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ANALYTICAL SOLUTION TO BLOCH EQUATIONS FOR CW NMR FLOW MAGNETIZATION SIGNAL WITH A NON VARYING B_1 rF FIELD USING LAPLACE TRANSFORM

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ABSTRACT

Close-form solutions of NMR Bloch equation with flow dependent magnetization are hitherto not available as such solutions are truly non-trivial. Using the Bloch equations and a time derivative of magnetization related to velocity gradient we have developed a second order differential equation whose solution using Laplace transform has provided a velocity dependent CW NMR signal. It is hoped that the practical application of our theoretical investigation will provide a better insight in blood vessel hemodynamics, estimation and elimination of static tissue signal in CW NMR flow measurement which can be afforded by people in developing countries.

1. Introduction

The understanding of blood flow dynamic is considered very important (Caro et al., 1978; Odoh, 2008) for a number of clinical applications. Nuclear magnetic resonance/magnetic resonance imaging is currently one useful technique for obtaining information on blood flow in human vessels. However, there has not been any satisfactory flow signal equation that has given a completely correct and adequate understanding of the dependence of CW NMR signal on blood flow characteristics in human even though it provides a cheaper mean of blood flow estimation (Odoh and De, 2008). It is desirable to find suitable NMR techniques so that the measurement of NMR signal strength can yield knowledge of blood flow rates in human patients despite the finite size of blood vessels, small field inhomogeneity ($\approx$ 1 mG), static tissue signal from tissues surrounding the blood vessel, variation of effective time from patient to patient (Caro et al., 1978; Battocletti et al., 1979). Using the techniques one would be able to obtain reliable estimate of blood flow rates and other medically relevant parameters, such as blood cross-section and relaxation time.

However, literature does not abound in close-form solutions of NMR Bloch equations with flow dependent magnetization. Such solutions are truly non-trivial. Several studies have attempted to solve the Bloch equations as ways of exploring the benefits continuous wave nuclear magnetic resonance has in structural studies of certain materials (Canet, 1996) and in quantitatively explaining the estimation of blood flow rates (Battocletti, 1986 and Awojoyogbe, 1997. In trying to do this they employed some assumptions and transformations. The use of integrating factor to get the particular and general solution of the resulting differential equation using certain boundary conditions (Stroud, 1996) arising from such transformation make their methods cumbersome and lead to some errors. So far no satisfactory solution has been given that could reliably be used to account for some physiological dysfunction of the heart and other cardiovascular disorders.

2. The Laplace Transform Solution To The Bloch Equations

In our approach, we have tried to solve the Bloch equations using a Laplace transform method which has the advantage of avoiding the problems stated above after the appropriate initial conditions have been defined for the magnetized blood bolus entering into the NMR system. We give below details of our approach.

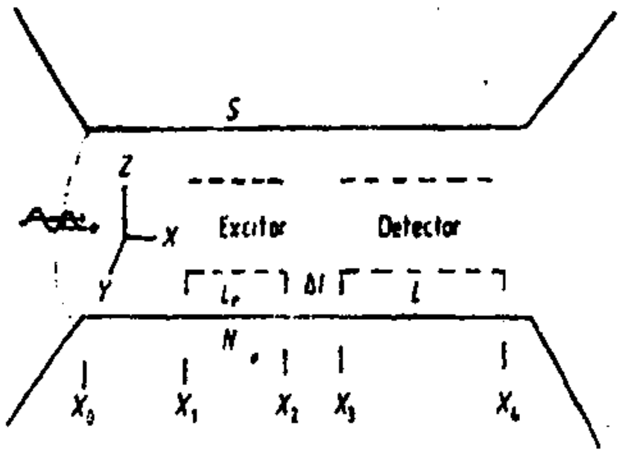

Figure 1 : Diagram Of The CW NMR Excitation Scheme With Separate Movable Detection System For Accurate Estimation Of Steay Blood Velocity, The Pulse Flow Velocity And Also The Total Cross-Section Of The Blood Vessel. L_e Is The Length Of The Excitor Coil; Δl Is The Separation Of The Excitor And The Detector Coil Whose Length Is L.

It is assumed that the blood or fluid protons, prior to entering the excitor coil (Figure 1) are magnetized by a static B_0 field to an equilibrium magnetization M_0, that is given by a well known Brillouin Functions (Odoh and De, 2008). A similar has been used by De (1990) to investigate pulsatile and steady flow of blood using CWNMR. The time dependent rF B_1 field is in the laboratory X direction which coincides with the axis of the excitor and detector coils.

The Bloch equations in the laboratory frame of axes are given by

$$\frac{dM}{dt} = \gamma(M \times B) + \text{Relaxation terms} \tag{1}$$

with

$$M = iM_X + jM_Y + kM_Z \tag{2}$$

and

$$B = kB_0 + iB_1(t) \tag{3}$$

The rF $B_1(t)$ field usually is of the form $B_1(t) = B_{10} \cos \omega t$

It can be viewed as two rotary magnetic fields (rotating with angular frequency ω) and amplitude $B_{10}/2$, in clockwise counterclockwise directions (Slichhter, 1978). One of this does not aid rF absorption and at resonance and is so discarded.

Even when M is not flow dependent solutions of the above equations in the laboratory frame of axes is not easy. One therefore resorts to a rotating frame of axes, whose z axis coincides with Z axis of the laboratory frame and xy axes of rotating frame rotates with angular frequency ω about the x axis. The x axis of the rotating frame makes an angle ωt with the X axis in the laboratory frame. In this rotating frame $B_1(t)$ is viewed as time independent and equation (3) is written as

$$B = kB_0 + B_{10}/2 = kB_0 + iB_1'$$

and in the absence of blood flow,

$$\left(\frac{dM}{dt}\right)_{lab} = \left(\frac{dM}{dt}\right)_{rot} + (\Omega \times M)_{rot} \tag{4}$$

where $\Omega = \omega - \gamma B_0$

At resonance $\Omega = 0$ and

$$\left(\frac{dM}{dt}\right)_{lab} = \left(\frac{dM}{dt}\right)_{rot} \tag{5}$$

with rF B_1 field time independent in the rotating frame. When flow velocity is time independent, i.e., for steady flow velocity, V,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V.grad \tag{6}$$

The B_1 field is time independent only when viewed from the rotating frame which is rotating about the Z axis of the fixed laboratory frame with the angular frequency ω of the field. The axis in the latter frame coincides with the laboratory z-axis, the rotating frame x makes an angle, ωt at any instant of time, t with the laboratory X axis. The x, y, z components (in the rotating frame) of magnetization of a fluid bolus are then given from equations (7) to (9)

$$\frac{dM_x}{dt} = V.gradM_x + \frac{\partial M_y}{\partial t} = \frac{-M_x}{T_2} \tag{7}$$

$$\frac{dM_y}{dt} = V.gradM_y + \frac{\partial M_y}{\partial t} = \gamma\, M_z B_1(x) - \frac{-M_y}{T_2} \tag{8}$$

$$\frac{dM_z}{dt} = V.gradM_z + \frac{\partial M_z}{\partial t} = -\gamma\, M_y B_1(x) + \frac{(M_0 - M_z)}{T_1} \tag{9}$$

To calculate M_x, M_y and M_z one needs to have initial boundary conditions. A reasonable boundary condition is that before entering the excitor coil, the blood bolus has magnetization

$$M_x = 0,\ M_y = 0$$

If $B_1(x)$ is large, $B_1(x) \cong 1\text{G}$ or more so that M_z of the fluid bolus changes appreciably from M_0, for steady flow in X direction: $\dfrac{\partial M_y}{dt} = 0$

From equation (8) we can write

$$M_z = \left(\frac{VdM_y}{dx} + \frac{M_y}{T_2}\right)\frac{1}{\gamma B_1(x)} \tag{10}$$

If we substitute for M_z in equation (9) with $B_1(x)$ written simply as B_1 for a non-varying field, we have

$$\frac{V^2 d^2 M_y}{dx^2} + V\left(\frac{1}{T_1} + \frac{1}{T_2}\right)\frac{dM_y}{dx} + \left(\gamma^2 B_1^2 + \frac{1}{T_1 T_2}\right)M_y = \frac{\gamma V_1 M_0}{T_1} \tag{11}$$

For convenience, we assume that $\gamma^2 B_1^2 >> \frac{1}{T_1 T_2}$

One can then write

$$\frac{d^2 M_y}{dx^2} + \frac{1}{V}\left(\frac{1}{T_1} + \frac{1}{T_2}\right)\frac{dM_y}{dx} + \frac{\gamma^2 B_1^2}{V^2} M_y = \frac{\gamma B_1 M_0}{V^2 T_1} \tag{12}$$

We define $R = \frac{1}{V}\left(\frac{1}{T_1} + \frac{1}{T_2}\right) = \frac{1}{VT_0};\ \frac{1}{T_0} = \frac{1}{T_1} + \frac{1}{T_2}$

Equation (12) can be written

$$\frac{d^2 M_y}{dx^2} + R\frac{dM_y}{dx} + PM_y = Q \tag{13}$$

where

$$P = \frac{\gamma^2 B_1^2}{V^2} \text{ and } Q = \frac{\gamma B_1 M_0}{V^2 T_1} \tag{14}$$

We shall solve equation (13) using Laplace Transform. We then use the following relation

$$\Im\{M_y\} = \overline{M}_y \tag{15}$$

$$\Im\{\dot{M}_y\} = s\overline{M}_y - M_{y_0} \tag{16}$$

$$\Im\{\ddot{M}_y\} = s^2\overline{M}_y - sM_{y_0} - M_{y_1} \tag{17}$$

Substituting equations (15), (16) and (17) in equation (13), we obtain

$$s^2\overline{M}_y - sM_{y_0} - M_{y_1} + Rs\overline{M}_y - RM_{y_0} + P\overline{M}_y = \frac{Q}{s} \tag{18}$$

We shall impose the following boundary conditions:

$$x = 0,\ M_{y_0} = 0 \text{ and } \frac{dM_y}{dx} = 0 \text{ at } M_y = 0$$

In this case, equation (18) becomes

$$s^2\overline{M}_y + Rs\overline{M}_y + P\overline{M}_y = \frac{Q}{s} \tag{19}$$

$$\overline{M}_y\left(s^2 + Rs + P\right) = \frac{Q}{s}$$

$$\overline{M}_y = \frac{Q}{s(s^2 + RS + P)} \tag{20}$$

Using partial fraction approach,

$$\overline{M}_y = \frac{Q}{s(s^2 + RS + P)} = \frac{A}{s} + \frac{Bs + C}{s^2 + Rs + P}$$

Or

$$\overline{M}_y = \frac{A}{s} + \frac{Bs + c}{s^2 + Rs + P} \tag{21}$$

Multiplying equation (21) through by $s(s^2 + Rs + P)$ gives

$$Q = A(s^2 + Rs + P) + s(Bs + C) \tag{22}$$

If $s = 0$, equation (22) becomes $Q = AP$

Therefore, $A = \frac{Q}{P}$

By inspection of coefficients involving s^2,

$0 = A + B$, so that $B = -A = -\frac{Q}{P}$

Similarly, by in inspection of coefficients involving s,

$ARs + Cs = 0$ or $s(AR + C) = 0$

Therefore,

$$C = -AR = -\frac{QR}{P}$$

Substituting the values of A, B and C in equation (15),

$$\overline{M}_y = \frac{Q}{Ps} + \frac{\left(-\frac{Q}{P}\right)s - \left(\frac{Q}{P}\right)R}{s^2 + Rs + P} \tag{23}$$

To have equation (23) in a form of Laplace Transform, we make the denominator of the second term on the right side a perfect square.

$$\overline{M}_y = \frac{Q}{Ps} + \frac{\left(-\frac{Q}{P}\right)s - \left(\frac{Q}{P}\right)R}{\left(s + \frac{R}{2}\right)^2 + P - \frac{R^2}{4}} \tag{24}$$

Equation (24) can be written as

$$\overline{M}_y = \frac{Q}{Ps} + \frac{\left(-\frac{Q}{P}\right)\left(s + \frac{R}{2}\right) - \left(\frac{Q}{P}\right)\left(R - \frac{R}{2}\right)}{\left(s + \frac{R}{2}\right)^2 + \frac{4P - R^2}{4}} \quad \text{Or} \quad \overline{M}_y = \frac{Q}{Ps} + \frac{\left(-\frac{Q}{P}\right)\left(s + \frac{R}{2}\right) - \left(\frac{Q}{P}\right)\left(\frac{R}{2}\right)}{\left(s + \frac{R}{2}\right)^2 + \frac{4P - R^2}{4}} \tag{25}$$

which also can be written as

$$\overline{M}_y = \frac{Q}{P}\left[\frac{1}{s} - \frac{\left(s+\frac{R}{2}\right)}{\left(s+\frac{R}{2}\right)^2 + \frac{4P-R^2}{4}} - \frac{\left(\frac{R}{2}\right)}{\left(s+\frac{R}{2}\right)^2 + \frac{4P-R^2}{4}}\right] \tag{26}$$

Let $\frac{R}{2} = a$ and $\frac{4P-R^2}{4} = b^2$ so that $b = \left(\frac{4P-R^2}{4}\right)^{\frac{1}{2}}$

Equation (26) will then be written as

$$\overline{M}_y = \frac{Q}{P}\left[\frac{1}{s} - \frac{s+a}{(s+a)^2+b^2} - \frac{ba}{b\left((s+a)^2+b^2\right)}\right] \tag{27}$$

Taking the inverse Laplace Transform of equation (27),

$$M_y = \frac{Q}{P}\left[1 - e^{-ax}\cos bx - \frac{a}{b}e^{-ax}\sin bx\right] \tag{28}$$

Using equation (14),

$$\frac{Q}{P} = \frac{M_0}{\gamma B_1 T_1} \text{ and } b = \left(\frac{4p-R^2}{4}\right)^{\frac{1}{2}} - \left(\frac{4\gamma^2 B_1^2 T_0^2 - 1}{4V^2 T_0^2}\right)^{\frac{1}{2}} \tag{29}$$

$\frac{1}{4V^2T_0^2} << 1$, so that

$$b = \left(\frac{4\gamma^2 B_1^2 T_0^2}{4V^2 T_0^2}\right)^{\frac{1}{2}} = \frac{\gamma B_1}{V}; \; a = \frac{R}{2} = \frac{1}{2VT_0}$$

Substituting for $\frac{Q}{P}$ and $\frac{a}{b}$ in equation (28),

$$M_y = \frac{M_0}{\gamma B_1 T_1}\left[1 - e^{-ax}\cos bx - \frac{1}{2\gamma B_1 T_0}e^{-ax}\sin bx\right] \tag{30}$$

The laboratory component, M_{y_0} is given by $M_{y_0} = M_y \cos \omega t$ (31)

The NMR signal detected in the receiver coil of length, L_D is defined

$$E.M.F = I_{FS} = c'\Omega_P \omega \int_0^{L_D} M_{y_0} dx \tag{32}$$

where c' = instrument factor, Ω_P is the blood vessel cross-section.

Substituting equation (30) in equation (32), gives

$$I_{FS} = \frac{\omega c' \Omega_P M_0 \cos\omega t}{\gamma B_1 T_1}\left[\int_0^{L_D}\left(1 - e\cos bx - Ke^{-ax}\sin bx\right)dx\right] \text{ where } K = \frac{1}{2\gamma B_1 T_0}$$

Or

$$I_{FS} = \frac{\omega c' \Omega_P M_0 \cos\omega t}{\gamma B_1 T_1}\left[\int_0^{L_D} dx - \int_0^{L_D} e^{-ax}\cos bx\, dx - K\int_0^{L_D} e^{-ax}\sin bx\, dx\right] \tag{33}$$

Equation (33) can now be integrated term by term with the last two terms on the right evaluated using integration by parts. These two terms are of the form

$$\int_0^{L_D} e^{\alpha x}\cos\beta x\, dx - K\int_0^{L_D} e^{\alpha x}\sin\beta x\, dx \tag{34}$$

with $\alpha = -a$, and $\beta = b$.

The result of equation (34) when substituted in equation (33) is

$$I_{FS} = \frac{\omega c' \Omega_P M_0 \cos\omega t}{\gamma B_1 T_1}\left[x - \frac{e^{\alpha x}(\beta\sin\beta x + \alpha\cos\beta x)}{\alpha^2+\beta^2} - K\frac{e^{\alpha x}(-\beta\cos\beta x + \alpha\sin\beta x)}{a^2+\beta^2}\right]_{x=0}^{x=L_D} \tag{35}$$

The last two terms in the square bracket of equation (35) can be written as

$$\begin{aligned}&\frac{e^{\alpha x}}{\alpha^2+\beta^2}[\beta(\sin\beta x - K\cos\beta x) + \alpha(\cos\beta x + K\sin\beta x)]\\ &= \frac{e^{\alpha x}}{\alpha^2+\beta^2}[\sin\beta x(\beta+\alpha K) + \cos\beta x(\alpha - \beta K)]\end{aligned} \tag{36}$$

Let $\beta + \alpha K = A\cos\theta$ (37)

$\alpha - \beta K = A\sin\theta$ (38)

Using equation (37) and (38) equation (36) becomes

$$\frac{e^{\alpha x}}{\alpha^2+\beta^2}[A\sin\beta x\cos\theta + a\cos\beta x\sin\theta] = \frac{e^{\alpha x}}{\alpha^2+\beta^2}[A\sin(\beta x+\theta)] \tag{39}$$

Squaring equation (37) and (38) and adding them, we can write that

$$A^2 = (\beta+\alpha K)^2 + (\alpha-\beta K)^2 = \beta^2+\alpha^2+K^2(\alpha^2+\beta^2) = (\alpha^2+\beta^2)(1+K^2) \tag{40}$$

$$A = \left(\sqrt{\alpha^2+\beta^2}\right)\left(\sqrt{1+K^2}\right) \tag{41}$$

We now substitute the value of A in equation (39) to get

$$\begin{aligned}\frac{e^{\alpha x}}{\alpha^2+\beta^2}[A\sin(\beta x+\theta)] &= \frac{\left(\sqrt{\alpha^2+\beta^2}\right)\left(\sqrt{1+K^2}\right)e^{\alpha x}\sin(\beta x+\theta)}{\alpha^2+\beta^2}\\ &= \frac{\sqrt{1+K^2}}{\sqrt{\alpha^2+\beta^2}}e^{\alpha x}\sin(\beta x+\theta)\end{aligned} \tag{42}$$

The signal as given by equation (35) can therefore be written as

$$I_{FS} = \frac{\omega c' \Omega_P M_0 \cos\omega t}{\gamma B_1 T_1} \left[x - \frac{\sqrt{1+K^2}}{\sqrt{\alpha^2+\beta^2}} e^{\alpha x} \sin(\beta x + \theta) \right]_{x=0}^{x=L_D} \tag{43}$$

$$I_{FS} = \frac{\omega c' \Omega_P M_0 \cos\omega t}{\gamma B_1 T_1} \left[L_D - \frac{\sqrt{1+K^2}}{\sqrt{\alpha^2+\beta^2}} \left\{ e^{\alpha L_D} \sin(\beta L_D + \theta) - \sin\theta \right\} \right] \tag{43}$$

From equation (38), $\sin\theta = \dfrac{\alpha - \beta K}{A}$

The NMR signal detedted in the receiver coil of length, L_D is therefore given as

$$I_{FS} = \frac{\omega c' \Omega_P M_0 \cos\omega t}{\gamma B_1 T_1} \left[L_D - \frac{\sqrt{1+K^2}}{\sqrt{\alpha^2+\beta^2}} \left\{ e^{\alpha L_D} \sin(\beta L_D + \theta) - \frac{\alpha - \beta K}{\sqrt{\alpha^2+\beta^2}\sqrt{1+K^2}} \right\} \right] \tag{44}$$

Equation (44) provides us with an NMR flow signal which clearly depends on the velocity of the blood flowing in the blood vessel. The first term in equation (44) represents flow independent CW NMR signal expected from static tissue from the blood vessels. The second term is flow dependent and is also seen to depend on L_D. The actual signal would consist of another term due to static tissue surrounding the blood vessels. The later part of the equation actually overwhelms the signal due to blood flow, i.e., the second term of equation (44). However, with the above expression, an algorithm can easily be formulated to eliminate the static tissue signal and quantify the blood flow rate. The blood steady flow velocity, the blood vessel cross-section whose knowledge is very important in case of atherosclerotic plaque for understanding of blood haemodynamic, can therefore readily be quantified. These are no doubt clinical parameters of important to human and their knowledge is very necessary any time in any patient.

3. Conclusion

Using the Bloch equations and a time derivative of magnetization related to velocity gradient, we have arrived at a second order differential equation. We have solved the developed equation using Laplace transform. Our calculation provided once more a CW NMR signal for steady flow of blood that relates to the velocity of blood in the human blood vessels. Using the derived relation one can easily form algorithm to eliminate static tissue signal from the measured signal and extract the signal that corresponds to blood flow rate. The blood flow information can then be obtained.

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