

Dynamic Analysis of Damped Driven Pendulum using Laplace Transform Method

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ABSTRACT

In many recent works, many researchers have demonstrated the usefulness of dynamical systems. In this paper, a damped driven pendulum, as a dynamical system, is considered. The effects of its angular displacement and angular driven force on the dynamics of the pendulum is analyzed. The Laplace transform method is used to transform the differential equation governing the motion of the pendulum into its algebraic form and the desired results obtained. It is observed that angular displacement and angular driven force affect the motion of the pendulum. Specifically it is noted that the lower the fixed value of the angular driving force the higher the angular velocity, at various values of the angular displacement.

Keywords: Dynamic analysis, damping, damped driven pendulum, angular velocity, angular displacement, angular driven force

Mathematics Subject Classification: 3JN15

1. INTRODUCTION

When a pendulum is acted on, both by a velocity dependent damping force, and a periodic driving force, it can display both ordered and chaotic behaviours, for certain ranges of parameters [3,6,12]. The pendulum is a dynamical system. Our investigation of pendulum dynamics begins with Newton's second law of motion; which states that the relationship between an object's mass m , its acceleration a , and the applied force F , is $F = ma$. If we hold damping and driving force in reserved; and the small angle approximation of a simplified, idealized pendulum is given by the following equation [3,8]:

$$I \frac{d^2\theta}{dt^2} + mgl \sin\theta = 0$$

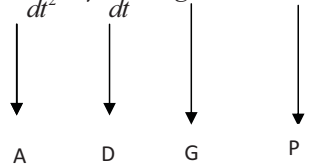
If we add a dissipative force proportional to the velocity, the governing equation of motion will now include the damping term, $\gamma \frac{d\theta}{dt}$, where γ is the dissipative coefficient. If we also drive the pendulum,

feeding in energy to resupply the energy dissipated, the governing equation will also include the driving force term, making the new situation a very interesting one [3,10]. Solving this governing equation and estimating the solution at different values of the parameters, using graphics give one an insight to the dynamics of the system.

The objective of this paper is to analyse the dynamic effects of the angular displacement and angular driving force on the damped pendulum.

2. THE GOVERNING EQUATION AND ITS DIMENSIONLESS FORM

The equation of motion for damped, driven pendulum of mass m and length l can be written as: [1,2]:

$$ml^2 \frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + mgl \sin \theta = C \cos(\omega_d t) \quad (1)$$


where A is acceleration, D is the damping, G is the gravitation and P is the driving force. ω_d is the angular driving force, θ is the angular displacement, t is the time, l is the length, m is the mass, γ is the dissipation coefficient, C is the amplitude of the driving force and g is the acceleration due to gravity.

The dimensionless form of the equation under study, which describes the damped driven pendulum of length l , and mass m , is [1, 2, 3]

$$\frac{d^2\theta}{dt^2} + \frac{1}{q} \frac{d\theta}{dt} + \sin \theta = a \cos(\omega_d t) \quad (2)$$

where the three terms on the left hand side represent acceleration, damping and gravitation respectively and the term on the right hand side is a sinusoidal driving torque, which is made up of an amplitude g and a frequency ω_d , q is the damping parameter [3]

Equation 2 can be written as

$$\ddot{\theta} + \frac{1}{q} \dot{\theta} + \sin \theta = a \cos(\omega_d t) \quad (3)$$

which can also be written as

$$\dot{w} + \frac{1}{q} w + \sin \theta = a \cos(\omega_d t) \quad (4)$$

where $w = \frac{d\theta}{dt}$ and $\omega_D = \frac{d\phi}{dt}$

$\phi = \omega_D t$ is the phase of the driving force term.

The three dimensions for this system represented by equation (1), become W , θ and ϕ . In order to simplify the results of the system, θ has been restricted to reside within $-\pi$ and π , while ϕ has been restricted to reside within 0 and 2π . Eq. (4) can now be written as

$$\dot{w} = \cos\phi - \sin\theta - \frac{1}{q} w \quad (5)$$

where

$$\dot{\theta} = w \quad (6)$$

$$\dot{\phi} = \omega_D \quad (7)$$

Depending on the values of the damping parameter and the forcing amplitude, the system can exhibit different dynamics.

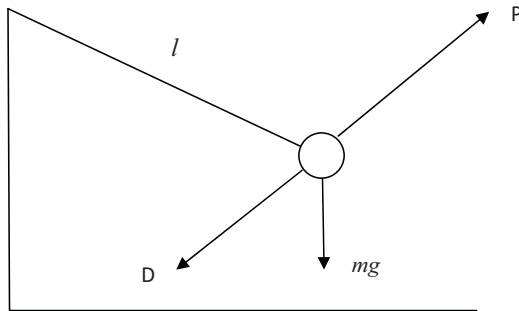


Fig. 1: The pendulum

2.1. Laplace transform method

The Laplace transform method was applied to solve the initial-value problem following three steps:

- (i) Taking the Laplace transforms of both side of the equation;
- (ii) Simplifying algebraically the result obtained.
- (iii) Finding the inverse transform in order to obtain the unknown function $y(t)$.

This inverse transform, $y(t)$ is the desired solution of the given differential equation.

3. ANALYSIS

From equation (5),

$$\dot{w} = \cos\phi - \sin\theta - \frac{1}{q}w$$

i.e. $\alpha = \cos\phi - \sin\theta - \frac{1}{q}\omega$, where α is the angular acceleration. (9)

We consider the value of W for different values of θ and ϕ at fixed parameters q and g . For this paper, we consider some values of θ and ϕ , then use Laplace transform method to find W in each case. This can be seen in table 1. Also the parameters are kept constant; taking $q = 2$, $g = 1$.

3.1 Initial Conditions:

Our initial value problem is formed by subjecting the differential equation in (9) to the following initial conditions:

$$w(0) = 0, \dot{w}(0) = 0. \tag{10}$$

The solution of the initial-value problem is obtained, using Laplace transform method, as

$$w = 2\left(e^{-\frac{1}{2}t} - 1\right) \tag{12}$$

when $\theta = 90^\circ$ and $\phi = 90^\circ$.

Different values of the angular velocity ω are obtained for different values of θ and ϕ where $0 \leq \phi \leq 2\pi$ and $-\pi \leq \theta \leq \pi$. In each case the initial conditions are satisfied.

4. Effectsof α , ω , θ , q , g on the pendulum motion

In order to compare the effects of angular displacement and angular driving force on the pendulum motion, the following cases are considered in sections (4.1), (4.2), (4.3) and (4.4):

4.1. Effects of Angular Driving force and angular displacement on the undamped pendulum :

From Eq. (1)

$$m l^2 \frac{d^2 \theta}{dt^2} + \gamma \frac{d\theta}{dt} + m g l \sin \theta = C \cos(\omega_b t)$$

where

$$\gamma \frac{d\theta}{dt} \text{ is the damping term.}$$

The equation, without damping, becomes

$$m l^2 \frac{d^2 \theta}{dt^2} + m g l \sin \theta = C \cos(\omega_b t) \tag{13}$$

The dimensionless form of the equation, without damping becomes

$$\frac{d^2 \theta}{dt^2} + \sin \theta = a \cos(\omega_b t) \tag{14}$$

which can be written as

$$\ddot{\theta} + \sin \theta = a \cos(\omega_b t) \tag{15}$$

that is,

$$\dot{w} - a \cos \phi + \sin \theta = 0 \tag{16}$$

or

$$\dot{w} = a \cos \phi - \sin \theta \tag{17}$$

i.e. $\alpha = a \cos \phi - \sin \theta = \cos \phi - \sin \theta$ (for $g = 1$)

4.2 Effect of Angular Displacement on the damped pendulum without the angular driving force

Writing Eq. (1) without the angular driving force, we have

$$m l^2 \frac{d^2 \theta}{dt^2} + \gamma \frac{d\theta}{dt} + m g l \sin \theta = 0 \tag{18}$$

The non-dimensional form becomes

$$\frac{d^2 \theta}{dt^2} + \frac{1}{q} \frac{d\theta}{dt} + \sin \theta = 0 \tag{19}$$

$$\ddot{\theta} + \frac{1}{q} \dot{\theta} + \sin \theta = 0 \quad (20)$$

$$\dot{w} + \frac{1}{q} w + \sin \theta = 0 \quad (21)$$

$$\dot{w} = -\left(\frac{1}{q} w + \sin \theta\right) \Rightarrow \alpha = -\left(\frac{1}{q} w + \sin \theta\right) \quad (22)$$

$$\Rightarrow \alpha = -\left[\frac{1}{q} w + \sin \theta\right] \quad (23)$$

4.3 Effect of Angular Displacement on the Undamped Pendulum without the angular Driving force

Writing Eq. (1) without the damping term and angular driving force, we have

$$m l^2 \frac{d^2 \theta}{dt^2} + m g l \sin \theta = 0 \quad (24)$$

The non-dimensional form is

$$\frac{d^2 \theta}{dt^2} + \sin \theta = 0 \quad (25)$$

$$\ddot{\theta} + \sin \theta = 0 \quad (26)$$

$$\dot{w} = -\sin \theta \quad (27)$$

$$\alpha = -\sin \theta \quad (28)$$

$$\alpha = \dot{w} = \frac{dw}{dt} = \frac{dw}{d\theta} \cdot \frac{d\theta}{dt} = \frac{dw}{d\theta} \cdot w \quad (29)$$

$$\alpha = w \frac{dw}{d\theta} = -\sin \theta \quad (30)$$

$$\Rightarrow \int w dw = \int -\sin \theta d\theta \quad (31)$$

$$\frac{w^2}{2} = \cos \theta, -\pi \leq \theta \leq \pi \quad (32)$$

$$\therefore w = \sqrt{2\cos\theta}, \quad -\pi \leq \theta \leq \pi \quad (33)$$

5. NUMERICAL RESULTS AND DISCUSSION

The numerical calculations have been carried out for a damped driven pendulum. The following values were used: $p=1$, $q=2$, $-\pi \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. In table 1 are shown the values, in terms of time t , of the angular velocity for the damped driven pendulum at different values of angular displacement and angular driven force phase. It is observed that the angular velocity has the same magnitude but opposite direction, for $\theta=30, \phi=30$ and $\theta=60, \phi=60$.

Figures 2(a)-2(g) are the angular velocity at various times, for particular values of angular displacement and angular driving force. Figure 3, on the other hand shows the angular velocity at various values of angular displacement, angular driving force and time. We can see clearly that the curves are asymptotic to particular values of the angular velocity, which implies at some point in time the rate of change of angular velocity becomes zero.

For various values of the angular displacement and fixed value of angular driving force, the angular velocity for the nine special cases considered, were calculated and are plotted in Figure 4, at a particular time. Also for various values of the angular driving force and fixed value of angular displacement, the angular velocity for the eight special cases considered, were calculated and were plotted in Figure 5, at a particular time. It can be seen that the minimum angular velocity occurred at $\phi = \pi$ for different values of θ .

Figure 6 shows the angular velocity of undamped, without any driving force, at different values of the angular displacement. Clearly the angular velocity is highest when the angular displacement is 0 degree, and lowest when it is -90 and 90 degrees.

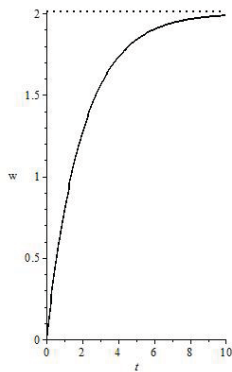
It can be noted from figure 5 that the lower the fixed value of the angular driving force the higher the angular velocity, at various values of the angular displacement. Also the smaller the fixed value of the angular displacement the smaller the angular velocity, at various values of angular driving force, at a particular time. This implies there is a negative correlation between the angular velocity and the angular driving force. Similarly, there is a positive correlation between the angular velocity and the angular displacement. Furthermore, the angular velocity is highest when the angular displacement is 0 degree.

Intuitively, we can imagine that the angular velocity occurs when the oscillating system (pendulum) has reached the equilibrium position (angular displacement equals zero) and is about to overshoot.

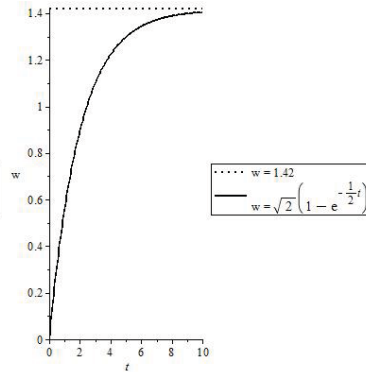
Table 1. The values of w for different values of θ and ϕ

θ	ϕ	w
0°	0°	$2(1 - e^{-\frac{1}{2}t})$
30°	30°	$(\sqrt{3} - 1)(1 - e^{-\frac{1}{2}t})$
45°	0°	$\sqrt{2}(1 - e^{-\frac{1}{2}t})$
45°	45°	0
30°	60°	0
60°	30°	0
60°	60°	$(1 - \sqrt{3})(1 - e^{-\frac{1}{2}t})$
0°	90°	0
90°	0°	0
90°	90°	$-2(1 - e^{-\frac{1}{2}t})$
180°	360°	$-2(1 - e^{-\frac{1}{2}t})$
-180°	180°	$-2(1 - e^{-\frac{1}{2}t})$
0°	180°	$2(1 - e^{-\frac{1}{2}t})$
-30°	30°	$-(\sqrt{3} + 1)(1 - e^{-\frac{1}{2}t})$
60°	270°	$-3(1 - e^{-\frac{1}{2}t})$

(a)



(b)



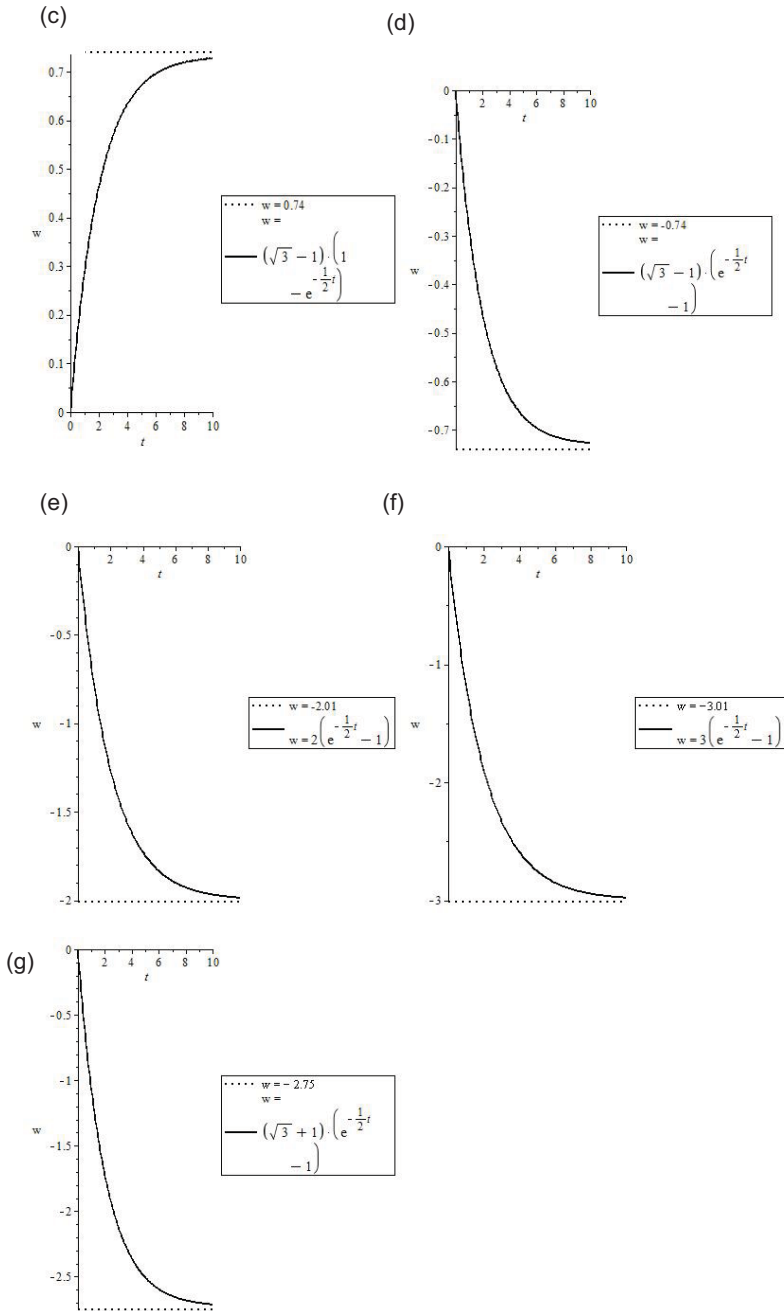


Figure2 (a) – (g): Graphs of angularvelocity w against time t .

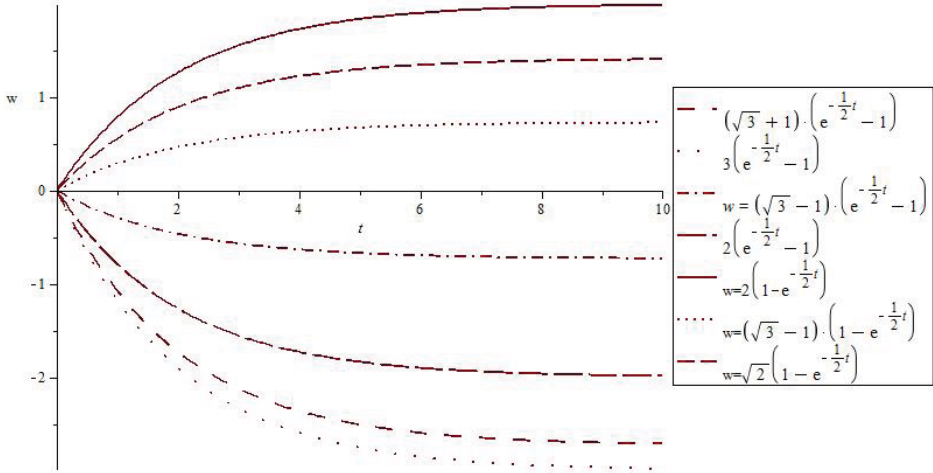


Figure 3. The angular velocity of the pendulum at different values of θ , ϕ and time t .

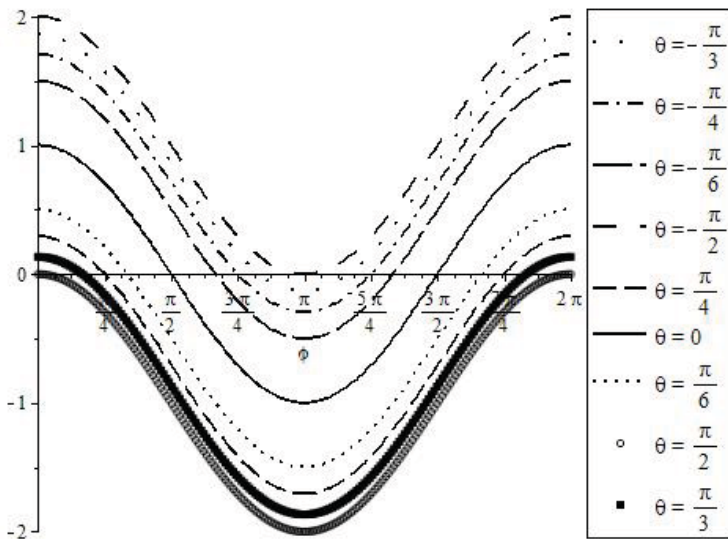


Figure 4. The angular velocity of the damped driven pendulum for fixed angular displacement and different values of angular driving force at a particular time

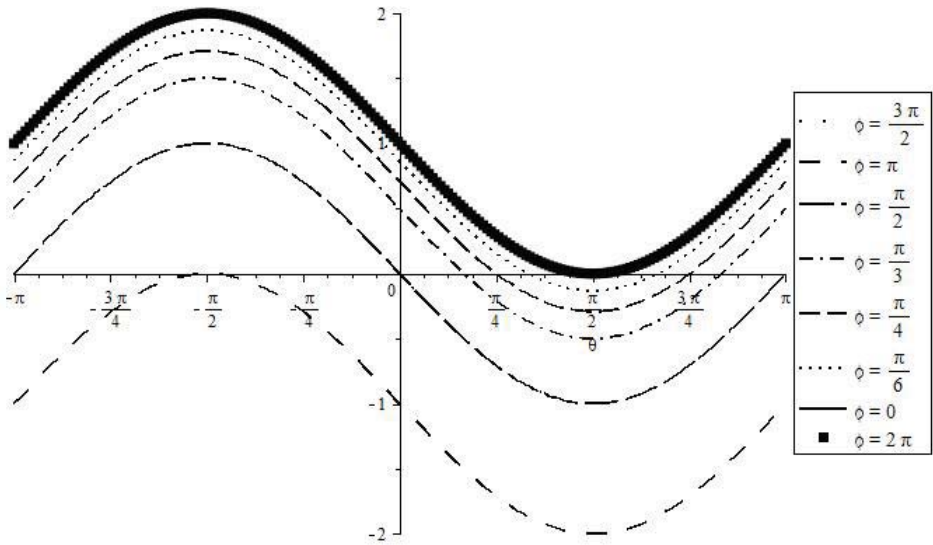


Figure 5. The angular velocity of the damped driven pendulum for fixed angular driving force and different values of angular displacement at a particular time

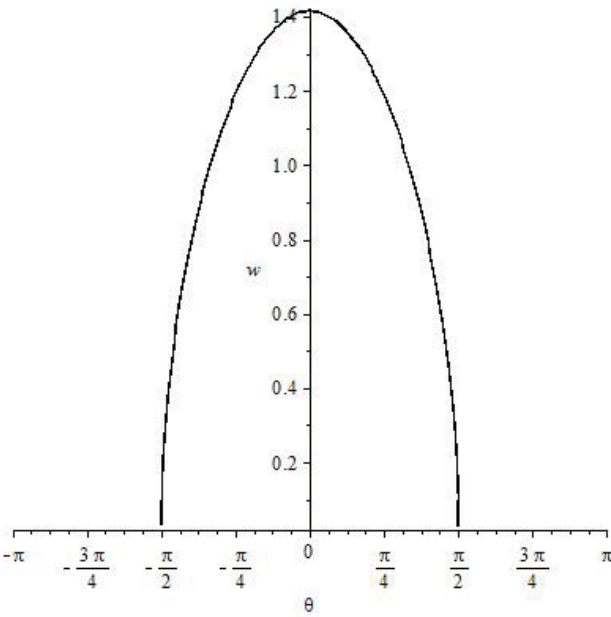


Figure 6. The angular velocity of undamped pendulum at different values of angular displacement

6. CONCLUSION

This paper has outlined a method to analyse the dynamics of damped driven pendulum. Also to compare the dynamics of the cases of undamped, driven and damped but not driven pendulums. It is shown that angular velocity and angular force phase are negatively correlated, while there is a positive correlation between the angular velocity and the angular displacement. In the case of when angular displacement is zero, the angular velocity assumed its highest value. It is also observed that the angular acceleration of the dynamic system becomes zero at some point in time. Concerning the angular velocity; it is noticed that the value of the angular velocity is minimum when the driving force is π radian. The study has contributed to scientific knowledge by showing the effects of damping and driving force on the dynamics of pendulum as a dynamical system. Also the effect of angle of displacement on the angular velocity, in addition to relationships of the parameters of a dynamical system have been outlined. This study has therefore shown that this dynamical system phenomenon is of great practical importance.

REFERENCES

- [1] Baker, G.L., Gollins, J.P., 1996, Chaotic Dynamics : An introduction, Cambridge University Press, NY.
- [2] Davies, B., 1999, Exploring Chaos: Theory and Experiment, Persues Books, MA.
- [3] Davidson, G.D., 2011, The damped driven pendulum: Bifurcation analysis of experimental data, The Division of Mathematics and Natural Sciences, Reed College.
- [4] Tsneq, Z.S., 2008, The Laplace Transform.
- [5] Russell, D.A., 2014, Acoustics and Vibration Animations, Graduate Program in Acoustics, The Pennsylvania State University.
- [6] Richard Fitzpatrick, 2006, The simple pendulum, farside.ph.utexas.edu/teaching/301/Lectures/node140.html
- [7] David Darling, Classical Mechanics, The Worlds of David Darling, Encyclopedia of Alternate Energy www.daviddarling.info/Encyclopedia of Science
- [8] Walter Lewin, 1999, Pendulum Equation, MIT OpenCourseWare, Massachusetts Institute of Technology
- [9] Peter Dourmashkin, J. David Litster, David Prtichad and Bernard Surrow, 2004, Pendulums and Collisions, MIT OpenCourseWare, Massachusetts Institute of Technology
- [10] Gray D. Davidso, 2011, The Damped Driven Pendulum : Bifurcation Analysis of Experimental Data. A Thesis for the Division of Mathematics and Natural Sciences, Reed College.
- [11] John H. Hubbard, The forced Damped Pendulum: Chaos, Complication and Control. <http://www.math.cornell.edu/~hubbard/pendulum.pdf>.
- [12] Dehui Qui, Min Rao, Qirun Huo, Jie Yang, 2013, Dynamical Modeling and Sliding Mode Control of Inverted Pendulum Systems, *International Journal of Applied Mathematics and Statistics*, **45(15)**, 325-333.