PERIODIC BOUNDARY VALUE PROBLEMS FOR FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS

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1 Introduction
The main motivation for this study is a recent paper by A.R. Aftabizadeh, Jian-Mingxu and Chaitan P. Gupta [1]. In that paper the authors study the third order periodic boundary value problem
\[ u'' + f(u)u' + h(u)u = g(t,u,u,u) - e(t) \]
\[ u(0) = u(1), \quad u'(0) = u'(1), \quad u''(0) = u''(1) \]
A uniqueness result was established for the equation
\[ u'' + Au + Bu = g(t,u) - e(t) \]
\[ u(0) = u(1), \quad u'(0) = u'(1), \quad u''(0) = u''(1) \]
where in each case \( g \) satisfies Caratheodory's conditions and \( A, k \) are constants.

Recently a number of papers have appeared dealing with the existence of solutions and in some cases, with the uniqueness of solutions of fourth order boundary value problems. For example see Aftabizadeh [2], Yang [7], Gupta [5], Agarwal [4]. All these papers considered the Non-resonance problems near the first eigenvalue.

In this paper we shall extend the study in [1] to periodic boundary value problems of the fourth order. We shall use Mawhin's version of Leray-Schauder continuation theorem and Wirtinger's type inequalities to obtain the necessary a-priori estimates.

We shall also establish a uniqueness result for the equation
\[ u'' + Au + Bu + Cu = g(t,u) - e(t) \]
\[ u(0) = u(1), \quad u'(0) = u'(1), \quad u''(0) = u''(1), \quad u'''(0) = u'''(1) \]

2 Existence Results
Let \( \mathbb{R} \) denote the reals, \( C^k[0,1] \) the space of all \( k \)-times continuously differentiable functions defined on \([0,1]\) with norm

\[ \| f \|_{C^k} = \max_{0 \leq t \leq 1} |f(t)| + \sum_{i=1}^{k} \max_{0 \leq t \leq 1} |f^{(i)}(t)| \]

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\[ L^p[0,1], 1 \leq p \leq \infty \text{ denote the usual Lebesgue Spaces with norm } \| \|_p. \]

Let \( L^p[0,1], 1 \leq p \leq \infty \) denote the usual Lebesgue Spaces with norm \( \| \|_p. \)

Also, let \( X \) and \( Z \) be real normed vector spaces.

\[ L: \text{dom } L \subset X \rightarrow Z \]
a Fredholm mapping of index zero, and \( D \) an open bounded subset of \( X. \) Then \( C_L(D) \)

will denote the class of mappings

\[ F: \text{dom } L \cap \mathbb{R} \rightarrow Z \]

which are of the form \( F = L + G \) with \( G: \mathbb{R} \rightarrow Z, \) L-compact on \( \mathbb{R} \) and which

satisfy \( 0 \in F(\text{dom } L \cap \mathbb{R}) \) where \( \mathbb{R} \) denotes the boundary of \( D. \)

**THEOREM 2.1 (J. Mawhin [4], Theorem IV.4)**

Let \( H \in C_L(D) \) be one-to-one on \( \mathbb{R} \) and such that for some \( z \in H(\text{dom } L \cap \mathbb{R}) \) and

for every \( (x, A) \in (\text{dom } L \cap \partial D) \times (0,1) \)

\[ \lambda Fx + (1 - \lambda)(fx - z) \neq 0 \]

Then the equation

\[ Fx = 0 \]

has at least one solution in \( \text{dom } L \cap \mathbb{R}. \)

We shall apply a version of theorem 2.1 to obtain existence results for the periodic
boundary value problem

\[ x'' + f(x) + h(x) - p(t) \]

\[ x(0) = x(1), \ x'(0) = x'(1), \ x(0) = x(1), \ x'(0) = x'(1) \]

(2.1)

where \( f, h \in C(\mathbb{R}, \mathbb{R}), \ p \in L^1[0,1] \) and \( g: [0,1] \times \mathbb{R}^4 \rightarrow \mathbb{R} \) satisfies Carathéodory’s
conditions.

**LEMMA 2.1**

The eigenvalue problem

\[ x'' = \lambda x \]

\[ x(0) = x(1), \ x'(0) = x'(1), \ x(0) = x(1), \ x'(0) = x'(1) \]

(2.2)

has \( \lambda = 0 \) as its only eigenvalue.

**PROOF**

Substituting \( x(t) = Ae^{\lambda t} \) where \( A \) and \( \lambda \) are constants into (2.2) we obtain the
solution of (2.2) in the form

\[ x(t) = Ae^{\sqrt{\lambda} t}, \]
Applying the boundary conditions we have that
\[ A \neq 0 \] if and only if \( \lambda = 0 \).

The above result implies that there is resonance at 0 and nowhere else. This suggests that in (2.1)
\[ g(t, x, i, x, i) \geq 0 \text{ or } g(t, x, i, x, i) \leq 0. \]

Let \( I = [0, 1] \).

**THEOREM 2.2**

Let \( g : I \times \mathbb{R}^4 \to \mathbb{R} \) be a Carathéodory's function. That is
i. \( g(\cdot, x) \) is measurable for each \( x \in \mathbb{R}^4 \).
ii. \( g(t, \cdot) \) is continuous for a.e. \( t \in I \).
iii. For every \( s > 0 \) there exists \( r_s \in L^1[0, 1] \) such that
\[ |g(t, x, i, x, i)| \leq r(t) \text{ for a.e. } t \in I \]
where \( |x| \leq s \).

Let \( r, R, m, M \) with \( r < 0 < R \) and \( m \leq M \) be such that
\[ g(t, x, i, x, i) \geq M \text{ for } x \geq R, (t, i, x, i) \in I \times \mathbb{R}^3 \]
and
\[ g(t, x, i, x, i) \leq m \text{ for } x \leq r, (t, i, x, i) \in I \times \mathbb{R}^3 \]
1. There exists \( u(t) \in L^2(0, 1) \), \( b(t), c(t), d(t) \in L^\infty(0, 1) \) and \( \epsilon \in L^1[0, 1] \) such that
\[ |g(t, x, i, x, i)| \leq u(t)|x| + b(t)|i| + c(t)|xi| + d(t)|xi| + \epsilon(t) \]
2. There exists \( k \in \mathbb{R} \) such that
\[ |u(t)| \leq k \]
Then for every \( p(t) \in L^1(0, 1) \) with
\[ m \leq \int_0^1 b(t) \, dt \leq M \]
the boundary value problem (2.1) has at least one solution provided
\[ |d_{\|2} + |b_{\|\infty} + 2\pi|d_{\|1} + 4\pi^2|d_{\|0} + 2\pi|\| < 8\pi^3. \]

**PROOF**

As a consequence of theorem 2.1 it suffices to show that the set of possible solutions of the family of equations
\[ x^{(4)} = (1 - \lambda)^{\|1} e(t) \, dt + \lambda^\|2(1) |x| + \lambda h(x) \, dt = \lambda g(t, x, i, x, i) - \lambda p(t) \]
\[ x(0) = x(1), x(0) = x(1), i(0) = i(1), x(0) = i(1) \]

(2.3)
is a-priori bounded in $C^3[0,1]$ by a constant independent of $\lambda \in [0,1]$.

Thus, the question about existence of periodic solutions of equation (2.1) is now translated into the existence of solutions of equation (2.3) when $\lambda = 1$.

Let
\[ q_3(t) = \rho(t) - \frac{M + m}{2} \]

Then
\[ q_3(t, x, \dot{x}, \ddot{x}, \dddot{x}) = g(t, x, \dot{x}, \ddot{x}, \dddot{x}) - \frac{M + m}{2} \]

Thus, the question about existence of periodic solutions of equation (2.1) is now translated into the existence of solutions of equation (2.3) when $\lambda = 1$.

Let
\[ (t, x, \dot{x}, \ddot{x}, \dddot{x}) \in I \times \mathbb{R}^3 \]

and
\[ m - \frac{M}{2} \leq \int_I \rho_1(t) dt \leq \frac{M - m}{2} \]

We then rewrite (2.3) in the form
\[ x'' - (1 - \lambda) \int_I x(t) dt + H(x)\ddot{x} + \lambda \delta(t, x, \dot{x}, \ddot{x})x = \lambda \rho_1(t) \]

Suppose that $x(t) \geq R > 0$ for every $t \in I$, then integrating (2.7) on $I$, we have
\[ (1 - \lambda) \int_I x(t) dt = \lambda \int_I \delta(t, x, \dot{x}, \ddot{x})x dt - \lambda \int_I \rho_1(t) dt \]

This and (2.4) - (2.6) imply that
\[ \frac{\lambda (M - m)}{2} \geq \lambda (M - m) \quad \text{for} \quad R \leq 0 \]

which is a contradiction.

If on the other hand $x(t) \leq r < 0$ for each $t \in I$, we arrive at a similar contradiction. So there exists $t_0 \in I$ such that
\[ x(t_0) = r \leq R. \]

Now, from $x(t) = x(t_0) + \int_{t_0}^t \dot{x}(s) ds$ and the inequalities
\[ |H_2| \leq \frac{1}{2\pi} |E_2| \]
\[ |H_2| \leq \frac{1}{2\pi} |E_2| \]
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we have

\[ \|x\|_\infty \leq \max(-r, R) + \frac{1}{2\pi} \|H\|_2 \]

\[ \leq \max(-r, R) + \frac{1}{4\pi^2} \|H\|_2 \]

\[ \leq \max(-r, R) + \frac{1}{4\pi^2} \|H\|_2 \]  \hspace{1cm} (2.10)

Multiplying (2.3) by \( x \) and integrating over \([0, 1]\) we get

\[ -\int_a^b x^2 dt + \lambda \int_a^b g(x, \dot{x}) x^2 dt = \lambda \int_a^b h(x, \dot{x}) x dt - \lambda \int_a^b p(t) \dot{x} dt \]

or

\[ \int_a^b x^2 dt = \lambda \int_a^b g(x, \dot{x}) x^2 dt + \lambda \int_a^b h(x, \dot{x}) x dt + \lambda \int_a^b p(t) \dot{x} \]

From conditions (1) and (2) we have

\[ \int_a^b x^2 dt \leq \int_a^b g(0, 0) x^2 dt + \int_a^b g(0, 0) \dot{x} \]

\[ \|x\|_2^2 \leq \|a\|_1 \|x\|_1 + \|b\|_1 \|x\|_1 + \|c\|_1 \|x\|_1 + \|d\|_1 \|x\|_1 \]

Using (2.8), (2.9), (2.10) and

\[ \|x\|_2 \leq \|x\|_1 \]  \hspace{1cm} (2.11)

we obtain

\[ \|x\|_2^2 \leq \|a\|_1 \frac{1}{2\pi} \|H\|_2 (\max(-r, R) + \frac{1}{4\pi^2} \|H\|_2) \]

\[ + \|b\|_1 \frac{1}{2\pi} \|H\|_2^2 \frac{1}{4\pi^2} \|H\|_2 \]

\[ + \|c\|_1 \|x\|_1 + \|d\|_1 \|x\|_1 \]

or

\[ \|x\|_2 \leq \frac{4\pi^2 \|a\|_1 \max(-r, R) + 8\pi^4 \|b\|_1} \]

\[ + 8\pi^4 \|b\|_1 + 2\pi\|c\|_1 + 4\pi^2 \|d\|_1 + 2\pi\|x\|_1 + \|x\|_1 \]

\[ = \beta_1 \]  \hspace{1cm} (2.17)

It follows from (2.10) that
\begin{align}
|\mathbf{x}_n| & \leq \max(-r,R) + \frac{\mu_1}{2\pi} = \beta_1 \\
\text{and from (2.12) we have} \quad |\mathbf{y}_n| & \leq \beta_1 \\
\text{Using the fact that} \quad |\mathbf{w}_n| & \leq \frac{1}{2\pi} |\mathbf{y}_n| \\
\text{implies} \quad |\mathbf{v}_n| & \leq \frac{\beta_1}{2\pi} = \beta_3
\end{align}

From (2.3) we have
\begin{align}
|\mathbf{w}_n| & \leq |\mathbf{x}_n| + \mathcal{A}_{f}(\mathbf{x}_n) + \mathcal{A}_{h}(\mathbf{x}_n) + \mathcal{A}_{l}(\mathbf{x}_n) + \mathcal{A}_{e}(\mathbf{x}_n)
\end{align}

Hence using condition (1) we get
\begin{align}
|\mathbf{w}_n| & \leq |\mathbf{x}_n| + \int_{0}^{1} |f(x,x,x,x,x)| dx + \int_{0}^{1} |h(x,x,x,x,x)| dx + |l| dx + |e(x)| dx \\
& \leq B_1 + \mathcal{A}_{f}(\mathbf{w}_n) + \mathcal{A}_{h}(\mathbf{w}_n) + \mathcal{A}_{l}(\mathbf{w}_n) + \mathcal{A}_{e}(\mathbf{w}_n)
\end{align}

Since \( \mathbf{x}(\cdot) = \mathbf{x}(\cdot) \) there exists \( \tau \in (0,1) \) such that \( \mathbf{x}(\tau) = 0 \).

Therefore
\begin{align}
|\mathbf{v}_n| & \leq \beta_3
\end{align}

From (2.13), (2.14), (2.15) and (2.17), we can see that the set of solutions (2.3) are a priori bounded in \( C^{(0,1]} \) by a constant independent of \( \lambda \in (0,1] \).

**Theorem 2.3**

Let \( g : I \times \mathbb{R}^3 \to \mathbb{R} \) be a function satisfying Caratheodory's conditions, \( f, h \in C(\mathbb{R}, \mathbb{R}) \) and there exists \( \delta \in \mathbb{R} \) such that \( h(x) \leq \delta \).

Let \( r, m, M \) with \( r < 0 < A \) and \( m \leq M \) be such that
\begin{align}
g(x, i, \hat{x}, \hat{\lambda}, \hat{x}, \hat{\lambda}, \hat{x}) & \leq M \quad \text{for} \quad x \geq R, \quad (i, \hat{x}, \hat{\lambda}, \hat{x}) \in I \times \mathbb{R}^3
\end{align}

and
\begin{align}
g(x, i, \hat{x}, \hat{\lambda}, \hat{x}) & \leq m \quad \text{for} \quad x \leq r, \quad (i, \hat{x}, \hat{\lambda}, \hat{x}) \in I \times \mathbb{R}^3.
\end{align}

Assume the following...
i. There exists function \( a(t) \in C([0,1]) \) with \( a(0) = a(1) \), \( b(t), c(t), e(t) \in C([0,1]) \), \( d(t) \in L^1([0,1]) \), and real numbers \( a_0, b_0, c_0, e_0 \) such that \( a'(t) \leq a_0, b(t) \leq b_0, c(t) \leq c_0, e(t) \leq e_0 \) for a.e. \( t \in [0,1] \) and for every \( x, \dot{x}, \ddot{x}, y, z \in \mathbb{R} \, \text{a.e.} \, \epsilon \in I \) we have

\[
\dot{x}(t, x, \dot{x}, \ddot{x}, y, z) \geq a(t)x + b(t)\dot{x}^2 + c(t)\ddot{x}^2 + e(t)\epsilon^2.
\]

ii. There exists \( \alpha \in C([0,1] \times \mathbb{R}^3, \mathbb{R}) \) and \( \beta \in L^1([0,1]) \) such that

\[
|\alpha(t, x, \dot{x}, \ddot{x})| \leq \|\alpha(t, x, \dot{x}, \ddot{x})\|_1 + \beta(t)
\]

for every \( x, \dot{x}, \ddot{x} \in \mathbb{R} \, \text{a.e.} \, t \in I \). Then for every \( \rho \in L^1([0,1]) \) with

\[
m \leq \int_0^1 \rho(t) dt \leq M
\]

the boundary value problem (2.1) has at least one solution provided

\[4n^2 a_0 + 4n^2 b_0 + 4n^2 c_0 + 4n^2 e_0 < 16n^4\]

**Proof**

Proceeding as in the proof of theorem 2.2, we can show that there exists \( \varphi \in I \) such that

\[r \leq x(t_0) \leq R\]

and hence

\[e_1 \leq \max(-r, R) + e_2\]

\[\leq \max(-r, R) + \frac{1}{2\pi} e_2\]

\[\leq \max(-r, R) + \frac{1}{4\pi} \|e_1\|\]

As in the proof of theorem 2.2, we show that the set of all possible solutions of the family of equations

\[
x'' - (1 - \lambda) \int_0^l x(t) dt + \lambda x(t) \dot{x} + \lambda \dot{x}(t, x, \dot{x}, \ddot{x}) - \lambda p(t) x = 0,
\]

\[
x(0) = x(1), \dot{x}(0) = \dot{x}(1), \ddot{x}(0) = x(1), \ddot{x}(1) = -x(1)
\]

is a-priori bounded in \( C^2([0,1]) \) by a constant independent of \( \lambda \in I \). Multiplying (2.18) by \( \ddot{x} \) and integrating over \([0,1]\) we obtain

\[-\int_0^1 \dot{x}^2 dt + \lambda \int_0^1 \dot{x}^2 dt \ddot{x} - \lambda \int_0^1 \dot{x}^2 dt = \lambda \int_0^1 g(t, x, \dot{x}, \ddot{x}) \ddot{x} dt - \int_0^1 f(t) \ddot{x} dt
\]

Since \( k(t) \leq k \) we get

\[-\int_0^1 \dot{x}^2 dt \geq \lambda \int_0^1 g(t, x, \dot{x}, \ddot{x}) \ddot{x} dt - \lambda \int_0^1 \dot{x}^2 dt - \lambda \int_0^1 f(t) \ddot{x} dt
\]

Using condition (i), we have
Using Wirtinger's inequality we get
\[ |\mathbf{u}_\alpha| \leq \frac{\alpha_0 |\mathbf{u}_\alpha| - \alpha_0 |\mathbf{u}_\alpha|^2 - \lambda \alpha_0 |\mathbf{u}_\alpha| |\mathbf{u}_\alpha| - \lambda |\mathbf{u}_\alpha| |\mathbf{u}_\alpha| \]

\[ -e_0 |\mathbf{u}_\alpha| - \lambda |\mathbf{u}_\alpha| |\mathbf{u}_\alpha| - \lambda |\mathbf{u}_\alpha| |\mathbf{u}_\alpha| \]

and

\[ |\mathbf{v}_\alpha| \leq |\mathbf{v}_\alpha| \]

Applying condition (ii) to equation (2.18) with \( \alpha \in \mathcal{C}(I \times \mathbb{R}^3, \mathbb{R}) \) we get
\[ |\mathbf{v}_\alpha| \leq |\mathbf{v}_\alpha| \]

\[ + |\mathbf{v}_\alpha| \]

\[ + |\mathbf{v}_\alpha| \]

Hence there exists \( c_5 \) independent of \( \lambda \) such that
\[ |\mathbf{v}_\alpha| \leq c_5. \]

Since \( \tilde{x}(0) = \tilde{x}(1) \), there exists \( \tau \in (0,1) \) such that \( x(\tau) = 0 \). Therefore
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Since the constants $c_2, c_3, c_4$ and $c_5$ are independent of $\lambda \in (0, 1)$ the theorem is proved.

3 Uniqueness Result

We shall consider the uniqueness of solutions for the fourth order periodic boundary value problem of the form

$$u^{iv} + Au'' + Bu = g(t, u) - e(t)$$
$$u(0) = u'(0) = u(1), u'(0) = u'(1),$$

Here $A, B$ and $C$ are constants, $g(t, u)$ satisfies Caratheodory's conditions.

**Theorem 3.1**

Assume that the following holds

i. For almost every $t \in I$, $g(t, .) : I \times R \to R$ is strictly increasing on $R$.

ii. For some $r, R, m, M$ with $r < 0 < R$ and $m < M$, $g(0, u) \geq M$ for $u \geq R, t \in I$ and $g(t, u) \leq m$ for $u \leq r, t \in I$.

iii. There exists a function $b(t) \in L^2[0, 1]$ and a real number $b_0$ such that

$$b(t) \leq b_0$$

for almost every $t \in I$ and

$$|g(t, u) - g(t, v)| \leq b(t)|u - v|$$

Suppose that $b_0 + 2MB < 2\pi$.

Then for every $e \in L^1[0, 1]$ with

$$m \leq \int_I |e(t)| dt \leq M$$

the problem (3.1) has a unique solution.

**Proof**

Existence of solution follows from theorem 2.2 or 2.3. Suppose $u_1, u_2$ are two solutions of (3.1).

Then

$$u_1^{iv} - u_2^{iv} + A(u_1'' - u_2'') + B(u_1' - u_2') + C(u_1 - u_2) = g(t, u_1) - g(t, u_2)$$

which gives on integration over the interval $I$ that

$$\int_I (g(t, u_1) - g(t, u_2))dt = 0$$

Since $g(t, .)$ is strictly increasing on $R$ for all $t \in I$, there exists $t_0 \in I$ such that

$$u_1(t_0) = u_2(t_0)$$

(3.3)
Multiply equation (3.2) by \((\tilde{u}_1 - \tilde{u}_2)\) and integrating over \(I\) we get
\[-\int_I (\tilde{u}_1 - \tilde{u}_2)^2 \, dt + B \int_I (\tilde{u}_1 - \tilde{u}_2)^2 \, dt = \int_I (g(t, \tilde{u}_1) - g(t, \tilde{u}_2))(\tilde{u}_1 - \tilde{u}_2) \, dt.
\]
From condition (iii) we have
\[-\int_I (\tilde{u}_1 - \tilde{u}_2)^2 \, dt + B \int_I (\tilde{u}_1 - \tilde{u}_2)^2 \, dt \geq -\int_I b(t)\tilde{u}_1 - \tilde{u}_2^2 \, dt
\geq -b_0\int I \|\tilde{u}_1 - \tilde{u}_2\|_1^2 \, dt
\geq -b_0\int I \|\tilde{u}_1 - \tilde{u}_2\|_1^2 \, dt.
\]
In view of (3.3) we have
\[\|\tilde{u}_1 - \tilde{u}_2\|_2 \leq \frac{1}{2\pi} \|\tilde{u}_1 - \tilde{u}_2\|_1, \tag{3.4}\]
Hence
\[-b_0 \int I \|\tilde{u}_1 - \tilde{u}_2\|_1^2 \geq -b_0 \int I \|\tilde{u}_1 - \tilde{u}_2\|_1^2.
\]
From (3.4) and Wirtinger's inequality
\[\|\tilde{u}_1 - \tilde{u}_2\|_2 \leq \frac{1}{2\pi} \|\tilde{u}_1 - \tilde{u}_2\|_1,\]
we obtain
\[-b_0 \int I \|\tilde{u}_1 - \tilde{u}_2\|_1^2 \geq -b_0 \int I \|\tilde{u}_1 - \tilde{u}_2\|_1^2.
\]
or
\[(2\pi - (b_0 + 2\pi B)) \|\tilde{u}_1 - \tilde{u}_2\|_1^2 \leq 0\]
and hence \(\|\tilde{u}_1 - \tilde{u}_2\|_1 = 0\) since \(b_0 + 2\pi B < 2\pi\) by assumption.
Hence \(u_1(t) = u_2(t)\) for \(a \leq t \leq b\). Since \(u_1, u_2\) are continuous on \(I\), \(u_1(t) = u_2(t)\) for every \(t \in I\).

References
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