PERIODIC SOLUTIONS OF A CLASS OF EVEN ORDER DIFFERENTIAL EQUATIONS *

H. O. Tejumola
(University of Ibadan, Ibadan, Nigeria)
S. A. Iyase
(University of Abuja, Abuja, Nigeria)

Abstract

We investigate sufficient conditions (theorem 1) for the non-existence of periodic solutions of equation (2.1) with $P \equiv 0$ and sufficient conditions (theorem 2) for existence of periodic solutions of equation (1.1.1).

Key words ω-periodic solution

1 Introduction

1.1

The aim of this paper is to provide sufficient conditions for the existence of periodic solutions of the 2rth order differential equation

$$x^{(2r)} + a_1 x^{(2r-1)} + a_2 x^{(2r-2)} + \cdots + a_{2r-3} \dddot{x} + f_{2r-2}(\dot{x})\ddot{x} + f_{2r-1}(x)\dot{x} + f_{2r}(x)$$
$$= p(t_2 x_2 \dot{x}_2 \ddot{x} \cdots x^{(2r-1)}) \qquad (1.1.1)$$

and the non-existence of non-trivial periodic solutions in the case $p \equiv 0$, where $a_1, a_2, a_3, \cdots, a_{2r-3}$ are constants and $f_{2r-2}, f_{2r-1}, f_{2r}, p$ are continuous real-valued functions depending only on the arguments shown, the function p is ω-periodic in t, that is $p(t, x_1, \cdots, x_{2r}) \equiv p(t+\omega, x_1, \cdots, x_{2r})$ for some $\omega > 0$ and for arbitrary $x_1, \cdots, x_{2r}$. Such periodic differential equations arise

*Manuscript received Nov. 10, 1996

From (1.2.4) we also derive that $\phi_2(\alpha) > 0$ if r is even and a_{2j-1}, $j = 1, 2, \cdots, r$ satisfy

$$a_1 \geq 0,\ a_3 \leq 0,\ a_5 \geq 0,\ \cdots,\ a_{2r-1} < 0 \tag{1.2.7}$$

or

$\phi_2(\alpha) < 0$ if r is odd and a_{2j-1}, $j = 1, 2, \cdots, r-1$ satisfy

$$a_1 \geq 0,\ a_3 \leq 0,\ a_5 \geq 0,\ \cdots,\ a_{2r-1} < 0 \tag{1.2.8}$$

Thus it follows from (1.2.5) that if r is even (resp. from (1.2.6) if r is odd) that (1.2.1) does not have periodic solutions except $x = 0$.

Therefore from the general theory, the perturbed equation

$$x^{(2r)} + a_1 x^{(2r-1)} + a_2 x^{(2r-2)} + \cdots + a_{2r-2}\ddot{x} + a_{2r-1}\dot{x} + a_{2r}x = p(t) \tag{1.2.9}$$

in which p is a continuous ω-periodic function of t has a unique ω-periodic solution. Our main object here is to obtain non-linear analogues of (1.2.5) and (1.2.6) from which we shall derive our non-existence and existence results. Similar non-existence and existence results using non-linear analogues of (1.2.7) and (1.2.8) are under preparation and will appear elsewhere.

2 Main results

2.1

We start here with the differential equation

$$x^{(2r)} + a_1 x^{(2r-1)} + a_2 x^{(2r-2)} + \cdots + a_{2r-3}\dddot{x} + f_{2r-2}(\dot{x})\ddot{x} + f_{2r-1}(x)\dot{x} + f_{2r}(x) = 0$$

$$(f_{2r}(0) = 0) \tag{2.1.1}$$

where as before a_1, a_2, a_3, $\cdots$, a_{2r-3} are constants and f_{2r-2}, f_{2r-1}, f_{2r} are continuous real valued functions depending only on the arguments shown. Our non-existence result is as follows:

Theorem 1 Let the constants a_{2j}, $j = 1, 2, \cdots, r-2$ and the function f_{2r-2} and f_{2r} satisfy

$$\left.\begin{array}{l} a_2 \leq 0,\ a_4 \geq 0,\ a_6 \leq 0,\ \cdots,\ a_{2r-4} \geq 0 \\ y\int_0^y f_{2r-2}(s)ds \leq 0,\ y f_{2r}(y) > 0\ (y \neq 0) \end{array}\right\} \tag{2.1.2}$$

To prove theorem 1, it suffices to show that every ω-periodic solution $(x_1(t),\cdots,x_{2r}(t))$ of (2.2.1) satisfies $(x_1(t),\ x_2(t),\ \cdots,\ x_{2r}(t))=(0,0,0,\cdots,0)$ for all t.

To this end we define a function $V(y_1,\cdots,y_{2r})$ which will play a crucial role in our proof as follows,

$$\begin{aligned} V \;=\; & (-1)^{r+1}y_1\textstyle\int_0^{y_2} f_{2r-2}(s)ds+(-1)^{r+1}\displaystyle\int_0^{y_1} sf_{2r-1}(s)ds \\ & +(-1)^{r+1}y_1(y_{2r}+\sum_{k=1}^{2r-3}a_k y_{2r-k}) \\ & +\sum_{k=2}^{r}(-1)^k y_k(y_{2r+1-k}+\sum_{j=1}^{2r+1-2k}a_j y_{2r+1-(k+j)}) \\ & +\frac{1}{2}\sum_{k=1}^{r-1}(-1)^k a_{2k-1}y^2_{(r+1)-k} \end{aligned} \tag{2.2.2}$$

and show that $\dot{v}=0$ and from this we conclude that $(y_1(t),\ y_2(t),\ \cdots,\ y_{2r}(t))=(0,0,0,\cdots,0)$ for all t.

To prove theorem 2, we will consider the non-linear vector differential equation.

$$\dot{x}=A(t)x+G(t,x) \tag{2.2.3}$$

where $A(t)$ is an $n\times n$ matrix continuously for all $t\in R$. $A(t)=A(t+\omega)$, $G(t,x)$ is continuous for all $(t,x)\in R\times R^n$, $G(T,x)\equiv G(t+\omega,x)$ and the following theorem.

Theorem[4] If

a. the homogeneous linear equation

$$\dot{x}=A(t)x \tag{2.2.4}$$

has no non-trivial ω-periodic solution and if

b. there exists an a-priori estimate independent of μ for the ω-periodic solutions of the equation

$$\dot{x}=A(t)x+\mu G(t,x),\ 0\le\mu\le 1, \tag{2.2.5}$$

then (2.2.3) admits at least one ω-periodic solution. In what follows for any

Since

$$V(t) = V(t + m\omega) \tag{3.1.3}$$

for any arbitrary m, we have

$$V(t) = V_0 \text{ for all t} \tag{3.1.4}$$

Thus

$$\dot{V}(t) = 0 \text{ for all t} \tag{3.1.5}$$

From (3.1.1) and (3.1.5) we conclude that $y_1 = 0$ for all t. Also since

$$\dot{y}_i = y_{i+1} \quad (i = 1, 2, 3, \cdots, 2r-1)$$

then

$$0 = \dot{y}_1 = y_2 = y_3 = \cdots = y_{2r}$$

3.2 Proof of Theorem 2

We rewrite (1.1.1) in the form

$$x^{(2r)} + a_1 x^{(2r-1)} + \cdots + a_{2r-3} \dddot{x} + \mu f_{2r-2}(\dot{x})\ddot{x} + \mu f_{2r-1}(x)\dot{x} + f_{2r}^*(x)$$
$$= \mu p(t, x, \dot{x}, \cdots, x^{(2r-1)}) \tag{3.2.1}$$

where

$$f_{2r}^*(x) = (1-\mu)b_{2r}x + \mu f_{2r}(x) \tag{3.2.2}$$

For $\mu = 1$, (3.2.1) coincides with (1.1.1), b_{2r} is a constant whose value can be fixed such that

$$\frac{f_{2r}(x)}{x} \geq b_{2r} > 0, \ (x \neq 0) \text{ if } r \text{ is even} \tag{3.2.3}$$

or

$$\frac{f_{2r}(x)}{x} \leq b_{2r} < 0, \ (x \neq 0) \text{ if } r \text{ is odd} \tag{3.2.4}$$

We now rewirte (3.2.1) in the form

$$\dot{x}_i = x_{i+1} \quad (i = 1, 2, 3, \cdots, 2r-1)$$
$$\dot{x}_{2r} = -\sum_{k=1}^{2r-3} a_k x_{2r+1-k} - \mu f_{2r-2}(x_2)x_3 - \mu f_{2r-1}(x_1)x_2$$

From (3.2.9)

$$a_{2r-2} = a_{2r-1} = 0 \tag{3.2.12}$$

Similarly from (3.2.3) or (3.2.4) we get

$$a_{2r} = b_{2r} > 0 \tag{3.2.13}$$

or

$$a_{2r} = b_{2r} < 0 \tag{3.2.14}$$

depending on whether r is even or odd. Combining (3.2.10)-(3.2.14) we get that (3.2.8) has no non-trivial ω-periodic solution. Hence assumption (a) of theorem of is satisfied. Since $\det(e^{-\omega A} - I) \neq 0$, I being the $2r \times 2r$ identify matrix all ω-periodic solutions $x(t)$ of (3.2.6) must satisfy the integral equation

$$x(t) = \mu T\{x(t)\} = \mu(e^{-\omega A} - I)^{-1} \int_t^{t+\omega} e^{-(s-t)A} G(s, x(s)) ds \tag{3.2.15}$$

To verify hypothesis (b) of theorem we must show that

$$\|x\| = \max(|x_1(t)| + |x_2(t)| + \cdots + |x_{2r-1}(t)|) \leq D_0 \tag{3.2.16}$$

It is sufficient in view of (3.2.7) and (3.2.15) to obtain the following estimates.

$$\max_{t\in[0,\omega]} |x_1(t)| \leq D_1, \quad \max_{t\in[0,\omega]} |x_2(t)| \leq D_2, \quad \max_{t\in[0,\omega]} |x_3(t)| \leq D_3 \tag{3.2.17}$$

For every ω-periodic solution $y(t)$ of (3.2.1), we define a function $w = w(y_1 \cdots y_{2r})$ by

$$\begin{aligned} w \;=\; & (-1)^{r+1} y_1 \int_0^{y_1} f_{2r-2}(s) ds + (-1)^{r+1} \mu \int_0^{y_1} s f_{2r-1}(s) ds \\ & +(-1)^{r+1} y_1 (y_{2r} + \sum_{k=1}^{2r-3} a_k y_{2r-k}) \\ & +\sum_{k=2}^{r} (-1)^k y_k [y_{2r+1-k} + \sum_{j=1}^{2r+1-2k} a_j y_{2r+1-(k+j)}] \\ & +\frac{1}{2} \sum_{k=1}^{r-1} (-1)^k a_{2k-1} y^2_{(r+1)-k} \end{aligned} \tag{3.2.18}$$

and by hypothesis (2.1.6) we have

$$|\mu xp| \leq k|x|$$

Thus

$$\begin{aligned}\dot{w} &\geq \sum_{k=1}^{r-2}((-1)^k a_{2k}x^2_{(r+1)-k}) + (-1)^{r+2}b_{2r}x^2 - k|x| \\ &\geq -a_2x_r^2 + a_4x_{r-1}^2 - a_6x_{r-2}^2 + \cdots + a_{2r-6}\dddot{x}^2 + a_{2r-4}\ddot{x}^2 - (-1)^{r+2}\frac{k^2}{4b_{2r}}\end{aligned}$$

Using condition (3.2.10) if r is even or (3.2.11) if r is odd, we get

$$\dot{w} \geq D_1(x_r^2 + x_{r-1}^2 + x_{r-2}^2 + \cdots + \dddot{x}^2 + \ddot{x}^2) - (-1)^{r+2}\frac{k^2}{4b_{2r}} \tag{3.2.22}$$

where $D_1 = \min(-a_2, a_4, -a_6, \cdots, -a_{2r-6}, a_{2r-4})$. Integrating (3.2.22) from $t = 0$ to $t = \omega$ and using the fact that $w(t) = w(t+\omega)$ we get

$$0 \geq D_1\int_0^\omega (x_r^2 + x_{r-1}^2 + x_{r-2}^2 + \cdots + \dddot{x}^2 + \ddot{x}^2)dt - (-1)^{r+2}\frac{k^2\omega}{4}b_{2r} \tag{3.2.23}$$

Hence

$$\int_0^\omega \dddot{x}^2\, dt \leq \frac{(-1)^{r+2}k^2\omega}{4b_{2r}} \equiv D_2 \tag{3.2.24}$$

$$\int_0^\omega \ddot{x}^2 dt \leq D_3$$

where

$$D_3 = \max[D_2, \frac{D_2\omega^2}{4\pi^2}]$$

Since

$$\int_0^\omega \ddot{x}^2 dt \leq \frac{\omega^2}{4\pi^2}\int_0^\omega \dddot{x}^2\, dt$$

The periodicity condition

$$\dot{x}(0) = \dot{x}(\omega) \text{ on } x \text{ implies that } \dot{x}(T_1) = 0$$

at some $T_1 \in (0, \omega)$.

Thus

$$\ddot{x}(t) = \ddot{x}(T_1) + \int_{T_1}^t \dddot{x}(s)ds$$

[2] Ezeilo, J. O. C. Periodic Solutions of certain sixth order Differential equations. Journal of the Nigerian Mathematical Society, Vol. 1 1-9 (1982)

[3] Iyase, S. A. Periodic solutions of certain eight order differential equations. Journal of the Nigerian Mathematical Society

[4] Reissing, R. Sansone, G. and Conti R. Non linear Differential Equations of Higher Order, Noordhoff, Leyden (1974)

[5] Tejumola, H. O. Boundedness and periodicity of solutions of certain fourth order Differential equations, Ann. Mat Pura Appl (4) 80 (1968)

[6] Tejumola, H. O. Periodic solutions of certain fourth order Differential Equation. Atti Acad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat-Nat. LVII (57) Nov. (1974).