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Solutions of the Klein-Gordon Equation with Equal Scalar and Vector Harmonic Oscillator plus Inverse Quadratic Potential

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Abstract. The solutions of the Klein-Gordon equation with equal scalar and vector harmonic oscillator plus inverse quadratic potential for *S*-waves have been presented using the Nikiforov-Uvarov method. The bound state energy eigenvalues and the corresponding un-normalized eigenfunctions are obtained in terms of the Laguerre polynomials.

Keywords: Klein-Gordon, harmonic oscillator potential, inverse quadratic potential, Nikiforov-Uvarov method, energy-eigenvalues, eigenfunctions
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INTRODUCTION

The bound state solutions of the Klein-Gordon (KG) equation are only possible for some potentials of physical interest [1-5]. These solutions could be exact or approximate and they normally contain all the necessary information for the quantum system. Quite recently, several authors have tried to solve the problem of obtaining exact or approximate solutions of the KG equation for a number of special potentials using different methods [6–20]. Some of these potentials are known to play very important roles in many fields of Physics such as Molecular Physics, Solid State and Chemical Physics [21]. When a particle is in a strong potential field, the relativistic effects must be considered, leading to the relativistic quantum mechanical description of such a particle [22–26]. In the relativistic limit, the particle's motions are very often described using either the KG equation or the Dirac equation depending on the spin character of the particle [23–24]. The spin-zero particles like the mesons are satisfactorily described by the KG equation while the spin-half particles such as the electrons are described by the Dirac equation. It is therefore of interest in nuclear and high energy physics to obtain exact solutions of the KG and Dirac equations. The solution of the Klein-Gordon equation under different potentials model plays an important role in physics and chemistry since their solutions contain all necessary information governing the quantum mechanical system under consideration. Among the most successful methods that have been used to solve the Schrödinger, Dirac and the Klein-Gordon equation are the NU [27], supersymmetric quantum mechanics methods[28], asymptotic interation method [29] and others [30-39]. The purpose of the present work is to present the solutions of the Klein-Gordon equation with the harmonic oscillator plus inverse quadratic (HO+IQ) potential of the form [27-28]:

$$V(r) = kr^2 + \frac{g}{r^2}, \tag{1}$$

where r represents spherical coordinate, k is arbitrary constant and $g$is the inverse quadratic potential strength. Dong and Lozada-Cassou [27] have used algebraic method to solve the Schrodinger equation in three dimensions with the potential in equation (1) and obtained eigen functions and eigen values of the Schrodinger equation. Also, Ikhdair and Sever [28] solved the D-dimensional radial Schrodinger equation with some molecular potentials and obtained the solution for (HO +IQ) potential as a special case of pseudoharmonic oscillator for $l > 0$ waves. However, not much has been achieved in the area of solving the Klein - Gordon equation for S-waves (*i.e.*, for $l = 0$) with (HO + IQ) potential using Nikiforov-Uvarov method in the literature.

Overview of the Nikiforov-Uvarov Method

The Nikiforov-Uvarov (NU) method is based on the solutions of a generalized second-order linear differential equation with special orthogonal functions [29]. The Schrodinger equation of the type as:

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$$\psi''(r)+[E-V(r)]\psi(r)=0 \tag{2}$$

can be solved by this method. This can be done by transforming equation (2) into an equation of hypergeometric type with appropriate coordinate transformation $z=z(r)$ to get

$$\psi''(z)+\frac{\bar{\tau}(z)}{\sigma(z)}\psi'(z)+\frac{\bar{\sigma}(z)}{\sigma^2(z)}\psi(z)=0. \tag{3}$$

To find the exact solution to equation (3), we write $\psi(z)$ as

$$\psi(z)=\phi(z)\chi(z). \tag{4}$$

Substitution of equation (4) into equation (3) yields equation (5) of hypergeometric type as

$$\sigma(z)\chi''(z)+\tau(z)\chi'(z)+\lambda\chi(z)=0. \tag{5}$$

In equation (4), the wave function $\phi(z)$ is defined as the logarithmic derivative [29]

$$\frac{\phi'(z)}{\phi(z)}=\frac{\pi(z)}{\sigma(z)} \tag{6}$$

with$\pi(z)$ being at most first order polynomials. Also, the hypergeometric-type functions in equation (5) for a fixed integer n is given by the Rodrigue relation as

$$\chi_n(z)=\frac{B_n}{\rho_n}\frac{d^n}{dz^n}[\sigma^n(z)\rho(z)] \tag{7}$$

where B_n is the normalization constant and the weight function $\rho(z)$ must satisfy the condition

$$\frac{d}{ds}[\sigma^n(z)\rho(z)]=\tau(z)\rho(z) \tag{8}$$

with

$$\tau(z)=\bar{\tau}(z)+2\pi(z). \tag{9}$$

In order to accomplish the condition imposed on the weight function $\rho(z)$ it is necessary that the polynomial $\tau(z)$ be equal to zero at some point of an interval (a,b) and its derivative at this interval at $\sigma(z)>0$ will be negative [30].

The function $\pi(z)$ and the parameter λ required for the NU method are then defined as [30]

$$\pi(z)=\frac{\sigma'-\bar{\tau}}{2}\pm\sqrt{\left(\frac{\sigma'-\bar{\tau}}{2}\right)^2-\bar{\sigma}+k\sigma}, \tag{11}$$

$$\lambda=k+\pi'(z). \tag{12}$$

The z-values in equation (11) are possible to evaluate if the expression under the square-root be square of polynomials. This is possible if and only if its discriminant is zero. Therefore, the new eigenvalue equation becomes [31-32]

$$\lambda=\lambda_n=-n\frac{d\tau}{dz}-\frac{n(n-1)}{2}\frac{d^2\sigma}{dz^2},\ \ n=0,1,2,\ldots \tag{13}$$

A comparison between equations (12) and (13) yields the energy eigenvalues.

Solutions of the Klein-Gordon Equation

The Klein-Gordon (KG) equation for equal scalar and vector harmonic oscillator plus inverse quadratic (HO+IQ) potential could be written as

$$\frac{d^2U(z)}{dz^2}+\frac{1}{2z}\frac{dU(z)}{dz}+\frac{1}{4z^2}(Cz-Dz^2-F)U(z) \quad =0, \tag{14}$$

where the wave function is $U(z)$ and

$$C=E^2-M^2, D=2(E+M)K, F=2(E+M)g. \tag{15}$$

We can also rewrite equation (14) as

$$\frac{d^2U(z)}{dz^2}+\frac{1}{2z}\frac{dU(z)}{dz}+\frac{1}{4z^2}(-Dz^2+Cz-F)U(z) \quad =0. \tag{16}$$

Equation (16) is then compared with equation (3) and the following expressions are obtained

$$\tilde{\tau}=1, \sigma=2z, \tilde{\sigma}=-Dz^2+Cz-F, \sigma'=2. \tag{17}$$

We then obtain the function π by substituting equation (17) into equation (11):

$$\pi=\frac{1}{2}\pm\sqrt{Dz^2+(k-C)z+(F+\frac{1}{4})}. \tag{18}$$

According to the NU method, the quadratic form under the square-root sign of equation (18) must be solved by setting the discriminant of this quadratic equation equal to zero, *i.e.*, $\delta=b^2-4ac=0$. This discriminant gives a new equation which can be solved for the constant k to get the two roots as

$$k_{\pm}=C\pm\frac{1}{2}\sqrt{D(4F+1)}. \tag{19}$$

Thus we have

$$k_{-}=C-\frac{1}{2}\sqrt{D(4F+1)}, \tag{20}$$

$$k_{+}=C+\frac{1}{2}\sqrt{D(4F+1)}. \tag{21}$$

When the two values of k given in equations (20) and (21) are substituted into equation (18), the four possible forms of $\pi(z)$ are obtained as

$$\pi(z)=\begin{cases}\frac{1}{2}\pm\sqrt{D}\left(z+\sqrt{\frac{4F+1}{4D}}\right) for\ k_{-}=C-\frac{1}{2}\sqrt{D(4F+1)} \\ \frac{1}{2}\pm\sqrt{D}\left(z-\sqrt{\frac{4F+1}{4D}}\right) for\ k_{+}=C+\frac{1}{2}\sqrt{D(4F+1)}\end{cases}. \tag{22}$$

One of the four values of the polynomial $\pi(z)$ is just proper to obtain the bound state solution since τ given in equation (1) must have negative derivative. Therefore, the most suitable expression of $\pi(z)$ is chosen as

$$\pi(z)=\frac{1}{2}-\sqrt{D}\left(z+\sqrt{\frac{4F+1}{4D}}\right). \tag{23}$$

For $k_- = C - \frac{1}{2}\sqrt{D(4F+1)}$. We obtain $\tau(z) = 2 - 2\sqrt{D}\left(z + \sqrt{\frac{4F+1}{4D}}\right)$ from equation (9) and the derivative of this expression would be negative, *i.e.*, $\tau'(z) = -2\sqrt{D} < 0$. From equations (12) and (13) we obtain

$$\lambda = C - \sqrt{D}\left(1 + \frac{1}{2}\sqrt{4F+1}\right), \quad \lambda_n = 2n\sqrt{D}. \tag{24}$$

When we compare these expressions, $\lambda = \lambda_n$, we obtain the energy of the harmonic oscilltor plus inverse quadratic potential for *S*-waves as

$$E^2 - M^2 = \sqrt{2(E+M)K\left(2n + 1 + \sqrt{2(E+M)g + \frac{1}{4}}\right)}. \tag{25}$$

If we choose $g = 0$ in equation (1), we obtain the harmonic oscillator potential and its energy for the KG equation becomes

$$E^2 - M^2 = \sqrt{2(E+M)K\left(2n + \frac{3}{2}\right)}. \tag{26}$$

Let us now calculate the radial wave function, $U(z)$. Using σ and π equations (6) and (8), the following expressions are obtained

$$\phi(z) = z^{\alpha_1/4} e^{-\frac{\sqrt{D}}{2}z}, \quad \alpha_1 = 1 - 2\sqrt{D}\left(\sqrt{\frac{4F+1}{4D}}\right), \quad \rho(z) = z^{\alpha_2/2} e^{-\sqrt{D}z}, \quad \alpha_2 = -2\sqrt{D}\left(\sqrt{\frac{4F+1}{4D}}\right). \tag{27}$$

Then from equation (7) one has

$$\chi_n(z) = B_n 2^n z^{-\alpha_2/2} e^{\sqrt{D}z} \frac{d^n}{dz^n}\left(e^{-\sqrt{D}z} z^{n+\alpha_2/2}\right) \tag{28}$$

$B_n$is a normalization constant. The wave function $U(z)$ can be obtained in terms of the generalized Laguerre polynomials as

$$U(z) = N_n z^{\alpha_1/4} e^{-\frac{\sqrt{D}z}{2}} L_n^{\alpha_2}(z). \tag{29}$$

N_n is the normalization constant.

TABLE 1. Energy Eigen Values of the Mixed Potential

n,l	**Energy *E* for *k*=5, *g*=0, *m*=0.5**
0,0	2.367460025
1,0	4.671641976
1,1	7.159601932
2,0	6,483289157
2,1	8.673485853
3,0	8.064098954
3,1	10.06294280
3,2	11.4053318

CONCLUSION

In conclusion, we have obtained the energy eigenvalues and the corresponding un-normalized wavefunction using the NU method for the Klein-Gordon equation with equal scalar and vectorharmonic oscillator plus inverse quadratic potential for *S*-waves. We have also obtained a special case for $g = 0$ giving the energy of the harmonic oscillator potential. In the non-relativistic limits, our result reduced to the harmonic potential for $g = 0$ reported by Ita *et al* [33].

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