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PERIODIC SOLUTIONS OF CERTAIN EIGHTH ORDER DIFFERENTIAL EQUATIONS

S.A. IYASE

1. Introduction

In this paper we give sufficient conditions for the existence of periodic solutions of the eight order differential equation

$$(1.1)\quad x^{(8)} + a_1x^{(7)} + a_2x^{(6)} + a_3x^{(5)} + a_4x^{(4)} + a_5x^{(3)} + f_6(x)\,\ddot{x} + f_7(x)\,\dot{x} + f_8(x) = p(t, x, \dot{x}, \ddot{x}, x^{(3)}, x^{(4)}, x^{(5)}, x^{(6)}, x^{(7)})$$

and also sufficient conditions for the non-existence of non-trivial periodic solutions for $p \equiv 0$, where a_1, a_2, a_3, a_4 and a_5 are constants and f_6, f_7 and f_8 are continuous functions depending only on the arguments shown. Furthermore, the function p is assumed to be ω-periodic in t, that is $p(t, x_1, \ldots, x_8) = p(t + \omega, x_1, \ldots, x_8)$ for some $\omega > 0$ and for arbitrary $x_1, \ldots, x_8$.

Recently Ezeilo [1] and Bereketoglu [4] studied the same problems for sixth and seventh order differential equations respectively. In order to formulate the main results we shall consider the eighth order constant-coefficient differential equation:

$$(1.2)\quad x^{(8)} + a_1x^{(7)} + a_2x^{(6)} + a_3x^{(5)} + a_4x^{(4)} + a_5x^{(3)} + a_6\,\ddot{x} + a_7\,\dot{x} + a_8\,x = 0.$$

The auxilliary equation of (1.2) is given by

$$(1.3)\quad \psi(\lambda) = \lambda^8 + a_1\lambda^7 + a_2\lambda^6 + a_3\lambda^5 + a_4\lambda^4 + a_5\lambda^3 + a_6\lambda^2 + a_7\lambda + a_8 = 0.$$

If β is an arbitrary real number, then the real part of $\psi(i\beta)$ is given by

$$(1.4)\quad \phi(\beta) = \beta^8 - a_2\beta^6 + a_4\beta^4 - a_6\beta^2 + a_8.$$

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If

$$a_2 \le 0,\ a_4 \ge 0,\ a_6 \le 0,\ a_8 > 0, \tag{1.5}$$

in which case $\phi(\beta) > 0$, then (1.2) cannot have any purely imaginary root whatever. It therefore follows from the general theory that (1.2) does not have periodic solutions except $x = 0$ and the perturbed equation

$$x^{(8)} + a_1 x^{(7)} + a_2 x^{(6)} + a_3 x^{(5)} + a_4 x^{(4)} + a_5 x^{(3)} + a_6\,\ddot{x} + a_7\,\dot{x} + a_8\,x = p(t) \tag{1.6}$$

in which p is a continuous ω-periodic function of t has an ω-periodic solution.

An analogous consideration of the imaginary part of $\psi(i\beta)$ also leads to conditions on a_1, a_3, a_5, a_7 for the non-existence of any periodic solution of (1.2) other than $x = 0$.

We shall now consider equation (1.2) when a_6, a_7 and a_8 are not necessarily constants. Our results are contained in the following theorem.

THEOREM 1. *Let us consider the differential equation*

$$\begin{aligned} x^{(8)} + a_1 x^{(7)} + a_2 x^{(6)} + a_3 x^{(5)} + a_4 x^{(4)} + a_5 x^{(3)} \\ + f_6(\dot{x})\,\ddot{x} + f_7(x)\,\dot{x} + f_8(x) = 0 \quad (f_8(0) = 0) \end{aligned} \tag{1.7}$$

If

$$\begin{cases} a_2 \le 0,\ a_4 \ge 0,\ y\int_0^y f_6(s)\,ds \le 0 \text{ for all } y \\ x f_8(x) > 0 \quad (x \ne 0) \end{cases} \quad \text{and} \tag{1.8}$$

then equation (1.7) *has no non-trivial periodic solutions.*

REMARK 1. There are no restrictions on a_1, a_3, a_5 as well as $f_7(x)$.

For the case $p \ne 0$, we have the following result.

THEOREM 2. *If*

(i) $a_2 < 0,\ a_4 > 0,\ y\int_0^y f_6(s)\,ds \le 0$ *for all* y
(ii) $x f_8(x) > 0$
(iii) $f_8(x)\,\mathrm{sgn}\,x \to \infty$ *as* $|x| \to \infty$
(iv) *There exists constant* k *such that* $|p(t, x_1, \ldots, x_8)| \le k$ *for all* $t, x_1, \ldots, x_8$.

Then equation (1.1) *admits at least one* ω*-periodic solution.*

2. Preliminaries

We consider the equivalent system

$$\begin{aligned} \dot{x}_i &= x_{i+1} \quad (i = 1, \ldots, 7) \\ \dot{x}_8 &= -a_1 x_8 - a_2 x_7 - a_3 x_6 - a_4 x_5 - a_5 x_4 \\ &\quad - f_6(x_2) x_3 - f_7(x_1) x_2 - f_8(x_1) \end{aligned} \tag{2.1}$$

obtained from (1.7) on setting

$$x_1 = x,\ x_2 = \dot{x},\ x_3 = \ddot{x},\ x_4 = x^{(3)},\ x_5 = x^{(4)},\ x_6 = x^{(5)},\ x_7 = x^{(6)}$$

To prove Theorem 1, we show that every ω-periodic solution $(x_1(t), \dots, x_8(t)) = (0, \dots, 0)$ for all t. We shall use the function $V = V(y_1, \dots, y_8)$ defined as follows

$$\begin{aligned} V = {} & y_1 \int_0^{y_2} f_6(s)\, ds - \int_0^{y_1} s f_7(s)\, ds - y_1 \left(y_8 + \sum_{j=1}^{5} a_j y_{8-j} \right) \\ & + \sum_{i=2}^{4} (-1)^i y_i \left[y_{9-i} + \sum_{j=1}^{9-2i} a_j y_{9-(i+j)} \right] \\ & + \frac{1}{2} \sum_{j=1}^{3} (-1)^j a_{2j-1} y_{5-j}^2 . \end{aligned} \tag{2.2}$$

To prove Theorem 2, we consider the nonlinear vector differential equation

$$\dot{X} = A(t)\, X + G(t, X) \tag{2.3}$$

where $A(t)$ is an $n \times n$ matrix continuous for all $t \in \mathbf{R}$, with $A(t) = A(t + \omega)$. $G(t, X)$ is continuous for all $(t, X) \in \mathbf{R} \times \mathbf{R}^n$ with $G(t, X) = G(t + \omega, X)$. We have the following theorem which will be useful in our proof.

THEOREM [2]. *If*

(a) *the homogeneous equation*

$$\dot{X} = A(t)\, X \tag{2.4}$$

has no non-trivial ω-periodic solution and if

(b) *there exists an a priori estimate independent of μ for the ω-periodic solutions of the equation*

$$\dot{X} = A(t)\, X + \mu G(t, X), \quad 0 \le \mu \le 1 \tag{2.5}$$

then the equation (2.3) *admits at least one ω-periodic solution.*

3. Proof of Theorem 1

Let $(y_1, \dots, y_8) = (y_1(t), \dots, y_8(t))$ be an arbitrary solution of (2.1) and differentiating the function $V(y_1, \dots, y_8)$ defined in (2.2) along solution paths of

(2.1) we get

$$V = -y_2 \int_0^{y_2} f_6(s)\,ds - y_1 f_6(y_2) y_3 - y_1 f_7(y_1) y_2$$
$$- y_1 \left(- \sum_{j=1}^{5} a_j y_{9-j} - y_3 f_6(y_2) - f_7(y_1) y_2) - f_8(y_1) \right)$$
$$- y_1 \left(\sum_{j=1}^{5} y_{9-j} \right) - y_2 \left(y_8 + \sum_{j=1}^{5} a_j y_{8-j} \right)$$
$$+ \sum_{i=2}^{4} (-1)^i y_i \left[y_{10-i} + \sum_{j=1}^{9-2i} a_j y_{10-(i+j)} \right]$$
$$+ \sum_{i=2}^{4} (-1)^i y_{i+1} \left[y_{9-i} + \sum_{j=1}^{9-2i} a_j y_{9-(i+j)} \right]$$

(3.1) $$+ \sum_{j=1}^{3} (-1)^j a_{2j-1} y_{5-j} y_{6-j}$$

(3.2) $$= -y_2 \int_0^{y_2} f_6(s)\,ds - a_2 y_4^2 + a_4 y_3^2 + y_1 f_8(y_1) + y_5^2$$

if (1.8) holds then $V(t) \geq 0$ for all t which implies $V(t)$ is monotone in t. Since V is continuous and $y_1, \ldots, y_8$ are periodic in t, $V(t)$ is bounded. Hence

(3.3) $$\lim_{t \to \infty} V(t) = V_0 \quad \text{(constant)}$$

Using (2.8) and the fact that $V(t) = V(t + m\omega)$ for any arbitrary fixed t and for arbitrary m we have

(3.4) $$V(t) = V_0 \quad \text{for all } t.$$

Thus

(3.5) $$V(t) = 0 \quad \text{for all } t$$

From (3.2) and (3.5) we have

(3.6) $$y_1 = 0 \quad \text{for all } t$$

Since $\dot{y}_i = y_{i+1}$ $(i = 1, \ldots, 7)$ implies that

(3.7) $$0 = \dot{y}_1 = y_2 = y_3 = y_4 = y_5 = y_6 = y_7 = y_8.$$

A combination of (3.6) and (3.7) gives us the desired result.

4. Proof of Theorem 2

We derive the differential equation

$$x^{(8)} + a_1 x^{(7)} + a_2 x^{(6)} + a_3 x^{(5)} + a_4 x^{(4)} + a_5 x^{(3)} + \mu f_6(\dot{x})\ddot{x} + \mu f_7(x)\dot{x} + f_8^*(x) = \mu p(t, x, \dot{x}, \dots, x^{(7)}) \quad (0 < \mu < 1) \tag{4.1}$$

from (1.1), where

$$f_8^*(x) = (1-\mu) b_8 x + \mu f_8(x).$$

b_8 is a constant whose value can be fixed such that

$$\frac{f_8(x)}{x} \geq b_8 > 0 \quad (x \neq 0) \tag{4.2}$$

We now rewrite (4.1) in the form

$$\dot{X} = A(t)\,X(t) + \mu G(t, X) \tag{4.3}$$

where

$$X = (x_1, x_2, \dots; x_8)^T$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -b_8 & 0 & 0 & -b_5 & -b_4 & -b_3 & -b_2 & -b_1 \end{pmatrix}$$

$$G = (0, 0, 0, 0, 0, 0, 0, \phi)$$

where

$$\phi = p(t, x_1, \dots, x_8) - f_6(x_2)x_3 - f_7(x_1)x_2 + b_8 x_1 - f_8(x_1). \tag{4.4}$$

$G(t, X)$ is continuous for all $t, x_1, \dots, x_8$ are ω-periodic in t. When $\mu = 0$ the system (4.3) reduces to the homogeneous equation

$$\dot{X} = AX \tag{4.5}$$

The eigenvalues of A are the roots of the equation

$$\lambda^8 + a_1\lambda^7 + a_2\lambda^6 + a_3\lambda^5 + a_4\lambda^4 + a_5\lambda^3 + b_8 = 0 \tag{4.6}$$

and by hypothesis

(4.7) $$a_2 < 0, \ a_4 > 0$$

and from (4.6),

(4.8) $$a_6 = a_7 = 0.$$

Similarly from (4.2) we have that

(4.9) $$b_8 > 0$$

combining (4.7) – (4.9), we get that (4.5) has no non-trivial ω-periodic solution. Hence assumption (a) of theorem is satisfied. Now, $\det(e^{-\omega A} - I) \neq 0$, I being the 8×8 identity matrix, hence all ω-periodic solutions $X(t)$ of (4.3) must satisfy the integral equation

$$X(t) = \mu T\{X(t)\} = \mu(e^{-\omega A} - I)^{-1} \int_t^{t+\omega} e^{-(s-t)A} G(s, X(s))\, ds.$$

To fulfill the assumption (b) of theorem we must show that

$$\|x\| = \max\left(|x_1(t)| + |x_2(t)| + \cdots + |x_7(t)|\right) \leq D_0$$

where D_0 is a constant. It is sufficient in view of (4.4) to obtain the following inequality

(4.10) $$\max_{t\in[0,\omega]} |x_1(t)| \leq D_1, \quad \max_{t\in[0,\omega]} |x_2(t)| \leq D_2, \quad \max_{t\in[0,\omega]} |x_3(t)| \leq D_3$$

where D_1, D_2, D_3 are constants. For every ω-periodic solution $(y_1(t), \ldots, y_8(t))$ of (4.1), define $W(y_1, \ldots, y_8)$ in the form

$$\begin{aligned} W = &-\mu y_1 \int_0^{y_2} f_6(s)\, ds - \mu \int_0^{y_1} s f_7(s)\, ds - y_1\left(y_8 + \sum_{j=1}^{5} a_j y_{8-j}\right) \\ &+ \sum_{i=2}^{4} (-1)^i y_i \left[y_{9-i} + \sum_{j=1}^{9-2i} a_j y_{9-(i+j)}\right] \\ &+ \frac{1}{2}\sum_{j=1}^{3} (-1)^j a_{2j-1} y_{5-j}^2 \end{aligned} \tag{4.11}$$

Then,

$$\dot{W} = -\mu y_2 \int_0^{y_2} f_6(s)\, ds - \mu y_1 f_6(y_2) y_3 - \mu y_1 f_7(y_1) y_2$$

$$
\begin{aligned}
& - y_1 \left(- \sum_{j=1}^{5} a_j y_{9-j} - \mu y_1 f_6(y_2) - \mu f_7(y_1) y_2 \right. \\
& \left. - (1-\mu) b_8 y_1 - \mu f_8(y_1) + \mu p(t, y_1, \ldots, y_8) \right) \\
& - y_1 \sum_{j=0}^{5} y_{9-j} - y_2 \left(y_8 + \sum_{j=1}^{5} a_j y_{8-j} \right) \\
& + \sum_{i=2}^{4} (-1)^i y_i \left[y_{10-i} + \sum_{j=1}^{9-2i} a_j y_{10-(i+j)} \right] \\
& + \sum_{i=2}^{4} (-1)^i y_{i+1} \left[y_{9-i} + \sum_{j=1}^{9-2i} a_j y_{9-(i+j)} \right] \\
& + \sum_{j=1}^{3} (-1)^j a_{2j-1} y_{5-j} y_{6-j} \\
= & -a_2 y_4^2 + y_5^2 - \mu y_2 \int_0^{y_2} f_6(s)\, ds + a_4 y_3^2 + \mu y_1 f_8(y_1) \\
& + (1-\mu) b_8 y_1^2 - \mu y_1 p(t, y_1, \ldots, y_8).
\end{aligned} \tag{4.12}
$$

From condition (i) of Theorem 2 we have $y_2 \int_0^{y_2} f_6(s)\, ds \leq 0$, $a_2 < 0$, $a_4 > 0$. Hence

$$
\begin{aligned}
\dot{W} \geq & -a_2 x_4^2 + x_5^2 + a_4 x_3^2 + (1-\mu) b_8 x^2 \\
& + \mu x f_8(x) - \mu x p(t, x, \dot{x}, \ldots, x^{(7)})
\end{aligned}
$$

From (4.2) we get

$$x f_8(x) - b_8 x^2 \geq 0$$

and by hypothesis (iv) of Theorem 2 we have

$$|\mu x p| \leq k\,|x|.$$

Thus,

$$
\begin{aligned}
\dot{W} & \geq -a_2 (x^{(3)})^2 + a_4 \ddot{x}^2 + b_8 x^2 - k\,|x| \\
& \geq -a_2 (x^{(3)})^2 + a_4 \ddot{x}^2 + b_8 x^2 - \frac{k^2}{4b_8} \\
& \geq D_5 \left((x^{(3)})^2 + \ddot{x}^2 \right) - \frac{k^2}{4b_8}
\end{aligned} \tag{4.13}
$$

where $D_5 = \min(-a_2, a_4)$.

Integrating (4.13) with respect to t from $t = 0$ to $t = \omega$ and using the fact that $W(t) = W(t+\omega)$ we get

$$0 \geq D_5 \int_0^\omega ((x^{(3)})^2 + \ddot{x}^2)\, dt - \frac{k^2\omega}{4b_8}$$

and

$$\int_0^\omega (x^{(3)})^2\, dt \leq \frac{k^2\omega}{4b_8 D_5} \equiv D_6$$

$$\int_0^\omega \ddot{x}^2\, dt \leq D_7$$

where $D_7 \equiv \max\left[D_6, D_6\frac{\omega^2}{4\pi^2}\right]$ and we have used the fact that

$$\int_0^\omega \ddot{x}^2\, dt \leq \frac{\omega^2}{4\pi^2}\int_0^\omega (x^{(3)})^2\, dt.$$

The periodicity condition $\dot{x}(0) = \dot{x}(\omega)$ on x implies $\ddot{x}(T_1) = 0$ at some $T_1 \in (0,\omega)$. Thus

$$\ddot{x}(t) = \ddot{x}(T_1) + \int_{T_1}^t x^{(3)}(s)\, ds.$$

Therefore

$$\max|\ddot{x}(t)| \leq \int_0^\omega |x^{(3)}|\, ds \leq \omega^{1/2}\left(\int_0^\omega |x^{(3)}|^2\, ds\right)^{1/2}$$

whence

(4.14) $$\max|\ddot{x}(t)| \leq D_6^{1/2}\omega^{1/2}.$$

Since $x(0) = x(\omega)$, there exists $\tau \in (0,\omega)$ such that $\dot{x}(\tau) = 0$. Therefore

$$\dot{x}(t) = \dot{x}(\tau) + \int_\tau^t \ddot{x}(s)\, ds.$$

Hence

(4.15) $$\max|\dot{x}(t)| \leq D_6^{1/2}\omega^{3/2}.$$

Integrating (4.1) from $t = 0$ to $t = \omega$ we get

(4.16) $$\int_0^\omega ((1-\mu)b_8 x + \mu f_8(x))\, dt = \mu\int_0^\omega p(t, x_1, \ldots, x_8)\, dt \leq k\omega.$$

Using hypothesis (iii) of Theorem 2, it follows that $x(t)$ must satisfy

$$|x_2(T_2)| \leq D_8 \quad \text{for some } T_2 \in (0,\omega)$$

and for sufficiently large D_8. Now,

$$x(t) = x(T_2) + \int_{T_2}^{t} \dot{x}(s)\,ds.$$

Using (4.15) we have

$$\max_{t\in[0,\omega]} |x(t)| \leq D_8 + D_6^{1/2}\omega^{5/2}.$$

This completes the verification of (4.10).

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Department of Mathematics, Statistics & Computer Science
University of Abuja
P.M.B. 117
Abuja, Nigeria.