AMERICAN SCIENTIFIC PUBLISHERS

Advanced Science, Engineering and Medicine
Vol. 5, pp. 1–9, 2013
(www.aspbs.com/asem)

ARTICLE

NMR/MRI Blood Flow Magnetization Equation in the Rotating Frame of Reference-Part I

Dilip K. De

Department of Physics, Kaduna State University, Kaduna, Nigeria

This paper describes thoroughly the need and the method of deriving the first of its kind the NMR/MRI blood flow magnetization (y component) equation in the rotating frame when rf B_1 field is applied along laboratory X direction.

$$\left[\left(v\cdot\nabla+\frac{\partial}{\partial t}\right)\frac{1}{\gamma B_1(x,t)}\left(v\cdot\nabla+\frac{\partial}{\partial t}+\frac{1}{T_2}\right)+\frac{1}{\gamma B_1(x,t)T_1}\left(v\cdot\nabla+\frac{\partial}{\partial t}+\frac{1}{T_2}+\gamma^2B_1^2(x,t)T_1\right)\right]\boldsymbol{M}_y=\frac{M_o}{T_1}$$

where

$$v\cdot\nabla=v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}$$

v_x and v_y are the components of blood flow velocity along the x and y directions of the rotating frame in an NMR experiment. The equation is expected to serve as the mother equation for accurate non invasive blood flow quantification through all NMR/MRI experiments. It is shown how Awojyogbe's equation of blood flow magnetization can be obtained from above equation under assumption of constant B_1 field and $v_y=0$. The method of deriving the equation can be applied to modify Bloch Torey Diffusion MRI equation to include relaxation times and flow and also to derive the NMR/MRI spin flow magnetization equation in the laboratory frame of reference. The derivation of the corresponding equation for magnetization of flowing blood spins in the laboratory frame of reference will be discussed in a separate paper.

KEYWORDS: NMR/MRI, Flow Magnetization Equation, Relaxation Times, Diffusion MRI, Rotating Frame, Laboratory Frame, Mother Equation.

1. INTRODUCTION

Nuclear Magnetic Resonance Imaging has advanced greatly the science of medical technology. As the image qualities are improving and as the amount of information that they can offer has been appreciated, the modality has become the primary topic of discussion among imaging scientists, Imaging physicians, Hospital administrators and Government officials responsible for funding health care in many countries around the world. The combining applications of NMR and MRI in medical studies in human—a vast complex and very promising prospect—have not been adequately explored and are being investigated by scientists, engineers, mathematicians and clinicians around the world. At the heart of all these investigatory efforts lie the Bloch NMR equations for the three components of Magnetizations, M_x, M_y and M_z of the nuclear spin. NMR/MRI offers non-invasive mesurements of blood flow in humans. This is very important in many clinical diagnosis. Magnetizations of flowing spins under given rf excitations are dependent on flow velocity of spins apart from rf B_1 field, magnetizing field, B_o and field gradients and rf frequency etc. This forms the basis of measuring blood flow rate (both velocity and volume flow rate) by NMR/MRI techniques.

Various techniques have been developed over the past two decades in quantifying blood flow velocity and rates. Some of these are discussed below. Magnetic resonance (MR) signals are usually generated in three ways:[1–5] Spin Echo (SE), Free induction decay (FID) and gradient Echo Technique. The MRI sequences depend on integration of these types of signals with gradient sequences necessary for spatial encoding. Flowing spins along an applied magnetic field gradient can acquire phase shift in the MR signal compared to the static tissue signal. The phase shift is proportional to the strength, duration of the gradient and motion of the spins. Phase contrast as a result of the phase shift enables us to achieve complete suppression of the static tissue and thus estimation of information on flowing spins (usually of blood). The final signal in a given

Emails: dlpd770@gmail.com, dipak61@yahoo.com
Received: 8 February 2013
Accepted: 23 April 2013

time frame should however depend not only on the phase shift but also on the flow dependent magnetizations. De[6] and Odoh and De[7] reported theoretical investigation and computation of time dependent CW NMR Blood flow signal for estimation of blood flow parameters using a new approach for continuous wave nuclear magnetic resonance (CW NMR). It has been shown that CW NMR can be a very useful technique that can be used for quantification of blood flow rate if the mathematical complexities involved can be handled successfully. However, in the analysis it was assumed that B_1 field is such that M_z component does not vary appreciably from M_o, the saturation magnetization at a given bias magnetic field. This is true only for low values of B_1 field.

The feasibility of measuring blood flow to the human retina using arterial spin labeling MRI, a quantitative, noninvasive tomographic technique was investigated by Maleki et al. (2011).[8] Quantification of cerebral blood flow (CBF) is very essential to diagnose many of the neurological pathologies that affect neonates and small infants as well as adults. CBF measurements are often performed through applications of radioactive tracers or other invasive methods which in turn can affect the human system. Varela et al. (2012),[9] developed technique using phase contrast MRI that can easily be appended to a neonatal MRI examination to provide rapid, robust, and non-invasive estimates of mean CBF, thus providing a means to monitor developmental or pathology-related alterations in cerebral perfusion and the impact of different treatment courses. Even though key hemodynamic factors for flow quantification, including arterial transit delay and the apparent decay time of the signal, were estimated by repeated measurements with different arterial spin labeling timing, it is expected that flow quantification by NMR/MRI in any part (through proper slice selection technique) of the entire human body would improve greatly if the Bloch NMR/MRI flow magnetization equations are properly framed. Awojoyogbe and his group's works (2000–2012)[10–20] in this direction are note worthy, some of which are discussed below:

Awojoyogbe[10–12] derived the following flow magnetization equation

$$V^2M'' + V(T_1^{-1} + T_2^{-1})M' + (\gamma^2 B_1^2(x\cdot t) + T_1^{-1}T_2^{-1})M = \gamma M_o B_1(x\cdot t)T_1^{-1} \quad \text{(A)}$$

Where V is the flow speed of the nuclear spin along the rotating x direction along which the rf B_1 field is applied. T_1 and T_2 are the relaxation times. $B_1(x,t)$ is the rf B_1 field. γ is the gyromagnetic ratio of the nuclear spin. M' and M'' are the first and second order derivative (w.r. to x) of the y component of magnetization. M_o is the saturation magnetization. Dada et al. (2008)[21] suggested application of the Boubaker-Turki polynomials to Magnetic Resonance (MR) blood flow imaging. Their suggestion was based on the comparison of the following second order differential Eq. (B):

$$4x(1-x^2)y'' + P(x,n)y' + Q(x,n) = 2Q(x,n)Tn(x) \quad \text{(B)}$$

Where $P(x,n) = -4x^2 + 2nx - 2n + 8$

$$Q(x,n) = -4x^2n + 6n - n^2 - 32$$

with the flow magnetization Eq. (A).

Even though the correctness of Eq. (B) is not checked in this present work, obviously Eq. (A) does not fit in to the form of Eq. (B), the reason being that V, the velocity of flowing spins (which is the object of investigation by NMR/MRI), cannot be forced to assume any chosen functions of x, so as to conform to Eq. (B). Thus application of Boubaker polynomial for solution of Eq. (1) is thus highly questionable. Moreover, as shown later Eq. (1) for NMR/MRI flow magnetization is true only in rotating frame of reference and has limited validity. It is valid when V is independent of x and t or in other words for steady blood flow through uniform vessel cross section and V is the flow speed of the spin along the rotating x direction along which the rf B_1 field is applied. For the equation to be valid other conditions that must hold are: rf B_1 field must be independent of x and t which is hard to achieve in any MRI/NMR sequence. A number of article were published by Awojoyogbe[10–13] and Awojoyogbe and his group[15–20] that discussed mathematical concept of the Bloch flow equations for general magnetic resonance imaging and exploring new dimensions cardiovascular flow and motion: application of Bloch NMR flow equations, Bessel and spherical harmonic functions based on the following ideas: Since molecular motion of water is significantly affected by macromolecules, the variation in the relaxation times between tissues is attributed to the effect of macromolecular interaction. The movement of water molecules during diffusion-driven random displacement is restricted by compartmental boundaries and other molecular obstacles in such a way that the actual diffusion distance is reduced, compared with what is expected in unrestricted diffusion. NMR/MRI sequences of diffusion driven spins should yield information on restricted compartmental boundaries through which diffusion of blood occurs in case of sicknesses such as sickle cell anemia. This is based on the idea that NMR/MRI signals in such cases would significantly differ (specially because of different relaxation times) from that of blood spin diffusion through unrestricted boundaries (as in healthy patients).

Jain et al. (2010)[22] discussed MRI estimation of global brain oxygen consumption rate. Their method of flow estimation like many other techniques are based on change of phase angle of magnetisation of flowing spins with magnetic field gradient pulse and flow velocity using the equation: $\Delta\varphi = \gamma v \Delta M_1$ where $M_1 = \gamma v[\int G(t)t\,dt]$. ΔM_1 is the difference in the first moment M_1 between two interleaves under magnetic field gradient. This equation is truly speaking an over simplification in the sense that flowing blood

magnetisation is itself velocity dependent and the equation does not take care of that and the equation could only yield good result when the spatial separation between the two interleaves as well as the flow velocity are quite small. Because the NMR/MRI magnetization is flow dependent, assumption of linear dependence of phase difference is ideally questionable.

Scientifically actual quantification of information on diseased vessels depends on correct Bloch NMR/MRI magnetization equation including flow, diffusion and relaxation effects of the spins. If this is not cast accurately from strict physics point of view it is prone to yield inaccurate information when quantification is made through application of such equations on the measured NMR/MRI signals or images. Moreover, extraction of flow information from NMR/MRI signals/images without the application of correct flow magnetization equation is bound to be associated with errors. Therefore, we have undertaken examination of the correct form of Bloch NMR flow magnetizations equation (the mother equation) from which NMR/MRI signals can be computed under various schemes of Imaging sequences for both steady and pulsatile blood (nuclear spin) flow. In this examination, we limit ourselves to the rotating frame of references. We also limit ourselves to the formulation of flow magnetizations equation without applied field gradient. We believe that the case of applied field gradients can be easily included following our scheme of formulations. The corresponding equations in Laboratory frames of references will be discussed in a subsequent paper Part II. We discuss the application of our equations to diffusion MRI.

In the following formulations and discussions, we assume that the bias magnetic field (B_o) is applied along laboratory Z_o axis (of the laboratory $X_oY_oZ_o$ frame of axes) which coincides with the z axis of the rotating xyz axes) as shown in Figure 1. The rf B_1 field in the form B_{10} coswt is applied along the laboratory X_o axis by an excitor coil as shown in Figure 2(a). A time independent magnetic field of magnitude $B_{10}/2$ can be considered to be along the rotating x axis making an angle wt with the laboratory X_o axis counter clockwise. The detector coil has plane perpendicular to the Y_o axis of the laboratory frame (Fig. 2(b)). Therefore, we have undertaken examination of the correct form of Bloch NMR flow magnetizations equation (the mother equation) from which NMR/MRI signals can be computed under various schemes of Imaging sequences for both steady and pulsatile blood (nuclear spin) flow. In this examination, we limit ourselves to the rotating frame of references. We also limit ourselves to the formulation of flow magnetizations equation without applied field gradient. We believe that the case of applied field gradients can be easily included following our scheme of formulations. The corresponding equations in Laboratory frames of references will be discussed in a subsequent paper Part II. We discuss the application of our equations to diffusion MRI.

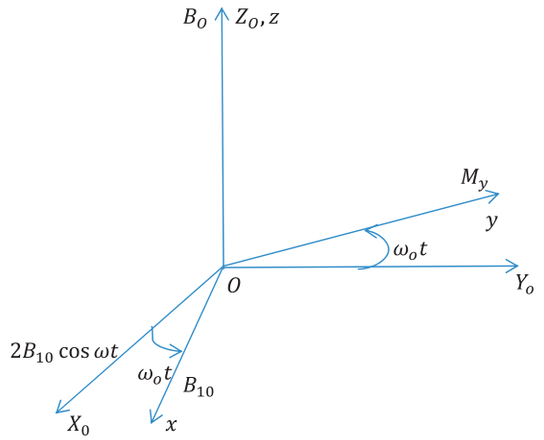

Fig. 1. Rotational frame of axes (xyz) in relation to the fixed Laboratory frame of axes ($X_0Y_0Z_0$). x axis of the rotational frame makes an angle $\omega_0 t$ with the X_0 axis of the laboratory frame at time t. $\omega_0 = \gamma B_o = 2\pi f_o$. f_o is the NMR resonance frequency. The oscillating rf B_1 field is applied along the X_0 axis along which the blood flow is considered to take place. Along the x axis of the rotating frame the rf B_{10} field is independent of time. The magnetization component M_y which is function of B_{10}, T_1, T_2, flow velocity etc. produces the signal in the detector coil placed in quadrature mode to the excitor coil (see Fig. 2).

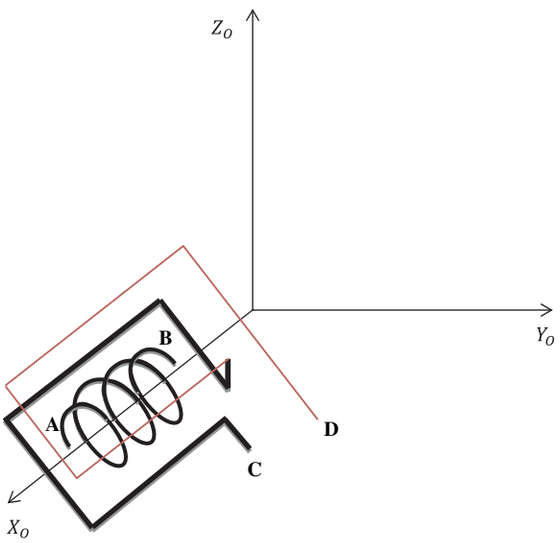

Fig. 2. The relative disposition of the excitor coil (A) (B) and the detector coil (C) (D) in quadrature mode to be employed in the blood flow estimation by NMR.

2. PRELIMINARY BASICS OF THE BLOCH NMR EQUATIONS

In NMR the nuclear spins are magnetized by an external magnetic field B_o (along laboratory Z directions). Radio frequency (rf) excitation B_1 field are applied along laboratory X directions. Rf B_1 field of frequency ω is given by

$$\mathbf{B}_1(\mathbf{t}) = \mathbf{B}_{1o}\sin(\omega t). \quad (1)$$

The net field is given by

$$\mathbf{B} = k_o B_o + i_o B_1(t) \tag{2}$$

as seen from the laboratory XYZ axes. i_o and k_o are the unit vectors along X and Z axes. Please note that small letters x and y are used for rotating frame coordinates. The rotation of the nuclear spins under the net field can be described by the fact that the net rate of change of angular momentum (dL/dt) is equal to the torque ($\tau = \mathbf{M} \times \mathbf{B}$ due) to the net magnetic field.

Thus

$$\frac{dL}{dt} = \mathbf{M} \times \mathbf{B} \tag{3}$$

In some MRI sequences a pulsed magnetic field gradient may be applied along Z direction for selecting particular slices for investigation. We are omitting that from our present discussion. Now

$$\mathbf{M} = \gamma \mathbf{L} \tag{4}$$

Using Eq. (4), the Eq. (3) becomes

$$\frac{dM}{dt} = \gamma \mathbf{M} \times \mathbf{B} \tag{5}$$

Equation (5) is correct only for a single nuclear spin in absence of any interaction with neighbouring spins). But when the spin is surrounded by an environment (as in water, tissues, fluids samples or in solids) etc.) the Z, X and Y components of the spin magnetization M relaxes due to the interaction with the environment. M_z relaxes with time T_1 (called spin-lattice relaxation times) according to

$$M_z = M_{zo}(1 - \exp(-t/T_1)). \tag{6}$$

M_{zo} is the magnetization acquired by the sample as given by

$$M_{zo} = M_o B_J(\alpha) \tag{7}$$

$$M_o = N_A \mu_B (j(j+1))^{0.5} \tag{8a}$$

$\mu_B =$ Bohr magneton and $N_A =$ Avogradro number.

Where $B_J(\alpha)$ is the Brillouin function given by

$$B_J(\alpha) = \left(\frac{2j+1}{2j} \coth \frac{2j+1}{2j}\alpha - \frac{1}{2j} \coth \frac{1}{2j}\alpha \right). \tag{8b}$$

$$\alpha = B_o \mu_B (j(j+1))^{0.5}/k_B T \tag{8c}$$

j is the total nuclear spin. M_x and M_y components relaxes with time T_2. With relaxation effects the Eq. (5) then takes the form

$$\frac{dM_z}{dt} = \gamma(\mathbf{M} \times \mathbf{B})_z + \frac{M_o - M_z}{T_1} \tag{9a}$$

$$\frac{dM_x}{dt} = \gamma(\mathbf{M} \times \mathbf{B})_x - \frac{M_x}{T_2} \tag{10a}$$

$$\frac{dM_y}{dt} = \gamma(\mathbf{M} \times \mathbf{B})_y - \frac{M_y}{T_2} \tag{11a}$$

Because of relations (1) and (2) solutions of (9a)–(11a) are difficult. A rotating frame of reference x, y, z is invoked where z axes of the rotating frame coincides with laboratory Z axis and the x–y axes rotate (with angular velocity ω) in the laboratory X–Y plane anticlockwise such that the rotating x axis makes an angle ωt at any time t with the X axis. Then the rf B_1 field is along the rotating x axis in the rotating frame. Then the Eqs. (9)–(11) become at NMR resonance conditions (i.e., at $\omega = \omega_o = \gamma \mathbf{B}_o$).

$$\frac{dM_z}{dt} = \gamma(\boldsymbol{M} \times \mathbf{B}_{1o})_z + \frac{M_o - M_z}{T_1} \tag{9b}$$

$$\frac{dM_x}{dt} = \gamma(\boldsymbol{M} \times \mathbf{B}_{1o})_x - \frac{M_x}{T_2} \tag{10b}$$

$$\frac{dM_y}{dt} = \gamma(\boldsymbol{M} \times \mathbf{B}_{1o})_y - \frac{M_y}{T_2} \tag{11b}$$

$$M = iM_x + jM_y + kM_z \tag{12}$$

where i, j and k are now unit vectors in the rotating frame of references as the components also are. In Eqs. (9b)–(11b) the rf B_1 field is independent of time in a CW NMR experiment and is dependent on time in a pulsed NMR experiment. The Eqs. (9b)–(11b) then finally become ($\omega_o = \gamma B_o$).

$$\frac{dM_z}{dt} = -\gamma M_y B_{1o} + \frac{M_o - M_z}{T_1} \tag{13}$$

$$\frac{dM_x}{dt} = -\frac{M_x}{T_2} \tag{14}$$

$$\frac{dM_y}{dt} = \gamma M_z B_{1o} - \frac{M_y}{T_2} \tag{15}$$

B_{1o} is the amplitude of rf B_1 field. It is to be noted that the M_y component produces the signals which are detected by coil placed usually at right angles to the rf B_1 field coil.

3. FORMULATIONS OF NMR FLOW MAGNETIZATION EQUATION

In this section we derive the expression for the component of flow magnetization that will produce the final signal in the detector coil (with its axis along the Laboratory X axis). The Eqs. (13)–(15) are true in the rotating frames of references when there is no translational motion of the spins. Let us now consider the effect of translational motion (flow) of the nuclear spins on the above equations. With flow $\boldsymbol{M}$ will become function of $x \cdot y \cdot z$ and t. i.e.,

$$\boldsymbol{M} = \boldsymbol{M}(x, y, z, t) \tag{16}$$

Then the total time derivative of $\boldsymbol{M}$ *is given by:*

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial x}\frac{dx}{dt} + \frac{\partial M}{\partial y}\frac{dy}{dt} + \frac{\partial M}{\partial z}\frac{dz}{dt} \tag{17}$$

where dx/dt, dy/dt, dz/dt, are the components of the fluid velocity V.

We can also write this equation in the form:

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial x}V_x + \frac{\partial M}{\partial y}V_y + \frac{\partial M}{\partial z}V_z \tag{18}$$

and

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + V \cdot \nabla M \tag{19}$$

$\nabla = i(\partial/\partial x) + j(\partial/\partial y) + k(\partial/\partial z)$ is the gradient vector. The total derivative dM/dt is also a function of x, y, z, and t. A similar relation holds between partial and total derivative of any quantity, and we may write, symbolically,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V \cdot \nabla$$

where V is the fluid velocity.

Let us assume that the fluid is moving only along x direction. Then $V_y = V_z = 0$ and $V_x = V_x$. Then the Eq. (19) reduces to

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + V(\nabla M)_x = \frac{\partial M}{\partial t} + V\frac{\partial M}{\partial x} \tag{20}$$

Now $\mathbf{M}$ is a vector quantity and is given by $\mathbf{M} = \mathbf{i}M_x + \mathbf{j}M_y + \mathbf{k}M_z$.

Then from Eq. (20) the Bloch flow NMR Equations for flowing spins then [DE (1990), Odoh and De [(2008), (2009)] as follows:

$$\frac{dM_x}{dt} = \frac{\partial M_x}{\partial t} + V\frac{\partial M_x}{\partial x} = -\frac{M_x}{T_2} \tag{21}$$

$$\frac{dM_y}{dt} = \frac{\partial M_y}{\partial t} + V\frac{\partial M_y}{\partial x} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \tag{22}$$

$$\frac{dM_z}{dt} = \frac{\partial M_z}{\partial t} + V\frac{\partial M_z}{\partial x} = -\gamma M_y B_1(x) + \frac{M_o - M_z}{T_1} \tag{23}$$

Omitting the total derivative on the left of Eqs. (21)–(23)

$$\frac{\partial M_x}{\partial t} + V\frac{\partial M_x}{\partial x} = -\frac{M_x}{T_2} \tag{21'}$$

$$\frac{\partial M_y}{\partial t} + V\frac{\partial M_y}{\partial x} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \tag{22'}$$

$$\frac{\partial M_z}{\partial t} + V\frac{\partial M_z}{\partial x} = -\gamma M_y B_1(x) + \frac{M_o - M_z}{T_1} \tag{23'}$$

The M_y component in the above equations produces the signal in the detector coil when its cross section is placed normal to either the X or Y direction (note that the y axis makes ωt angle with the Y axis. Therefore, the task is how to solve for M_y from Eqs. (7) and (8) above. In this paper a new operator formalism is adopted to arrive at a differential equation involving M_y only from the above equations. The above equations are true at NMR/MRI resonance conditions (i.e., the frequency, ω of the rf B_1 field is equal to i.e., $\omega = \gamma B_o$) in the rotating frame of reference (x, y, z) in which rotating frame **x** *axis* coincides with the rotating $B_1(x)$ axis; **z** *axis* coincides with the applied B_o direction (LABORATORY Z axis) y axis is along $\mathbf{z} \times \mathbf{x}$ direction. B_1 field in a CW experiment is time independent in a CW NMR experiment. However, in pulsed NMR experiment B_1 field is time dependent even in a rotating frame. Hence forth $B_1(x)$ will be replaced by $B_1(x, t)$ in the following equations so as to represent a general case.

4. OPERATOR FORMALISM METHOD FOR OBTAINING BLOCH NMR EQUATIONS OF FLOW MAGNETIZATION, M_Y IN THE ROTATING FRAME OF REFERENCE

From the above equations, a partial differential equation of second order is derived following an operator formalism method which is very valuable in the analyses of space and time dependence of the NMR magnetization components of flowing spins.

In Eq. (21') M_x can be solved by assuming $M_x = f(x)g(t)$ and using separation of variable when V is steady velocity or time dependent but independent of X. Below we use the symbol v for the velocity V in above equations. To solve for M_y we see that

From Eq. (23'), we have

$$v\frac{\partial M_z}{\partial x} + \frac{\partial M_z}{\partial t} + \frac{M_z}{T_1} = -\gamma M_y B_1(x, t) + \frac{M_o}{T_1}$$

or

$$\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{1}{T_1}\right)M_z = -\gamma M_y B_1(x, t) + \frac{M_o}{T_1}$$

Or,

$$M_z = \frac{1}{(v(\partial/\partial x) + \partial/\partial t + 1/T_1)} \times \left(-\gamma M_y B_1(x, t) + \frac{M_o}{T_1}\right) \tag{24}$$

In Eq. (24) $1/(v(\partial/\partial x) + \partial/\partial t + 1/T_1)$ is the inverse operator of $(v(\partial/\partial x) + \partial/\partial t + 1/T_1)$. Substituting M_z from Eq. (24) in Eq. (22'), it follows that

$$\frac{\partial M_y}{\partial t} + v\frac{\partial M_y}{\partial x} + \frac{M_y}{T_2} = \gamma\left\{\frac{1}{(v(\partial/\partial x) + \partial/\partial t + 1/T_1)} \times \left(-\gamma M_y B_1(x, t) + \frac{M_o}{T_1}\right)\right\}B_1(x, t) \tag{25}$$

Or,

$$\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{1}{T_2}\right)M_y = \gamma\left\{\frac{1}{(v(\partial/\partial x) + \partial/\partial t + 1/T_1)} \times \left(-\gamma M_y B_1(x, t) + \frac{M_o}{T_1}\right)\right\}B_1(x, t) \tag{26}$$

Equation (26) contains only M_y. In Eq. (26) it is not easy to evaluate the effect of $1/(v(\partial/\partial x)+\partial/\partial t+1/T_1)$ on the function $(-\gamma M_y B_1(x,t)+M_o/T_1)$.

We use the following method to eliminate the effect of the inverse operator and obtain a differential equation in M_y.

Let Ω_1 and Ω_2 represent the differential operators $(v(\partial/\partial x)+\partial/\partial t+1/T_1)$ and $(v(\partial/\partial x)+\partial/\partial t+1/T_2)$ and then $1/(v(\partial/\partial x)+\partial/\partial t+1/T_1)$ be denoted by the symbol Ω_1^{-1} (i.e., inverse of Ω_1. In Eq. (25)) it is important to note that Ω^{-1} acts only on $(-\gamma M_y B_1(x,t)+M_o/T_1)$ and not on $B_1(x,t)$ outside the curly bracket in Eqs. (25) and (26). Equation (26) is then written in terms of the operators as

$$\Omega_2 M_y = \gamma\left\{\Omega_1^{-1}\left(-\gamma M_y B_1(x)+\frac{M_o}{T_1}\right)\right\}B_1(x,t) \quad (27)$$

We note that the main problem in solving the Bloch NMR differential equations is handling the inverse operators of Ω_1 and Ω_2. We operate both sides of Eq. (27) by Ω_1. Then we get

$$\Omega_1[\Omega_2 M_y] = \gamma\Omega_1\left[\left\{\Omega_1^{-1}\left(-\gamma M_y B_1(x,t)+\frac{M_o}{T_1}\right)\right\}B_1(x,t)\right] \quad (28)$$

In the RHS of Eq. (28) we note that Ω_1 acts also on $B_1(x,t)$ outside the curly bracket. Let

$$-\gamma M_y B_1(x,t)+\frac{M_o}{T_1} = \boldsymbol{f(x)} \quad \text{and} \quad \Omega_1^{-1}(f(x)) = \boldsymbol{g(x)} \quad (29)$$

Then Eq. (28) becomes

$$\Omega_1[\Omega_2 M_y] = \gamma\Omega_1[\{\Omega_1^{-1}(f(x))\}B_1(x,t)] \quad (30)$$

Or

$$\Omega_1[\Omega_2 M_y] = \gamma\Omega_1[\{g(x)\}B_1(x,t)] \quad (31)$$

We now expand the RHS of Eq. (30) first. We get after expanding

$$\Omega_1[\Omega_2 M_y] = \gamma\left[B_1(x)\Omega_1\{g(x)\}+g(x)\left(\Omega_1-\frac{1}{T_1}\right)B_1(x,t)\right] \quad (32)$$

Now we substitute the expression for $\boldsymbol{g(x)}$ in Eq. (32) We get

$$\Omega_1[\Omega_2 M_y] = \gamma\left[B_1(x,t)\Omega_1\{\Omega_1^{-1}f(x)\}+(\Omega_1^{-1}f(x))\times\left(\Omega_1-\frac{1}{T_1}\right)B_1(x,t)\right] \quad (33)$$

$$\Omega_1[\Omega_2 M_y] = \gamma\left[B_1(x,t)f(x)+\{\Omega_1^{-1}f(x)\}\times\left(\Omega_1-\frac{1}{T_1}\right)B_1(x,t)\right] \quad (34)$$

In Eq. (34) we have utilised the fact that $\Omega_1\Omega_1^{-1}=1$

Equation (27) is written interms of $f(x)$ as

$$\Omega_2 M_y = \gamma\{\Omega_1^{-1}f(x)\}B_1(x,t) \quad (27)$$

Eliminating $\{\Omega_1^{-1}f(x)\}$ from the Eqs. (34) and (27) we get

$$\Omega_1[\Omega_2 M_y] = \gamma[B_1(x,t)f(x)]+\{\Omega_2 M_y\}\times\left\{\left(\Omega_1-\frac{1}{T_1}\right)B_1(x,t)\right\}\Big/B_1(x,t) \quad (34)$$

Substituting $\boldsymbol{f(x)}$ from Eq. (29) into Eq. (34) we get

$$\Omega_1[\Omega_2 M_y] = \gamma\left[B_1(x,t)\left\{-\gamma M_y B_1(x,t)+\frac{M_o}{T_1}\right\}\right]+\{\Omega_2 M_y\}\left\{\left(\Omega_1-\frac{1}{T_1}\right)B_1(x,t)\right\}\Big/B_1(x,t) \quad (35)$$

Substituting the expressions for Ω_1, Ω_2 into Eq. (35) we get (noting that the terms in the curly brackets of the second term of Eq. (35) are multiplicative (hence commutative)

$$\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_1}\right)\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_2}\right)\boldsymbol{M_y} = \gamma\left[B_1(x,t)\left\{-\gamma M_y B_1(x,t)+\frac{M_o}{T_1}\right\}\right]+\left\{\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)B_1(x,t)\right\}\times\left\{\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_2}\right)\boldsymbol{M_y}\right\}\Big/B_1(x,t) \quad (36)$$

The differential Eq. (36) can be written in a compact form as given below:

$$\left[B_1(x,t)\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_1}\right)\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_2}\right)+\gamma^2 B_1^3(x,t)-\left\{\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)B_1(x,t)\right\}\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_2}\right)\right]\boldsymbol{M}_y = \gamma\left[B_1^2(x,t)\frac{M_o}{T_1}\right] \quad (37)$$

The operators appearing in Eqs. (36) and (37) are all commutative in the realm of NMR/MRI sequences, i.e., $v(\partial/\partial x)(\partial/\partial t)=(\partial/\partial t)v(\partial/\partial x)$, specially when v corresponds to steady flow situation and independent of time. The solution of Eq. (37) when v is time dependent or pulsatile flow as in humans is out of the scope of this paper. Determination of T_2, T_1 relaxation times and MRI images are based on pulsed sequences when $B_1(x,t)$ is time dependent and usually given by a Fourier series in ωt. We believe that the above Eqs. (36) and (37) should hold in such case also and the Eq. (37) represents the true diffrential equation for M_y which produces the detectable NMR signal under any NMR/MRI sequence.

5. DISCUSSION

In absence of flow ($v = 0$), the equation of Bloch NMR magnetization M_y is then given from Eq. (37) as:

$$\left[B_1(x,t)\left(\frac{\partial}{\partial t}+\frac{1}{T_1}\right)\left(\frac{\partial}{\partial t}+\frac{1}{T_2}\right)+\gamma^2 B_1^3(x,t) - \left\{\left(\frac{\partial}{\partial t}\right)B_1(x,t)\right\}\left(\frac{\partial}{\partial t}+\frac{1}{T_2}\right)\right]\boldsymbol{M_y} = \gamma\left[B_1^2(x,t)\frac{M_o}{T_1}\right] \quad (38)$$

Equation (38) can describe magnetization component M_y of stationary spin in the rotating frame for all NMR/MRI pulse sequences in which $B_1(x,t)$ can be in the form of pulsed rf. For CW NMR (without flow) the explicit time derivative part can be dropped from Eq. (38).

In such situation $B_1(x,t) = B_{1o}$ and we see from Eq. (38) that

$$\left(\frac{1}{T_1T_2}+\gamma^2B_1^2\right)M_y = \frac{\gamma B_{1o}M_o}{T_1}$$

$$M_y = \frac{\gamma B_{1o}T_2M_o}{1+\gamma^2B_1^2T_1T_2} \quad (39)$$

Where B_{1o} is the magnitude of rf B_1 field along the x axis of the rotating frame of reference. This makes an angle ωt with the laboratory X_o axis. We note that the solution of Eq. (21) for M_x in absence of flow gives, $M_x = M_{xo}\exp(-t/T_2)$. M_x becomes neglible when $t >> T_2$ in CW NMR case with no flow of spins. Then one can easily obtain the expressions for M_{Xo} and M_{Yo} in the Laboratory frame as follows:

$$M_{Xo} = \frac{-\mathrm{Sin}(wt)\gamma B_1T_2M_o}{1+\gamma^2B_1^2T_1T_2} \quad (40)$$

$$M_{Yo} = \frac{\mathrm{Cos}(wt)\gamma B_1T_2M_o}{1+\gamma^2B_1^2T_1T_2} \quad (41)$$

Fortunately Eqs. (40) and (41) are the exact solutions of Bloch CW NMR magnetizations at resonance condition in the laboratory frames with no flow of spins (i.e., for stationary nuclear spins). This shows the correctness of our approach in formulating a coherent differntial Eq. (37) that can describe the M_y component of magnetization of flowing spins in the rotating frame under any form of rf B_1 field excitation. We believe that the Eq. (37) can be applicable to all cases of NMR/MRI excitations of flowing spins whether pulsed rf or CW NMR. We believe that such formulations (Eqs. (36) or (37)) is not seen in literature before.

If an ideal imaginary situation can be assumed when $B_1(x,t)$ is both independent of time and x then the above Eq. (37) reduces to:

$$\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_1}\right)v\frac{\partial M_y}{\partial x}+\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_1}\right)\frac{\partial M_y}{\partial t} + \left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_1}\right)\frac{M_y}{T_2} = \gamma\left(-\gamma M_yB_1+\frac{M_o}{T_1}\right)B_1 \quad (42)$$

Assuming flowing spin velocity v to be both independent of time and x, and assuming $\partial/\partial x(\partial M_y/\partial t) = \partial/\partial t(\partial M_y/\partial x)$ one gets from (42)

$$v^2\frac{\partial^2M_y}{\partial x^2}+v\frac{\partial^2M_y}{\partial x\partial t}+\frac{v}{T_1}\frac{\partial M_y}{\partial x}+v\frac{\partial^2M_y}{\partial x\partial t}+\frac{\partial^2M_y}{\partial t^2}+\frac{1}{T_1}\frac{\partial M_y}{\partial t} + \frac{v}{T_2}\frac{\partial M_y}{\partial x}+\frac{1}{T_2}\frac{\partial M_y}{\partial t}+\frac{1}{T_1T_2}M_y = -\gamma^2B_1^2M_y+\frac{\gamma B_1M_o}{T_1}$$

or,

$$v^2\frac{\partial M_y}{\partial x^2}+2v\frac{\partial^2M_y}{\partial x\partial t}+v\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{\partial M_y}{\partial x}+\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{\partial M_y}{\partial t} + \frac{\partial^2M_y}{\partial t^2}+\left(\frac{1}{T_1T_2}+\gamma^2B_1^2\right)M_y = \frac{\gamma B_1M_o}{T_1} \quad (43)$$

Equation (43) describing flow magnetization M_y in the rotating frame is obtained from Eqs. (36)–(37) when B_1 field is independent of both time and x. From it at any given time t, we could get information about the system, provided that appropriate boundary conditions are applied. If in a NMR experiment the time dependence $\partial M_y/\partial t$ of M_y can be dropped in Eq. (39) and one gets the flow magnetization equation:

$$v^2\frac{\partial M_y}{\partial x^2}+v\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{\partial M_y}{\partial x}+\left(\frac{1}{T_1T_2}+\gamma^2B_1^2\right)M_y = \frac{\gamma B_1M_o}{T_1} \quad (44)$$

as derived by Awojyogbe[10–12] earlier as the original flow magnetization equation.[11] They had assumed B_1 to be function of both x and t. We see from our above derivations that in Eq. (44) the B_1 field must be independent of both x and t. Equation (44) has been recently been discussed theoretically in several publications recently. We have shown above that Eq. (44) can be derived from the mother Eqs. (36)–(37) of NMR flow magnetization, based on conditions described above. Experimental situations conforming to such conditions of Eq. (44) could be a situation in which in a CW NMR the B_1 field is nearly uniform over the area of flow interest.

Thus we see that our Eqs. (36)/(37) becomes the mother equation for NMR/MRI flow magnetization from which all equations of flow magnetization (in the rotating frame at resonance, i.e., when $\omega = \omega_o$) can be obtained depending on experimental situations i.e., rf pulse sequences, slice selection gradient pulses etc. In a given NMR/MRI experiment a particular slice is selected through the application of magnetic field gradient and rf pulse is applied and signal is recorded. Based on form of the rf pulse $B_1(x,t)$ *field* the flow magnetization and hence the blood flow signal across the slice or along a selected vessel can be computed from Eqs. (37)–(38). This would enable us to obtain non-invasive flow information (appropriate for the medical diagnosis) accurately using the signal data. The approach given above could be extended to obtaining correct NMR/MRI spin flow magnetization equations in the Laboratory frame of reference. Such equations will be reported in our next work.

5.1. For Blood Flow Velocity Along Laboratory *X* Direction

In the above formulations (Eqs. (36) and (37)) it was assumed that the flow velocity is along x direction, the direction of the rotating rf field. In a realistic MRI sequence of blood flow estimation the artery (through which flow is occuring) could be selected through the slice encoding pulsed gradient fields and the rf B_1 field could be aligned along or perpendicular to the flow direction (through the selected vessel). Let us assume that in such situation the flow velocity V_o as well as the rf B_1 field is along the laboratory X direction. The components of the flow velocity along the rotationg x and y directions are:

$$v_x = V_o \cos \omega t \quad \text{and} \quad v_y = V_o \sin \omega t \tag{44a}$$

Then it can be shown easily by following above methods the Bloch NMR/MRI flow magnetization equation (in the rotating frame) takes the form with magnetizing field B_o being along Z direction,

$$\left[\left(v\cdot\nabla+\frac{\partial}{\partial t}\right)\frac{1}{\gamma B_1(x,t)}\left(v\cdot\nabla+\frac{\partial}{\partial t}+\frac{1}{T_2}\right)+\frac{1}{\gamma B_1(x,t)T_1}\right.$$
$$\left.\times\left(v\cdot\nabla+\frac{\partial}{\partial t}+\frac{1}{T_2}+\gamma^2 B_1^2(x,t)T_1\right)\right]\boldsymbol{M}_y=\frac{M_o}{T_1} \tag{45}$$

where

$$v\cdot\nabla = v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}$$

Equation (45) describes the flow magnetization in the rotating frame of reference with v_x and v_y obtained through the velocity V_o in the Laboratory frame (see Eq. (44a)). One can easily see from Eq. (45) that for no flow conditions one obtains Eq. (39) and consequently Eqs. (40) and (41) for CW NMR on static tissue or samples. Also Eqs. (36) and (37) emerge from (45) when v_y is dropped (i.e., only v_x is retained) as a special case. Solutions of Bloch NMR/MRI flow magnetizations in the rotating frame as given by Eqs. (36), (37) and (45) are beyond the scope of the present article and would be discussed subsequently. It is to be noted that the solutions depend on specific MRI sequences used in the blood flow estimation. The solutions of Eqs. (36)–(37) and (45) may be attempted in line with the solutions proferred by De and Odoh (2009).[23]

5.2. Slice Selection in MRI

Questions may arise how to include magnetic field gradient used normally for tissue/vessel site selection? With the slice selection field gradients the resonance frequencies vary with tissue coordinates (x, y, z) depending on the bias magnetic field at the time of resonance. So, if the rf B_1 field is known at the point of consideration, then from the signal obtained one should be able to use the above equations to obtain the flow velocity (assumed to be along $\times$ direction) at that point.

In all the derivations above it is assumed that $B_1(x, t)$ is uniform in the yz plane This may not be the case in actual MRI imaging sequences or blood flow mapping. However, we believe that Eqs. (36)–(37) or (45) can be applied in such cases also for computation of the magnetization over and hence the MRI signal from a given pixel, if the distributions of $B_1(x, t)$ over the yz plane is known.

5.3. Applications of the Above Method of Deriving Eqs. (36), (37) and (45) to Diffusion MRI

Diffusion NMR/MRI equations are gaining importance recently[24] (Faromika, Ph.D Thesis 2012).[13] Diffusion MRI was introduced in the mid-1980s[25,26] shows that the most successful application of diffusion MRI since the early 1990s has been in brain ischemia, following the discovery in cat brain by Moseley et al. (1990)[27] that water diffusion drops at a very early stage of the ischemic event. This pioneering work on diffusion anisotropy really took off with the introduction of the more rigorous formalism of the *diffusion tensor* by Basser et al.[28–30] With diffusion tensor imaging (DTI), diffusion anisotropy effects in diffusion MRI data could be fully extracted, characterized, and exploited, providing even more exquisite details of tissue microstructure. Many studies have been published dealing with the optimization of the MRI sequences necessary to gain access to the diffusion tensor, the processing and display of DTI data, and, of course, potential applications. The most advanced application is certainly that of fibre tracking in the brain, which, in combination with functional MRI, might open a window onto the important issue of connectivity. Basser and Jones[31] shows that when implanted cells proliferate and become organized into tissue, the overall diffusion of water decreases because the molecules encounter additional physical barriers. Diffusion weighted MRI provides a means to identify the emerging tissue structure through the reduction of the apparent diffusion coefficient. However, so far diffusion MRI/NMR or diffusion magnetization with and without flow has not been given thorough theoretical foundation, specially, when relaxation times are to be considered. We expect that our above methodology will provide the correct theoretical equation describing diffusion magnetization with flow and relaxation effects. It may be mentioned that Bloch Torrey equation[32] describes diffusion magnetization without relaxation effects and flow and therefore, from the physics point of view it is not applicable to a realistic situation that can be studied by MRI. The reason is that whether magnetization undergoes diffusion with or without flow velocity it must be governed by relaxation effects (when in NMR/MRI both rf B_1 field and bias B_o field are present). We shall report later the equations for M_y and M_x magnetizations when both diffusion

and flow are present along with relaxation effects. Such formulations (not reported yet) then can open door for correct assessment of relevant diffusion parameters from the measured signals for application to medical diagnostics and petroleum industry.

6. CONCLUSION

We have derived NMR/MRI blood flow magnetization equation in the rotating frame of reference. This equation which is the first of its kind is expected to serve as the mother equation for the nuclear spin flow magnetization for all NMR/MRI experiments applicable to non-invasive blood flow estimation. We have shown that the corresponding equation that is in literature and which is mostly due to the work of Awojyogbe and his group, can be derived from our original equation under correct assumptions. The method of deriving the equation can be extended to NMR/MRI spin flow magnetization equation in the Laboratory frame and to Bloch-Torey equation for inclusion of flow of spins and relaxation times. It is needless to mention that the correct signal expression can be derived and the flow signal computed, once the correct dependence of magnetization components of flowing nuclear spins on flow velocity, rf B_1 field, spatial coordinates, and relaxation times etc. is known. This is expected to enhance the accuracy of blood flow estimation by NMR/MRI experiments.

Acknowledgment: The author acknowledges the facilities provided by the Physics Department of University of JOS to carry out the above work. The author (D. K. De) also acknowledges the opportunity provided by Professor O. B. Awojoyogbe to examine his student's (Faromika O. P.) Ph.D. thesis (University of Akure, Nigeria) that prompted the above work.

References and Notes

1. M. B. Scheidegger, S. E. Maier, and P. Boesiger, *Magn. Reson. Imaging* 9, 517 **(1992)**.
2. P. Boesiger, M. B. Scheidegger, S. E. Maier, K. Liu, and D. Maier, *J. Biomed. Engn.* 25, 55 **(1992)**.
3. K. Liu: Ph.D. Thesis, Magnetic Resonance Imaging for Imaging for the Acquisition of Vectorial Flow Velocity Pattern and Accurate Vessel Geometry, Inst. of Biomedical Eng. and Medical Informatics, University and ETH Zurich **(1992)**.
4. F. Stahlberg, *Magn. Reson. Imaging* 10, 13 **(1992)**.
5. P. Schmalbrock, C. Yuan, D. W. Chakers, J. Koli, and N. J. Pelc, *MR Angiography: Radiology* 175, 861 **(1990)**.
6. D. K. De, *Physics in Medicine and Biology* 35, 197 **(1990)**.
7. E. O. Odoh and D. K. De, *African Physical Review* 3, 0012 **(2009)**.
8. M. Nasim, D. Weiying, and A. C. David, *NMR in Biomedicine* 24, 104 **(2011)**.
9. M. Varela, A. M. Groves, T. Arichi, and J. V. Hajnal, *NMR in Biomedicine* 25, 1063 **(2012)**.
10. O. B. Awojoyogbe, *Physica A* 303, 163 **(2002)**.
11. O. B. Awojoyogbe, *Physica A* 323c, 534 **(2003)**.
12. O. B. Awojoyogbe, *Physica* 339, 437 **(2004)**.
13. O. B. Awojoyogbe, *Physica Scripta*. 75, 788 **(2007)**.
14. O. B. Awojoyogbe and K. A. Boubaker, *Curr. Appl. Phys.* 9, 271 **(2008)**.
15. O. B. Awojoyogbe, M. Dada, O. P. Faromika, O. E. Dada, *Concepts in Magnetic Resonance Part A* 38A, 85 **(2011)**.
16. O. B. Awojoyogbe and K. A. Salako, *ICTP Publication, Preprint Miramare-Trieste* 1–12, (Available at: http://www.ictp.it/~pub_off) IC/2005/064 **(2005)**.
17. O. B. Awojoyogbe, M. Dada, and M. Agida, *J. Math. Model App. Comput.* 1, 788 **(2008)**.
18. O. B. Awojoyogbe, M. Dada, O. P. Faromika, O. F. Moses, and I. A. Fuwape, *Open Magnetic Resonance Journal* 2, 46 **(2009)**.
19. O. B. Awojoyogbe, O. P. Faromika, Folorunsho, M. Dada, I. A. Fuwape, and K. Boubaker, *Current Applied Physics* 10, 289 **(2010)**.
20. O. B. Awojoyogbe, O. P. Faromika, M. Dada, O. S. Ojambati, and K. Boubaker, *J. Med. Syst.* DOI: 10.1007/s10916-009-9428-9.
21. D. Micahel, O. B. Awojoyogbe, M. Hasler, K. B. Mahmoud, and B. Amine, *Applications and Applied Mathematics An International Journal* (*AAM*) 3, 329 **(2008)**.
22. V. Jain, M. C. Langham, and F. W. Wehrli, *Journal of Cerebral Blood Flow and Metabolism* 30, 1598 **(2010)**.
23. E. O. Odoh and D. K. De, *Journal of Institute of Mathematics and Computer Sciences* (*India*) 22, 77 **(2009)**.
24. O. P. Faromika, Ph.D. Thesis, Department of Physics, University of Akure, Nigeria **(2012)**.
25. D. G. Taylor and M. C. Bushell, *Phys. Med. Biol.* 30, 345 **(1985)**.
26. S. Warach, D. Chien, W. Li, M. Ronthal, and R. R. Edelman, *Neurology* 42, 1717 **(1992)**.
27. M. E. Moseley, Y. Cohen, and J. Mintorovitch, *Magn. Reson. Med.* 14, 330 **(1990)**.
28. P. J. Basser, J. Mattiello, R. Turner, and L. D. Bihan, *Proceedings of the SMRM* 584 **(1993)**.
29. P. J. Basser, J. Mattiello, and L. D. Bihan, *J. Magn. Reson.* 103, 247 **(1994)**.
30. P. J. Basser, J. Mattiello, and L. D. Bihan, *Biophysical Journal* 66, 259 **(1994)**.
31. P. J. Basser and D. K. Jones, *NMR Biomed.* 15, 456 **(2002)**.
32. H. C. Torrey, *Physical Review* 104, 563 **(1956)**.

ARTICLE