\mathbf{R}, \mathbf{R})$, $p \in L^1[0,1]$ and $g : [0,1] \times \mathbf{R}^4 \to \mathbf{R}$ satisfies Caratheodory's conditions.

LEMMA 2.1

The eigenvalue problem

$$\begin{aligned} &x^{iv} = \lambda x \\ &x(0) = x(1),\ \dot{x}(0) = \dot{x}(1),\ \ddot{x}(0) = \ddot{x}(1),\ \dddot{x}(0) = \dddot{x}(1) \end{aligned} \tag{2.2}$$

has $\lambda = 0$ as its only eigenvalue.

PROOF

Substituting $x(t) = Ae^{mt}$ where A and m are constants into (2.2) we obtain the solutions of (2.2) in the form

$$x(t) = Ae^{\sqrt[4]{\lambda}\, t}$$

Applying the boundary conditions we have that

$A \neq 0$ if and only if $\lambda = 0$

The above result implies that there is resonance at 0 and nowhere else. This suggests that in (2.1)

$g(t,x,\dot{x},\ddot{x},\dddot{x}) \geq 0$ or $g(t,x,\dot{x},\ddot{x},\dddot{x}) \leq 0$.

Let $I = [0,1]$.

THEOREM 2.2

Let $g : I \times \mathbf{R}^4 \to \mathbf{R}$

be a Caratheodory's function. That is

i. $g(\cdot,x)$ is measurable for each $x \in \mathbf{R}^4$.

ii. $g(t,\cdot)$ is continuous for a.e. $t \in I$.

iii. For every $s > 0$ there exists $\gamma_s \in L^1[0,1]$ such that $|g(t,x)| \leq \gamma_s(t)$ for a.e. $t \in I$ where $\|x\| \leq s$.

Let r,R,m,M with $r < 0 < R$ and $m \leq M$ be such that

$g(t,x,\dot{x},\ddot{x},\dddot{x}) \geq M$ for $x \geq R$, $(t,\dot{x},\ddot{x},\dddot{x}) \in I \times \mathbf{R}^3$

and

$g(t,x,\dot{x},\ddot{x},\dddot{x}) \leq m$ for $x \leq r$, $(t,\dot{x},\ddot{x},\dddot{x}) \in I \times \mathbf{R}^3$

1. There exists $a(t) \in L^2[0,1]$, $b(t)$, $c(t), d(t) \in L^\infty[0,1]$ and $e \in L^1[0,1]$ such that

$$|g(t,x,\dot{x},\ddot{x},\dddot{x})| \leq a(t)|x| + b(t)|\dot{x}| + c(t)|\ddot{x}| + d(t)|\dddot{x}| + e(t)$$

2. There exists $k \in \mathbf{R}$ such that

$h(\dot{x}) \leq k$

Then for every $p(t) \in L^1[0,1]$ with

$$m \leq \int_0^1 p(t)dt \leq M$$

the boundary value problem (2.1) has at least one solution provided

$$\|a\|_2 + \|b\|_\infty + 2\pi\|c\|_\infty + 4\pi^2\|d\|_\infty + 2\pi|k| < 8\pi^3.$$

PROOF

As a consequence of theorem 2.1 it suffices to show that the set of possible solutions of the family of equations

$$x^{iv} - (1-\lambda)\int_0^1 x(t)dt + \lambda f(\ddot{x})\dddot{x} + \lambda h(\dot{x})\ddot{x} = \lambda g(t,x,\dot{x},\ddot{x},\dddot{x}) - \lambda p(t)$$

$$x(0) = x(1),\ \dot{x}(0) = \dot{x}(1),\ \ddot{x}(0) = \ddot{x}(1),\ \dddot{x}(0) = \dddot{x}(1) \tag{2.3}$$

is a-priori bounded in $C^3[0,1]$ by a constant independent of $\lambda \in [0,1]$.

Thus, the question about existence of periodic solutions of equation (2.1) is now translated into the existence of solutions of equation (2.3) when $\lambda = 1$.

Let

$$g_1(t,x,\dot{x},\ddot{x},\dddot{x}) = g(t,x,\dot{x},\ddot{x},\dddot{x}) - \frac{M+m}{2}$$

$$p_1(t) = p(t) - \frac{M+m}{2}$$

Then

$$g_1(t,x,\dot{x},\ddot{x},\dddot{x}) \geq \frac{M-m}{2} \geq 0 \text{ for } x \geq R,\ (t,\dot{x},\ddot{x},\dddot{x}) \in I \times \mathbf{R}^3 \qquad (2.4)$$

$$g_1(t,x,\dot{x},\ddot{x},\dddot{x}) \leq \frac{m-M}{2} \leq 0 \text{ for } x \leq r,\ (t,\dot{x},\ddot{x},\dddot{x}) \in I \times \mathbf{R}^3 \qquad (2.5)$$

and

$$\frac{m-M}{2} \leq \int_0^1 p_1(t)dt \leq \frac{M-m}{2} \qquad (2.6)$$

We then rewrite (2.3) in the form

$$\begin{aligned} &x^{iv} - (1-\lambda)\int_0^1 x(t)dt + \lambda f(\ddot{x})\dddot{x} + \lambda h(\dot{x})\ddot{x} = \lambda g_1(t,x,\dot{x},\ddot{x},\dddot{x}) - \lambda p_1(t) \\ &x(0) = x(1),\ \dot{x}(0) = \dot{x}(1),\ \ddot{x}(0) = \ddot{x}(1),\ \dddot{x}(0) = \dddot{x}(1). \end{aligned} \qquad (2.7)$$

Suppose that $x(t) \geq R > 0$ for every $t \in I$. Then integrating (2.7) on I, we have

$$-(1-\lambda)\int_0^1 x(t)dt = \lambda\int_0^1 g_1(t,x,\dot{x},\ddot{x},\dddot{x})dt - \lambda\int_0^1 p_1(t)dt$$

or

$$\lambda\int_0^1 p_1(t)dt = \lambda\int_0^1 g_1(t,x,\dot{x},\ddot{x},\dddot{x})dt + (1-\lambda)\int_0^1 x(t)dt$$

This and (2.4) - (2.6) imply that

$$\frac{\lambda(M-m)}{2} \geq \frac{\lambda(M-m)}{2} + (1-\lambda)R \text{ or } R \leq 0$$

which is a contradiction.

If on the other hand $x(t) \leq r < 0$ for each $t \in I$ we arrive at a similar contradiction. So there exists a $t_0 \in I$ such that

$$r \leq x(t_0) \leq R.$$

Now, from $x(t) = x(t_0) + \int_{t_0}^t \dot{x}(s)ds$ and the inequalities

$$\|\dot{x}\|_2 \leq \frac{1}{2\pi}\|\ddot{x}\|_2 \qquad (2.8)$$

$$\|\ddot{x}\|_2 \leq \frac{1}{2\pi}\|\dddot{x}\|_2 \qquad (2.9)$$

we have

$$\|x\|_\infty \le \max(-r,R)+\|\dot{x}\|_2$$
$$\le \max(-r,R)+\frac{1}{2\pi}\|\ddot{x}\|_2$$
$$\le \max(-r,R)+\frac{1}{4\pi^2}\|\dddot{x}\|_2 \qquad (2.10)$$

Multiplying (2.3) by $\dddot{x}$ and integrating over [0, 1] we get

$$-\int_0^1 \dddot{x}^2 dt+\lambda\int_0^1 h(\dot{x})\dddot{x}^2 dt=\lambda\int_0^1 g(t,x,\dot{x},\ddot{x},\dddot{x})\dddot{x}dt-\lambda\int_0^1 p(t)\dddot{x}dt$$

or

$$\int_0^1 \dddot{x}^2 dt=\lambda\int_0^1 p(t)\dddot{x}dt-\lambda\int_0^1 g(t,x,\dot{x},\ddot{x},\dddot{x})\dddot{x}dt+\lambda\int_0^1 h(\dot{x})\dddot{x}^2 dt$$
$$\le \lambda\int_0^1 |g(t,x,\dot{x},\ddot{x},\dddot{x})||\dddot{x}|dt+\lambda\int_0^1 h(\dot{x})\dddot{x}^2 dt+\lambda\int_0^1 |p(t)||\dddot{x}|dt$$

From conditions (1) and (2) we have

$$\int_0^1 \dddot{x}^2 dt\le\int_0^1 a(t)|x||\dddot{x}|dt+\int_0^1 b(t)|\dot{x}||\dddot{x}|dt+\int_0^1 c(t)|\dddot{x}|^2 dt+\int_0^1 d(t)|\ddot{x}||\dddot{x}|dt$$
$$+\int_0^1 |p(t)||\dddot{x}|dt+k\int_0^1 \dddot{x}^2 dt+\int_0^1 e(t)|\dddot{x}|dt$$

or

$$\|\dddot{x}\|_2^2\le\|a\|_2\|\dddot{x}\|_2\|x\|_\infty+\|b\|_\infty\|\dot{x}\|_2\|\dddot{x}\|_2+\|c\|_\infty\|\dddot{x}\|_2^2+\|d\|_\infty\|\ddot{x}\|_2\|\dddot{x}\|_2$$
$$+\|p\|_1\|\dddot{x}\|_\infty+k\|\dddot{x}\|_2^2+\|e\|_1\|\dddot{x}\|_\infty$$

Using (2.8), (2.9), (2.10) and

$$\|\dddot{x}\|_\infty\le\|\dddot{x}\|_2 \qquad (2.11)$$

we obtain

$$\|\dddot{x}\|_2^2\le\|a\|_2\frac{1}{2\pi}\|\dddot{x}\|_2[\max(-r,R)+\frac{1}{4\pi^2}\|\dddot{x}\|_2]$$
$$+\|b\|_\infty\frac{1}{8\pi^3}\|\dddot{x}\|_2^2+\|c\|_\infty\frac{1}{4\pi^2}\|\dddot{x}\|_2^2+\|d\|_\infty\frac{1}{2\pi}\|\dddot{x}\|_2^2$$
$$+\|p\|_1\|\dddot{x}\|_2+|k|\frac{1}{4\pi^2}\|\dddot{x}\|_2^2+\|e\|_1\|\dddot{x}\|_2$$

or

$$\|\dddot{x}\|_2\le\frac{4\pi^2\|a\|_2\max(-r,R)+8\pi^3\|e\|_1+8\pi^3\|p\|_1}{8\pi^3-(\|a\|_2+2\pi\|c\|_\infty+4\pi^2\|d\|_\infty+2\pi|k|+\|b\|_\infty)}$$
$$=\beta_1 \qquad (2.12)$$

It follows from (2.10) that

$$\|x\|_\infty \le \max(-r, R) + \frac{\beta_1}{4\pi^2} = \beta_2 \qquad (2.13)$$

and from (2.12) we have

$$\|\ddot{x}\|_\infty \le \beta_1 \qquad (2.14)$$

Using the fact that

$$\|\dot{x}\|_\infty \le \|\ddot{x}\|_2 \le \frac{1}{2\pi}\|\dddot{x}\|_2$$

implies

$$\|\dot{x}\|_\infty \le \frac{\beta_1}{2\pi} = \beta_3 \qquad (2.15)$$

From (2.3) we have

$$|x^{iv}| \le (1-\lambda)\|x\|_1 + \lambda|f(\ddot{x})||\dddot{x}| + \lambda|h(\dot{x})||\ddot{x}| + \lambda|g(t, x, \dot{x}, \ddot{x}, \dddot{x})| + \lambda|p(t)|.$$

Hence using condition (1) we get

$$\begin{aligned}\|x^{iv}\|_1 &\le \|x\|_1 + \int_0^1 |f(\ddot{x})||\dddot{x}|dt + \int_0^1 |h(\dot{x})||\ddot{x}|dt + \|p\|_1 \\ &\quad + \int_0^1 [|a(t)||x| + |b(t)||\dot{x}| + |c(t)||\ddot{x}| + |d(t)||\dddot{x}| + e(t)]dt \\ &\le \|x\|_\infty + \|f\|_{C[-\beta_1,\beta_1]}\|\dddot{x}\|_2 + \|h\|_{C[-\beta_3,\beta_3]}\|\ddot{x}\|_2 + \|a\|_2\|x\|_\infty \\ &\quad + \|b\|_\infty\|\dot{x}\|_2 + \|c\|_\infty\|\ddot{x}\|_2 + \|d\|_\infty\|\dddot{x}\|_2 + \|e\|_1 + \|p\|_1 \\ &= \beta_4 \end{aligned} \qquad (2.16)$$

Since $\ddot{x}(0) = \ddot{x}(1)$ there exists $\tau \in (0,1)$ such that $\dddot{x}(\tau) = 0$.
Therefore

$$\|\dddot{x}\|_\infty \le \beta_5 \qquad (2.17)$$

From (2.13), (2.14), (2.15) and (2.17), we can see that the set of solutions (2.3) are a priori bounded in $C^3[0,1]$ by a constant independent of $\lambda \in [0,1]$.

THEOREM 2.3

Let $g : I \times \mathbf{R}^4 \to \mathbf{R}$ be a function satisfying Caratheodory's conditions, $f, h \in C(\mathbf{R}, \mathbf{R})$ and there exists $k \in \mathbf{R}$ such that $h(\dot{x}) \le k$.
Let r, m, M with $r < 0 < R$ and $m \le M$ be such that

$$g(t, x, \dot{x}, \ddot{x}, \dddot{x}) \ge M \text{ for } x \ge R,\ (t, \dot{x}, \ddot{x}, \dddot{x}) \in I \times \mathbf{R}^3$$

and

$$g(t, x, \dot{x}, \ddot{x}, \dddot{x}) \le m \text{ for } x \le r,\ (t, \dot{x}, \ddot{x}, \dddot{x}) \in I \times \mathbf{R}^3.$$

Assume the following

i. There exists function $a(t)\in C^1[0,1]$ with $a(0)=a(1)$, $b(t), c(t), e(t)\in C[0,1]$, $d(t)\in L^1[0,1]$ and real numbers a_0, b_0, c_0, and e_0 such that $a'(t)\le a_0$, $b(t)\ge -b_0$, $c(t)\ge -c_0$, $e(t)\ge -e_0$ for a.e $t\in[0,1]$ and for every $x,\dot{x},\ddot{x},\dddot{x}\in\mathbf{R}$, a.e. $t\in I$ we have

$$\ddot{x}g(t,x,\dot{x},\ddot{x},\dddot{x})\ge a(t)\ddot{x}\dddot{x}+b(t)\ddot{x}^2+c(t)|\dot{x}\ddot{x}|-d(t)|\ddot{x}|+e(t)|x\ddot{x}|.$$

ii. There exists $\alpha\in C[I\times\mathbf{R}^3,\mathbf{R}]$ and $\beta\in L^1[0,1]$ such that

$$|g(t,x,\dot{x},\ddot{x},\dddot{x})|\le|\alpha(t,x,\dot{x},\ddot{x})||\dddot{x}|^2+\beta(t)$$

for every $x,\dot{x},\ddot{x},\dddot{x}\in\mathbf{R}$, a.e. $t\in I$. Then for every $p\in L^1[0,1]$ with

$$m\le\int_0^1 p(t)dt\le M$$

the boundary value problem (2.1) has at least one solution provided

$$4\pi^2a_0+4\pi^2b_0+4\pi^2c_0+e_0+4\pi^2|k|<16\pi^4$$

PROOF

Proceeding as in the proof of theorem 2.2, we can show that there exists $t_0\in I$ such that

$$r\le x(t_0)\le R$$

and hence

$$\begin{aligned}\|x\|_\infty&\le\max(-r,R)+\|\dot{x}\|_2\\&\le\max(-r,R)+\frac{1}{2\pi}\|\ddot{x}\|_2\\&\le\max(-r,R)+\frac{1}{4\pi^2}\|\dddot{x}\|_2.\end{aligned}$$

As in the proof of theorem 2.2, we show that the set of all possible solutions of the family of equations

$$\begin{gathered}x^{iv}-(1-\lambda)\int_0^1 x(t)dt+\lambda f(\ddot{x})\dddot{x}+\lambda h(\dot{x})\ddot{x}=\lambda g(t,x,\dot{x},\ddot{x},\dddot{x})-\lambda p(t)\\x(0)=x(1),\ \dot{x}(0)=\dot{x}(1),\ \ddot{x}(0)=\ddot{x}(1),\ \dddot{x}(0)=\dddot{x}(1)\end{gathered}\qquad(2.18)$$

is a-priori bounded in $C^3[0,1]$ by a constant independent of $\lambda\in I$.

Multiplying (2.18) by $\ddot{x}$ and integrating over [0,1] we obtain

$$-\int_0^1\dddot{x}^2dt+\lambda\int_0^1 h(\dot{x})\ddot{x}^2dt=\lambda\int_0^1 g(t,x,\dot{x},\ddot{x},\dddot{x})\ddot{x}dt-\lambda\int_0^1 p(t)\ddot{x}dt.$$

Since $h(\dot{x})\le k$ we get

$$-\|\dddot{x}\|_2^2\ge\lambda\int_0^1 g(t,x,\dot{x},\ddot{x},\dddot{x})\ddot{x}dt-\lambda k\int_0^1\ddot{x}^2dt-\lambda\int_0^1 p(t)\ddot{x}dt.$$

Using condition (i), we have

$$-\|\dddot{x}\|_2^2 \geq \lambda\int_0^1 a(t)\ddot{x}\,\dddot{x}\,dt + \lambda\int_0^1 b(t)\ddot{x}^2 dt + \lambda\int_0^1 c(t)|\dot{x}\ddot{x}|dt - \lambda\int_0^1 d(t)|\ddot{x}|dt$$

$$+\int_0^1 e(t)|x\dot{x}|dt - \lambda k\int_0^1 \ddot{x}^2 dt - \lambda\int_0^1 p(t)\ddot{x}dt$$

$$\geq -\frac{\lambda}{2}a_0\|\ddot{x}\|_2^2 - b_0\lambda\|\ddot{x}\|_2^2 - \lambda c_0\|\dot{x}\|_\infty\|\ddot{x}\|_2 - \lambda\|d\|_1\|\ddot{x}\|_\infty$$

$$-e_0\|x\|_\infty\|\dot{x}\|_2 - \lambda k\|\ddot{x}\|_2^2 - \lambda\|p\|_1\|\dot{x}\|_\infty$$

Using

$$\|\dot{x}\|_\infty \leq \|\ddot{x}\|_2$$

$$\|\ddot{x}\|_\infty \leq \|\dddot{x}\|_2$$

and Wirtinger's inequality we get

$$\|\dddot{x}\|_2^2 \leq \frac{a_0}{4\pi^2}\|\dddot{x}\|_2^2 + \frac{b_0}{4\pi^2}\|\dddot{x}\|_2^2 + \frac{c_0}{4\pi^2}\|\dddot{x}\|_2^2 + \|d\|_1\|\dddot{x}\|_2$$

$$+e_0[\max(-r,R) + \frac{1}{4\pi^2}\|\dddot{x}\|_2]\frac{\|\dddot{x}\|_2}{4\pi^2} + |k|\frac{\|\dddot{x}\|_2^2}{4\pi^2} + \|p\|_1\|\dddot{x}\|_2 .$$

Thus

$$\|\dddot{x}\|_2 \leq \frac{16\pi^4\|d\|_1 + 4\pi^2 e_0\max(-r,R) + 16\pi^4\|p\|_1}{16\pi^4 - (4\pi^2 a_0 + 4\pi^2 b_0 + 4\pi^2 c_0 + e_0 + 4\pi^2|k|)} = c_1$$

It then follows that

$$\|\ddot{x}\|_\infty \leq c_2$$

$$\|x\|_\infty \leq \max(-r,R) + \frac{1}{4\pi^2}\|\dddot{x}\|_2$$

$$\leq \max(-r,R) + \frac{c_1}{4\pi^2} = c_3 .$$

Also

$$\|\dot{x}\|_\infty \leq \|\ddot{x}\|_2 \leq \frac{1}{2\pi}\|\dddot{x}\|_2 \leq \frac{c_1}{2\pi} = c_4$$

Applying condition (ii) to equation (2.18) with $\alpha \in C[I \times \mathbf{R}^3, \mathbf{R}]$ we get

$$\left\|x^{iv}\right\|_1 \leq \|\alpha\|_{C[-c_3,c_3]\times[-c_4,c_4]\times[-c_2,c_2]}\|\dddot{x}\|_2^2 + \|\beta\|_1$$

$$+\|f\|_{C[-c_2,c_2]}\|\ddot{x}\|_1 + \|h\|_{C[-c_4,c_4]}\|\dot{x}\|_1 + \|p\|_1 + \|x\|_1 .$$

Hence there exists c_5 independent of λ such that

$$\left\|x^{iv}\right\|_1 \leq c_5 .$$

Since $\ddot{x}(0) = \ddot{x}(1)$, there exists $\tau \in (0,1)$ such that $\dddot{x}(\tau) = 0$.

Therefore

$$\|\ddot{x}\|_\infty \le \|x^{iv}\|_1 \le c_5.$$

Since the constants c_2, c_3, c_4 and c_5 are independent of $\lambda \in [0,1]$ the theorem is proved.

3 Uniqueness Result

We shall consider the uniqueness of solutions for the fourth order periodic boundary value problem of the form

$$\begin{aligned} &u^{iv} + A\dddot{u} + B\ddot{u} + C\dot{u} = g(t,u) - e(t) \\ &u(0) = u(1), \dot{u}(0) = \dot{u}(1), \ddot{u}(0) = \ddot{u}(1), \dddot{u}(0) = \dddot{u}(1) \end{aligned} \tag{3.1}$$

Here A, B and C are constants, $g(t,u)$ satisfies Caratheodory's conditions.

THEOREM 3.1

Assume that the following holds

i. For a.e $t \in I$, $g(t,\cdot): I \times \mathbf{R} \to \mathbf{R}$ is strictly increasing on $\mathbf{R}$.

ii. For some r, R, m, M with $r < 0 < R$ and $m \le M$, $g(t,u) \ge M$ for $u \ge R$, $t \in I$ and $g(t,u) \le m$ for $u \le r$, $t \in I$.

iii. There exists function $b(t) \in L^2[0,1]$ and a real number b_0 such that

$b(t) \le b_0$ for $t \in I$

and

$|g(t,u) - g(t,v)| \le b(t)|u - v|$ for $u, v \in \mathbf{R}$ and $t \in I$.

Suppose that $b_0 + 2\pi B < 2\pi$.

Then for every $e \in L^1[0,1]$ with

$$m \le \int_0^1 e(t)dt \le M$$

the problem (3.1) has a unique solution.

PROOF

Existence of solution follows from theorem 2.2 or 2.3. Suppose u_1, u_2 are two solutions of (3.1).

Then

$$u_1^{iv} - u_2^{iv} + A(\dddot{u}_1 - \dddot{u}_2) + B(\ddot{u}_1 - \ddot{u}_2) + C(\dot{u}_1 - \dot{u}_2) = g(t,u_1) - g(t,u_2) \tag{3.2}$$

which gives on integration over the interval I that

$$\int_0^1 (g(t,u_1) - g(t,u_2))dt = 0$$

Since $g(t,\cdot)$ is strictly increasing on R for a.e $t \in I$, there exists $t_0 \in I$ such that

$$u_1(t_0) = u_2(t_0) \tag{3.3}$$

Multiply equation (3.2) by $(\ddot{u}_1 - \ddot{u}_2)$ and integrating over I we get

$$-\int_0^1 (\ddot{u}_1 - \ddot{u}_2)^2\,dt + B\int_0^1 (\ddot{u}_1 - \ddot{u}_2)^2\,dt = \int_0^1 (g(t,u_1) - g(t,u_2))(\ddot{u}_1 - \ddot{u}_2)\,dt.$$

From condition (iii) we have

$$-\int_0^1 (\ddot{u}_1 - \ddot{u}_2)^2\,dt + B\int_0^1 (\ddot{u}_1 - \ddot{u}_2)^2\,dt \geq -\int_0^1 b(t)|u_1 - u_2||\ddot{u}_1 - \ddot{u}_2|\,dt$$

$$\geq -b_0\int_0^1 |u_1 - u_2||\ddot{u}_1 - \ddot{u}_2|\,dt$$

$$\geq -b_0\|u_1 - u_2\|_\infty \|\ddot{u}_1 - \ddot{u}_2\|_2.$$

In view of (3.3) we have

$$\|u_1 - u_2\|_\infty \leq \|\dot{u}_1 - \dot{u}_2\|_2 \leq \frac{1}{2\pi}\|\ddot{u}_1 - \ddot{u}_2\|_2. \tag{3.4}$$

Hence

$$-\|\ddot{u}_1 - \ddot{u}_2\|_2^2 + B\|\ddot{u}_1 - \ddot{u}_2\|_2^2 \geq \frac{-b_0}{2\pi}\|\ddot{u}_1 - \ddot{u}_2\|_2^2.$$

From (3.4) and Wirtinger's inequality

$$\|\ddot{u}_1 - \ddot{u}_2\|_2 \leq \frac{1}{2\pi}\|\dddot{u}_1 - \dddot{u}_2\|_2$$

we obtain

$$-\|\ddot{u}_1 - \ddot{u}_2\|_2^2 + B\|\ddot{u}_1 - \ddot{u}_2\|_2^2 \geq \frac{-b_0}{2\pi}\|\ddot{u}_1 - \ddot{u}_2\|_2^2.$$

or

$$[2\pi - (b_0 + 2\pi B)]\|\ddot{u}_1 - \ddot{u}_2\|_2^2 \leq 0$$

and hence $\|u_1 - u_2\|_\infty = 0$ since $b_0 + 2\pi B < 2\pi$ by assumption.

Hence $u_1(t) = u_2(t)$ for a.e $t \in I$. Since u_1, u_2 are continuous on I, $u_1(t) = u_2(t)$ for every $t \in I$.

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