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Resonant Oscillation of Certain Fourth Order Nonlinear Differential Equations with Delay

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Abstract

We prove the existence of periodic solutions for equation (1.1) using degree theoretic methods. The uniqueness of periodic solutions is also examined.

Key words and phrases: periodic solutions, resonant oscillations, caratheodory conditions

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1. Introduction

This paper is devoted to the study of existence and uniqueness of periodic solutions to the fourth order differential equation with delay of the form

$$x^{iv}(t) + a\dddot{x} + b\ddot{x} + h(x)\dot{x} + g(t, x(t-\tau)) = p(t) \qquad (1.1)$$
$$x^{(i)}(0) = x^{(i)}(2\pi), i = 0,1,2,3.$$

where a,b are constants, $\tau \in [0,2\pi)$ is a fixed time delay.
$h : \Re \to \Re$ is continuous, $p \in L^1_{2\pi}$ and $g : [0,2\pi] \times \Re \to \Re$ is 2π periodic in t and satisfies certain caratheodory conditions. The unknown function $x : [0,2\pi] \to \Re$ is defined for $0 \le t \le \tau$ by $x(t-\tau) = x(2\pi - (t-\tau))$.

In a recent paper [2] we studied the above equation with h(x) = c, a constant, with g(t,y) satisfying certain non-resonant conditions. The method of proof used was based on coincidence degree theory [1]. In our present study, we will allow g(t, y) to satisfy certain resonant conditions and the technique of proof utilises the Leray-Schauder degree theory It is pertinent to note that fourth order boundary value problems with delay occur in a variety of physical problems (see [2], [3]).

In section 2 of this paper, we study the linear part of (1.1). Section 3 deals with the problem of existence of periodic solutions of (1.1) and in section 4 we obtain uniqueness

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results. We use the following notations and definitions. Let $\Re$ denote the real line and I the interval [0, 2π]. The following spaces will be used: $L^p{}_{2\pi} = L^p(I, \Re)$ are the usual Lebesgue spaces, $1 \le p < \infty$ with $x \in L^p{}_{2\pi}$, 2π - periodic

$$H^k{}_{2\pi} = \begin{cases} x : I \to \Re, x, \dot{x}, \ldots, x^{k-1} \text{ are absolutely continuous, } x^k \in L^2{}_{2\pi} \text{ and} \\ x^{(i)}(0) = x^{(i)}(2\pi), \ i = 0, 1, 2, 3, \ldots, k\text{-}1 \end{cases}$$

with norm $\|x\|^2{}_{H^k{}_{2\pi}} = \left(\frac{1}{2\pi}\int_0^{2\pi} x(t)dt\right)^2 + \frac{1}{2\pi}\sum_{i=1}^{k}\int_0^{2\pi} |x^{(i)}(t)|^2 dt$

and $$W^{k,2}{}_{2\pi} = \begin{cases} x : I \to \Re, x, \dot{x}, \ldots, x^{k-1} \text{ are absolutely continuous, } x^k \in L^2{}_{2\pi} \text{ and} \\ x^{(i)}(0) = x^{(i)}(2\pi), \ I = 0, 1, 2, \ldots, k\text{-}1 \end{cases}$$

with norm

$$\|x\|^2{}_{W^{k,L}{}_{2\pi}} = \frac{1}{2\pi}\sum_{i=0}^{k}\int_0^{2\pi} |x^{(i)}(t)|^2 dt.$$

A function $x \in W^{4,2}{}_{2\pi}$ is a solution of (1.1) if it satisfies (1.1) almost everywhere on $\Re$.

2. Some Results on the Linear Part

We shall consider here the linear delay differential equation of the form

$$x^{iv} + a\dddot{x} + b\ddot{x} + c\dot{x} + d(t)x(t-\tau) = p(t) \tag{2.1}$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \ i = 0,1,2,3.$$

The coefficient d is not necessarily a constant. We have the following results which apart from being of independent interest are also useful in the non-linear cases involving (1.1).

Theorem 2.1: Let b < 0 and let $\Gamma(t) = b^{-1}d(t) \in L^2{}_{2\pi}$. Suppose that

$0 < \Gamma(t) < 1$ a.e $t \in [0, 2\pi]$ The for arbitrary constant a and c the boundary value problem (2.1) admits in $W^{4,2}{}_{2\pi}$ only the trivial solution.

Proof: Let $x \in W^{4,2}{}_{2\pi}$ be any solution of (2.1) and let $x(t) = \bar{x} + \tilde{x}(t)$ where

$\bar{x} = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt$ and $\tilde{x}(t) = x(t) - \bar{x}$ so that $\frac{1}{2\pi}\int_0^{2\pi} \tilde{x}(t)dt = 0.$

We consider (2.1) in the form

$$b^{-1}\left[x^{iv} + a\dddot{x} + c\dot{x}\right] + \left[\ddot{x} + \Gamma(t)x(t-\tau)\right] = 0 \tag{2.2}$$

Then on multiplying (2.2) by $\bar{x} - \tilde{x}(t)$ and integrating over $[0,2\pi]$ and noting that

$$\frac{1}{2\pi}\int_0^{2\pi} (\bar{x} - \tilde{x}(t))(a\dddot{x} + c\dot{x})dt = 0 \tag{2.3}$$

$$\frac{1}{2\pi}\int_0^{2\pi} (\bar{x} - \tilde{x}(t))x^{iv}(t)dt = -\int_0^{2\pi} \ddot{\tilde{x}}^2(t)dt \tag{2.4}$$

We have on using

$$-ab = \frac{(a-b)^2}{2} - \frac{a^2}{2} - \frac{b^2}{2}$$

that

$$0 = \frac{1}{2\pi}\int_0^{2\pi} (\bar{x} - \tilde{x}(t))\left[b^{-1}(x^{iv} + a\dddot{x} + c\dot{x}) + (\ddot{x} + \Gamma(t)x(t-\tau))\right]dt$$

$$= -\frac{b^{-1}}{2\pi}\int_0^{2\pi} \ddot{\tilde{x}}^2 dt + \frac{1}{2\pi}\int_0^{2\pi} (\bar{x} - \tilde{x}(t))\{\ddot{x}(t) + \Gamma(t)x(t-\tau)\}dt$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} (\bar{x} - \tilde{x}(t))(\ddot{x}(t) + \Gamma(t)x(t-\tau))dt$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \frac{\Gamma(t)}{2}\left[(x(t-\tau) - \tilde{x}(t)) + 2\bar{x}^2\right]dt$$

From the periodicity of x it follows that

$$\int_0^{2\pi} \dot{\tilde{x}}^2(t)dt = \int_0^{2\pi} \dot{\tilde{x}}^2(t-\tau)dt$$

Therefore from the positivity of Γ and by Lemma1of [4] we have

$$0 \geq \frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi}\left[\dot{\tilde{x}}(t) - \Gamma(t)\tilde{x}^2(t)\right]dt\right) + \frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi}\left[\dot{\tilde{x}}^2(t-\tau) - \Gamma(t)\tilde{x}^2(t-\tau)\right]\right)dt$$

$$\geq \delta |\tilde{x}|^2_{H^1_{2\pi}}$$

for some $\delta > 0$. This implies $\tilde{x} = 0$ a.e and hence $x = \bar{x}$, however since $\Gamma(t) \neq 0$ we have $x = 0$.

Corollary 2.1: Let Γ be as in theorem 2.1. Then for every fixed $\tau \in [0,2\pi)$ and for every $u \in L^2{}_{2\pi}$ the problem

$$x^{iv} + a\dddot{x} + b\ddot{x} + c\dot{x} + d(t)x(t-\tau) = u(t) \tag{2.5}$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3$$

admits in $W^{4,2}{}_{2\pi}$ one and only one solution which depends continuously on u.

Proof: The operator

$$T : x \in W^{4,2}{}_{2\pi} \to x^{iv} \in L^2{}_{2\pi}$$

is Fredholm of index zero. The operator $F : x \in W^{4,2}{}_{2\pi} \to a\dddot{x} + b\ddot{x} + c\dot{x} + d(t)x(t-\tau) \in L^2{}_{2\pi}$ is completely continuous. Hence T + F is Fredholm of index zero. Since $\ker(T+F) = \{0\}$ we conclude that (2.5) has a unique solution. The continuous dependence of the solution on u follows from the Banach continuous inverse theorem.

Theorem 2.2: Let all the conditions of Theorem 2.1 hold and let δ be related to Γ by Theorem 2.1. Suppose further that $V \in L^2{}_{2\pi}$ satisfies $0 \le V(t) \le \Gamma(t) + \varepsilon$ a.e $t \in [0,2\pi]$ where $\varepsilon > 0$. Then

$$\tfrac{1}{2\pi}\int_0^{2\pi} (\bar{x} - \tilde{x}(t))\left[b^{-1}\left(x^{iv} + a\dddot{x} + c\dot{x}\right) + \ddot{x} + V(t)x(t-\tau)\right]dt \ge (\delta - \xi)\,|\tilde{x}\,|_{11}{}'_{2\pi} \tag{2.6}$$

Proof: Using (2.3) and (2.4) we have

$$\tfrac{1}{2\pi}\int_0^{2\pi} (\bar{x} - \tilde{x}(t))\left[b^{-1}\left(x^{iv} + a\dddot{x} + c\dot{x}\right) + \ddot{x} + V(t)x(t-\tau)\right]dt$$

$$= -\tfrac{b^{-1}}{2\pi}\int_0^{2\pi} \ddot{\tilde{x}}^2(t)dt + \tfrac{1}{2\pi}\int_0^{2\pi} (\bar{x} - \tilde{x}(t))\left[\ddot{x} + V(t)x(t-\tau)\right]dt$$

$$\ge \tfrac{1}{2\pi}\int_0^{2\pi} (\bar{x} - \tilde{x}(t))\left[\ddot{x}(t) + V(t)x(t-\tau)\right]dt$$

Proceeding as in the proof of theorem 2.1, we get

$$\geq \int_0^{2\pi} \left(\dot{\tilde{x}}^2(t) - V(t)\tilde{x}^2(t)\right)dt \geq \frac{1}{2\pi}\int_0^{2\pi} \left(\dot{\tilde{x}}^2(t) - \Gamma(t)\tilde{x}^2(t)\right)dt - \frac{\varepsilon}{2\pi}\int_0^{2\pi} \dot{\tilde{x}}^2(t)dt$$

$$\geq \delta |\tilde{x}|^2{}_{H^1{}_{2\pi}} - \varepsilon |\tilde{x}|^2{}_{H^1{}_{2\pi}} = (\delta - \varepsilon)|\tilde{x}|^2{}_{H^1{}_{2\pi}}$$

3. Main Result

Definition 3.1: Let $g : [0,2\pi] \times \Re \to \Re$ be such that g(. x) is measurable on [0, 2 π] for each $x \in \Re$ and g(t .) is continuous for a.e $t \in [0,2\pi]$. Assume moreover that for each r > 0, there exists $Y_r \in L^2{}_{2\pi}$ such that |g(t, x)| ≤ Y_r for a.e $t \in [0,2\pi]$ and all $x \in [-r,r]$. Then such a g is said to satisfy caratheodory's conditions.

We shall establish the existence of periodic solutions to the non-linear differential equation (1.1) when the non-linear term g(t, y) is a caratheodory function with respect to $L^2{}_{2\pi}$ and satisfies certain resonant conditions stated below.

Theorem 3.1: Let b < 0 and suppose g is a caratheodory function satisfying the inequalities

$$b\,x\,g(t, x) \geq 0 \quad (|x| \geq r) \tag{3.1}$$

$$\operatorname*{Lim\,Sup}_{|x| \to \infty} \frac{g(t,x)}{bx} \leq \Gamma(t) \tag{3.2}$$

uniformly a.e $t \in [0,2\pi]$ where r > 0 is a constant and $\Gamma \in L^2{}_{2\pi}$ is such that

$$0 < \Gamma(t) < 1 \tag{3.3}$$

Suppose further that

$$\overline{P} = \frac{1}{2\pi}\int_0^{2\pi} p(t)dt = 0$$

Then for arbitrary constant a and arbitrary continuous function h, the boundary value problem (1.1) has at least one 2π periodic solution for every fixed $\tau \in [0,2\pi)$.

Proof: Let $\delta > 0$ be related to Γ as in theorem (2.1). By hypothesis (3.1) and (3.2) there exists r > 0 such that

$$0 \leq \frac{g(t,x)}{bx} \leq \Gamma(t) + \frac{\delta}{2} \tag{3.4}$$

for a.e $t \in [0,2\pi]$ and $|x| \geq r$.

Define

$$\widetilde{Y}(t,x) = \begin{cases} (bx)^{-1} g(t,x), & |x| \geq r \\ (br)^{-1} g(t,r), & 0 < x < r \\ -(br)^{-1} g(t,-r), & -r < x < 0 \\ \Gamma(t), & x = 0 \end{cases}$$

We have $0 \leq \widetilde{Y}(t,x) \leq \Gamma(t) + \delta/2$ (3.5)

For a.e $t \in [0,2\pi]$ and $x \in \Re$. Moreover the function $\widetilde{Y}(t,y)$ satisfies caratheodory's conditions and $\widetilde{g} : [0,2\pi] \times \Re \to \Re$ defined by

$$\widetilde{g}(t,x(t-\tau)) = g(t,x(t-\tau)) - bx(t-\tau)\widetilde{Y}(t,x(t-\tau))$$

is such that for a.e $t \in [0,2\pi]$ and all $x \in \Re$, $|\widetilde{g}(t,x(t-\tau))| \leq \alpha(t)$ for a.e $t \in [0,2\pi]$, $x \in \Re$ and some $\alpha(t) \in L^2{}_{2\pi}$. Let $\lambda \in [0,1]$ and $x \in H^1{}_{2\pi}$ be such that

$$b^{-1}x^{iv} + b^{-1}a\dddot{x} + \ddot{x} + \lambda b^{-1}h(x)\dot{x} + (1-\lambda)\Gamma(t)x(t-\tau) + b^{-1}(1-\lambda)c\dot{x}$$
$$\lambda b^{-1} g\big(t,x(t-\tau)\big) - b^{-1}\lambda p(t) = 0 \tag{3.6}$$

For $\lambda = 0$, we obtain equation (2.1) which by theorem (2.1) admits only the trivial solution. For $\lambda = 1$, we get the original equation (1.1). To prove that equation (3.6) or equivalently (1.1) has at least one 2π periodic solution, it suffices according to the Leray-Schauder method to show that the possible solutions of the family of equations

$$b^{-1}\left[x^{iv} + a\dddot{x} + \lambda h(x)\dot{x}\right] + \ddot{x} + (1-\lambda)\Gamma(t)x(t-\tau) + \lambda\widetilde{Y}\big(t,x(t-\tau)\big)x(t-\tau)$$
$$\lambda b^{-1}\widetilde{g}\big(t,x(t-\tau)\big) + b^{-1}(1-\lambda)c\dot{x} - b^{-1}\lambda p(t) = 0 \tag{3.7}$$

are a priori bounded in $C^3[0,2\pi]$ independently of $\lambda \in [0,2\pi]$. Notice that by inequalities (3.5) one has

$$0 \leq (1-\lambda)\Gamma(t) + \lambda\widetilde{Y}(t,x(t-\tau)) \leq \Gamma(t) + \delta/2 \tag{3.8}$$

for a.e $t \in [0,2\pi]$ and all $x \in \Re$ and for $\lambda = 0$ equation (3.7) has only the trivial solution. Hence using theorem (2.2) with

$$V(t) = (1-\lambda)\Gamma(t) + \lambda\widetilde{Y}(t,x(t-\tau))$$

and Cauchy-Schwartz inequality we get

$$0 = \frac{1}{2\pi}\int_0^{2\pi} (\bar{x} - \widetilde{x}(t))\{b^{-1}\left[x^{iv} + a\dddot{x} + \lambda h(x)\dot{x}\right] + \ddot{x} + ((1-\lambda)\Gamma(t)x(t-\tau) + \lambda\widetilde{Y}(t,x(t-\tau))x(t-\tau)$$

$$\lambda b^{-1}\tilde{g}(t,x(t-\tau)) + b^{-1}(1-\lambda)c\dot{x} - b^{-1}\lambda p(t)\}$$

$$\geq \delta/2\,|\tilde{x}|^2_{H^1_{2\pi}} - |b|^{-1}(|\alpha|_2 + |p|_2)\,(|\bar{x}| + |\tilde{x}|_2)$$

$$\geq \delta/2\,|\tilde{x}|^2_{H^1_{2\pi}} - \beta\,(|\bar{x}| + |\tilde{x}|^2_{H^1_{2\pi}})$$

Thus

$$|\tilde{x}|^2_{H^1_{2\pi}} \leq \tfrac{2\beta}{\delta}\,(|\bar{x}| + |\tilde{x}|^2_{H^1_{2\pi}})$$

with $\beta > 0$ independently of x. Integrating (3.6) over $[0,2\pi]$ we obtain

$$(1-\lambda)\int_0^{2\pi}\Gamma(t)x(t-\tau)dt = -b^{-1}\lambda\int_0^{2\pi} g(t,(x(t-\tau))dt \qquad (3.9)$$

Since $\Gamma(t) > 0$ we derive that $\frac{1}{2\pi}\int_0^{2\pi}\Gamma(t)dt = \bar{\Gamma} > 0.$

Hence if $x(t) \geq r$ for all $t \in [0,2\pi]$, (3.1) and (3.9) imply that $(1-\lambda)\bar{\Gamma}r < 0$ contradicting $\bar{\Gamma} > 0.$

Similarly, if $x(t) \leq -r$ for all $t \in [0,2\pi]$ we reach a similar contradiction. Consequently, there exists a $t_1 \in [0,2\pi]$ such that $x(t_1) < r$. From this point, we use exactly the arguments in [4] to obtain

$$|x|_{H^1_{2\pi}} \leq C_1, \qquad C_1 > 0 \qquad (3.10)$$

Thus

$$|\dot{x}|_2 \leq C_2, \qquad C_2 > 0$$

Now,

$$x(t) = x(t_1) + \int_{t_1}^{t}\dot{x}(s)ds \qquad (3.11)$$

Hence $\quad |x|_\infty \leq C_3, \qquad C_3 > 0 \qquad (3.12)$

Multiplying (3.7) by $\dot{x}(t)$ and integrating over $[0,2\pi]$ we have

$$|\ddot{x}|^2_2 \leq |a^{-1}|\,|h(x)|_\infty|\dot{x}|^2_2 + |1 + \delta/2|\,|x|_\infty|\dot{x}|_2|a^{-1}| + |a^{-1}|\,|\alpha|_2|\dot{x}|_2 + |p|_2|\dot{x}|_2$$

Therefore there exists $C_4 > 0$ such that

$$|\ddot{x}|_2 \leq C_4, \qquad C_4 > 0 \qquad (3.13)$$

Hence $\quad |\dot{x}|_\infty \leq C_5, \qquad C_5 > 0 \qquad (3.14)$

Multiplying (3.7) by $-\dddot{x}(t)$ and integrating over $[0,2\pi]$ we get

$$|\dddot{x}|^2_2 \leq C_6, \qquad C_6 > 0 \qquad (3.15)$$

Hence $\quad |\ddot{x}|_\infty \leq C_7, \qquad C_7 > 0 \qquad (3.16)$

Finally we multiply (3.7) by $x^{iv}(t)$ and integrating over $[0,2\pi]$ using (3.13), (3.15) to get

$$|x^{iv}|_2 \le C_8, \qquad C_8 > 0 \tag{3.17}$$

$$|\ddot{x}|_\infty \le C_9, \qquad C_9 > 0 \tag{3.18}$$

From (3.12), (3.14), (3.16) and (3.18), we conclude that the set of solutions of (3.7) are a priori bounded in $C^3[0,2\pi]$ by a constant independent of solutions and $\lambda \in [0,2\pi]$.

4. Uniqueness of Solution

In the special case of (1.1) in which h(x) = C, C a constant, the following uniqueness result holds.

Theorem 4.1: Let a, b, c be constants with b < 0 and suppose that g is a caratheodory function satisfying

$$0 < \frac{g(,x_1) - g(t,x_2)}{b(x_1 - x_2)} \le \Gamma(t) \tag{4.1}$$

for a.e $t \in [0,2\pi]$ and all x_1, $x_2 \in \Re$, $x_1 \neq x_2$ where $\Gamma \in L^2{}_{2\pi}$ is such that

$$0 < \Gamma(t) < 1$$

Then for arbitrary constants a and c and every fixed $\tau \in [0,2\pi)$ the bvp

$$x^{iv} + a\dddot{x} + b\ddot{x} + c\dot{x} + g(t, x(t-\tau)) = p(t) \tag{4.2}$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3$$

has at most one solution.

Proof: Let $u = x_1 - x_2$ for any two solutions x_1, x_2 of (4.2). Then u satisfies the bvp

$$b^{-1}\left[u^{iv} + a\dddot{u} + c\dot{u}\right] + \ddot{u} + \beta(t)u(t-\tau) = 0$$

$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1, 2, 3$$

where $\beta(t) \in L^2{}_{2\pi}$ is defined by

$$\beta(t) = \frac{g(t, x_2(t-\tau)) - g(t, x_1(t-\tau))}{b(x_1 - x_2)}$$

Since $0 < \beta(t) \le \Gamma(t)$ for a.e $t \in [0,2\pi]$ we use the arguments of theorem 2.1 to show that $x_1 = x_2$ a.e

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