PERIODIC SOLUTIONS OF A CLASS OF EVEN ORDER DIFFERENTIAL EQUATIONS

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Abstract

We investigate sufficient conditions (Theorem 1) for the non-existence of periodic solutions of equation (2.1) with $P \equiv 0$ and sufficient conditions (Theorem 2) for existence of periodic solutions of equation (1.1.1).

Key words $\omega$-periodic solution

1 Introduction

1.1

The aim of this paper is to provide sufficient conditions for the existence of periodic solutions of the 2th order differential equation

$$x^{(2r)} + a_1 x^{(2r-1)} + a_2 x^{(2r-2)} + \cdots + a_{2r-1} x + f_{2r-1}(\xi) \xi + f_{2r-1}(x) \xi + f_p(x) = 0,$$

(1.1.1)

and the non-existence of non-trivial periodic solutions in the case $P \equiv 0$, where $a_1, a_2, \cdots, a_{2r-1}$ are constants and $f_{2r-1}, f_{2r-1}, f_p, p$ are continuous real-valued functions depending only on the arguments shown, the function $p$ is $\omega$-periodic in $t$, that is $p(t, x_1, \cdots, x_{2r}) \equiv p(t + \omega, x_1, \cdots, x_{2r})$ for some $\omega > 0$ and for arbitrary $x_1, \cdots, x_{2r}$. Such periodic differential equations arise

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From (1.2.4) we also derive that $\phi_1(\alpha) > 0$ if $r$ is even and $a_{2j-1}$, $j = 1, 2, \ldots, r$ satisfy
\[ a_1 \geq 0, a_3 \leq 0, a_5 \geq 0, \ldots, a_{2r-1} < 0 \] or
\[ \phi_1(\alpha) < 0 \text{ if } r \text{ is odd and } a_{2j-1}, j = 1, 2, \ldots, r-1 \text{ satisfy} \]
\[ a_1 \geq 0, a_3 \leq 0, a_5 \geq 0, \ldots, a_{2r-3} < 0 \] 
Thus it follows from (1.2.5) that if $r$ is even (resp. from (1.2.6) if $r$ is odd) that (1.2.1) does not have periodic solutions except $x = 0$.

Therefore from the general theory, the perturbed equation
\[ x^{(r)} + a_1 x^{(r-1)} + a_2 x^{(r-2)} + \cdots + a_{2r-2}\dot{x} + a_{2r-1}\dot{x} + a_{2r}x = p(t) \] in which $p$ is a continuous $\omega$-periodic function of $t$ has a unique $\omega$-periodic solution. Our main object here is to obtain non-linear analogues of (1.2.5) and (1.2.6) from which we shall derive our non-existence and existence results. Similar non-existenee and existence results using non-linear analogues of (1.2.7) and (1.2.8) are under preparation and will appear elsewhere.

2 Main results

2.1

We start here with the differential equation
\[ x^{(r)} + a_1 x^{(r-1)} + a_2 x^{(r-2)} + \cdots + a_{2r-3}\dot{x} + f_{2r-2}(x)\dot{x} + f_{2r}(x) = 0 \]
\[ (f_{2r}(0) = 0) \] (2.1.1)
where as before $a_1, a_2, a_3, \ldots, a_{2r-3}$ are constants and $f_{2r-2}, f_{2r-1}, f_{2r}$ are continuous real valued functions depending only on the arguments shown. Our non-existence result is as follows:

Theorem 1 Let the constants $a_{2j}$, $j = 1, 2, \ldots, r-2$ and the function $f_{2r-2}$ and $f_{2r}$ satisfy
\[ a_2 \leq 0, a_4 \geq 0, a_6 \leq 0, \ldots, a_{2r-4} \geq 0 \]
\[ y \int_{0}^{y} f_{2r-2}(s)ds \leq 0, y f_{2r}(y) > 0 \quad (y \neq 0) \] (2.1.2)
To prove theorem 1, it suffices to show that every $\omega$-periodic solution $(x_1(t), \ldots, x_\omega(t))$ of (2.2.1) satisfies $(x_1(t), x_2(t), \ldots, x_\omega(t)) = (0, 0, \ldots, 0)$ for all $t$.

To this end we define a function $V(y_1, \ldots, y_\omega)$ which will play a crucial role in our proof as follows:

$$V = (-1)^{r+1} y_1 \int_0^\infty F_0(s) ds + (-1)^{r+1} \int_0^\infty s F_0(s) ds$$

$$+ \sum_{k=2}^{2r} (-1)^k y_1(y_2 + \sum_{k=1}^{2r-k} a_k y_2 - k)$$

$$+ \sum_{k=2}^{2r} (-1)^k y_1(y_2 + \sum_{k=1}^{2r-k} a_k y_2 - k)$$

$$+ \sum_{k=1}^{2r} (-1)^{k+1} a_{2k-1} y_2^{2k-1}$$

and show that $\dot{V} = 0$ and from this we conclude that $(y_1(t), y_2(t), \ldots, y_\omega(t)) = (0, 0, \ldots, 0)$ for all $t$.

To prove theorem 2, we will consider the non-linear vector differential equation.

$$\dot{x} = A(t)x + G(t,x)$$

where $A(t)$ is an $n \times n$ matrix continuously for all $t \in R$. $A(t) = A(t+\omega)$, $G(t,x)$ is continuous for all $(t,x) \in R \times R^n$, $G(T,x) \equiv G(t+\omega,x)$ and the following theorem.

Theorem: If

a. the homogeneous linear equation

$$\dot{z} = A(t)x$$

has no non-trivial $\omega$-periodic solution and if

b. there exists an a-priori estimate independent of $\mu$ for the $\omega$-periodic solutions of the equation

$$\dot{z} = A(t)x + \mu G(t,x), 0 \leq \mu \leq 1,$$

then (2.2.3) admits at least one $\omega$-periodic solution. In what follows for any
\[ \int_{a}^{b} x^2 \, dx - \int_{a}^{b} (x-1)^2 \, dx - \int_{a}^{b} (x-2)^2 \, dx - \cdots - \int_{a}^{b} (x-n)^2 \, dx = \frac{n(n+1)(n+2)}{6} \]

\[ \left( \left[ x^3 \right]_{a}^{b} - \int_{a}^{b} 3x^2 \, dx \right) - \left( \left[ x^3 \right]_{a}^{b} - \int_{a}^{b} 3x^2 \, dx \right) - \cdots - \left( \left[ x^3 \right]_{a}^{b} - \int_{a}^{b} 3x^2 \, dx \right) = \frac{n(n+1)(n+2)}{6} \]

For \( n = 1 \), \( \int_{a}^{b} x^3 \, dx - \int_{a}^{b} 3x^2 \, dx = 0 \)

\[ \frac{1}{(x)^2f} > \frac{x}{(x)^2f} \]

\[ \frac{1}{(x)^2f} + \frac{x}{(x)^2f} = \frac{(x)^2f + x}{(x)^2f} \]

\[ \frac{(x)^2f + x}{(x)^2f} + \frac{(x)^2f + x}{(x)^2f} + \cdots + \frac{(x)^2f + x}{(x)^2f} = \frac{n(n+1)(n+2)}{6} \]

\[ \frac{1}{(x)^2f} + \cdots + \frac{1}{(x)^2f} = \frac{n(n+1)(n+2)}{6} \]

**Proof of Theorem 2**

\[ \frac{n(n+1)(n+2)}{6} = 0 \]

When \( n = 0 \), \( \int_{a}^{b} f(x) \, dx = 0 \)

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Therefore, if \( n = 0 \), \( f(x) \) is odd or even.

\[ \int_{a}^{b} f(x) \, dx = (i)A \]

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From (3.2.9)

\[ a_{2r-2} = a_{2r+1} = 0 \]  

similarly from (3.2.3) or (3.2.4) we get

\[ a_{2r} = b_{2r} > 0 \]

or

\[ a_{2r} = b_{2r} < 0 \]

depending on whether \( c \) is even or odd. Combining (2.2.10)-(3.2.14) we get that (3.2.8) has no non-trivial \( \omega \)-periodic solution. Hence assumption (a) of theorem is satisfied. Since \( \text{det}(e^{-\omega A} - I) \neq 0 \), \( I \) being the \( 2\omega \times 2\omega \) identity matrix all \( \omega \)-periodic solutions \( x(t) \) of (3.2.6) must satisfy the integral equation

\[ x(t) = \mu T(x(t)) = \mu (e^{-\omega A} - I)^{-1} \int_0^{2\omega} e^{-(s-t)A} G(s, x(s)) ds \]  

To verify hypothesis (b) of theorem we must show that

\[ \|x\| = \max \{ |x_1(t)| + |x_2(t)| + \cdots + |x_{2\omega-1}(t)| \} \leq D_0 \]  

It is sufficient in view of (3.2.7) and (3.2.15) to obtain the following estimates.

\[ \max_{t \in [0,\omega]} |x_1(t)| \leq D_1, \quad \max_{t \in [0,\omega]} |x_2(t)| \leq D_2, \quad \max_{t \in [0,\omega]} |x_3(t)| \leq D_3 \]

For every \( \omega \)-periodic solution \( y(t) \) of (3.2.1), we define a function \( w = w(y_1 \cdots y_{2\omega}) \) by

\[ w = (-1)^{r+1} y_1 \int_0^\omega f_{2\omega-2}(s) ds + (-1)^{r+1} \mu \int_0^\omega s f_{2\omega-1}(s) ds + \cdots \]

\[ + (-1)^{r+1} y_1 (y_3 + \sum_{k=1}^{2r-3} a_k y_{2r-1-k}) \]

\[ + \sum_{k=2}^r (-1)^k y_k \left[ f_{2r+1-k} + \sum_{j=1}^{2r+1-2k} a_j y_{2r+1-k-j} \right] + \frac{1}{2} \sum_{k=1}^{r-1} (-1)^k a_{2r-2k} y_{2r+1-k} \]  

(3.2.18)
and by hypothesis (2.1.6) we have

\[ |p(x) - k| \leq k|x| \]

Thus

\[ \vartheta \geq \sum_{l=1}^{r-2} ((-1)^{r-2} a_k x^{r-1-k}) + (-1)^{r+1} b_2 x^2 - k|x| \]
\[ \geq -a_k x^2 + a_k x^{r-2} - a_k x^{r-2} + \cdots + a_{r-2} x^{2} + a_{r-2} x^{2} - (-1)^{r+1} \frac{k^2}{4b_2} \]

Using condition (3.2.10) if \( r \) is even or (3.2.11) if \( r \) is odd, we get

\[ w \geq D_1 (x^2 + \cdots + x^{r-2} + x^2 + \cdots + \frac{x^2}{2} - (-1)^{r+1} \frac{k^2}{4b_2} \) \tag{3.2.22} \]

where \( D_1 = \min(-a_k, a_k, \cdots, a_{r-2} x^{r-2}) \). Integrating (3.2.22) from \( t = 0 \) to \( t = \omega \) and using the fact that \( w(t) = w(t + \omega) \) we get

\[ 0 \geq D_3 \int_0^\omega (x^2 + \cdots + x^{r-2} + x^2 + \cdots + \frac{x^2}{2}) dt - (-1)^{r+1} \frac{k^2 \omega}{4b_2} \] \tag{3.2.23} \]

Hence

\[ \int_0^\omega x^2 dt \leq \frac{(-1)^{r+1} k^2 \omega}{4b_2} \equiv D_3 \] \tag{3.2.24} \]

where

\[ D_3 = \max \{ D_2, \frac{D_2 \omega^2}{4e^2} \} \]

Since

\[ \int_0^\omega x^2 dt \leq \frac{\omega^2}{4e^2} \int_0^\omega x^2 dt \]

The periodicity condition

\[ \dot{x}(0) = \dot{x}(\omega) \text{ on } x \text{ implies that } \dot{x}(T_1) = 0 \]

at some \( T_1 \in (0, \omega) \).

Thus

\[ \dot{x}(t) = \dot{x}(T_1) + \int_{T_1}^t \dot{x}(s) ds \]