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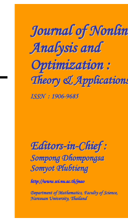

CONVERGENCE THEOREMS ON ASYMPTOTICALLY GENERALIZED Φ-PSEUDOCONTRACTIVE MAPPINGS IN THE INTERMEDIATE SENSE

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ABSTRACT. In this study, we introduce the class of asymptotically generalized Φ- pseudocontractive mappings in the intermediate sense and prove the convergence of Mann type iterative scheme to their fixed points. Our results improves and generalizes the results of Kim *et al.* [J. K. Kim, D. R. Sahu, Y. M. Nam, Convergence theorem for fixed points of nearly L-Lipschitzian mappings, Nonlinear Analysis 71 (2009) 2833-2838] and several others.

KEYWORDS : Asymptotically generalized Φ-pseudocontractive mappings in the intermediate sense, Banach spaces, Mann type iterative scheme, strong convergence, unique fixed point.
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1. INTRODUCTION

Let E be an arbitrary real normed linear space with dual E^*. We denote by J the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) := \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \tag{1.1}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing.

In the sequel, we give the following definitions which will be useful in this study

Definition 1.1. Let C be a nonempty subset of real normed linear space E. A mapping $T : C \longrightarrow E$ is said to be

(1) *strongly pseudocontractive* [12] if for all $x, y \in C$, there exist constant $k \in (0, 1)$ and

$j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y) \rangle \le k\|x - y\|^2, \tag{1.2}$$

(2) ϕ-*strongly pseudocontractive* [12] if for all $x, y \in C$, there exist strictly increasing

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function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|, \tag{1.3}$$

The class of ϕ-strongly pseudocontractive mappings includes the class of strongly pseudocontractive mappings by setting $\phi(s) = ks$ for all $s \in [0,\infty)$. However, the converse is not true (see, e.g. Hirano and Huang [10]).

(3) *generalized Φ-pseudocontractive* [1, 6] if for all $x, y \in C$, there exist strictly increasing

function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ with $\Phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|), \tag{1.4}$$

(4) *asymptotically generalized Φ-pseudocontractive* [12] with sequence $\{k_n\}$ if for each $n \in \mathbb{N}$ and $x, y \in C$, there exist constant $k_n \geq 1$ with $\lim_{n\longrightarrow\infty} k_n = 1$, strictly increasing function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ with $\Phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n\|x-y\|^2 - \Phi(\|x-y\|), \tag{1.5}$$

The class of asymptotically generalized Φ-pseudocontractive was introduced by Kim *et al.* [12].

Definition 1.2. [19]. A mapping $T : C \longrightarrow C$ is said to be *asymptotically pseudocontractive mapping in the intermediate sense* if

$$\limsup_{n\longrightarrow\infty} \sup_{x,y\in C} \left(\langle T^n x - T^n y, x-y\rangle - k_n\|x-y\|^2\right) \leq 0, \tag{1.6}$$

where $\{k_n\}$ is a sequence in $[1,\infty)$ such that $k_n \longrightarrow 1$ as $n \longrightarrow \infty$. Put

$$\nu_n = \max\left\{0, \sup_{x,y\in C} \left(\langle T^n x - T^n y, x-y\rangle - k_n\|x-y\|^2\right)\right\}. \tag{1.7}$$

It follows that $\nu_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then, (1.6) is reduced to the following:

$$\langle T^n x - T^n y, x-y\rangle \leq k_n\|x-y\|^2 + \nu_n,\ \forall n \geq 1,\ x, y \in C. \tag{1.8}$$

Qin *et al.* [19] introduced the class of asymptotically pseudocontractive mappings in the intermediate sense. They proved weak convergence theorems for this class of nonlinear mappings. They also established some strong convergence results without any compact assumption by considering the hybrid projection methods. Olaleru and Okeke [17] in 2012 proved a strong convergence of Noor type scheme for a uniformly L-Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense without assuming any form of compactness.

Motivated by the above facts, we now introduce the following class of nonlinear mappings

Definition 1.3. Let C be a nonempty subset of real normed linear space E. A mapping $T : C \longrightarrow C$ is said to be *asymptotically generalized Φ-pseudocontractive mapping in the intermediate sense* with sequence $\{k_n\}$ if for each $n \in \mathbb{N}$ and $x, y \in C$, there exists constant $k_n \geq 1$ with $\lim_{n\longrightarrow\infty} k_n = 1$ and strictly increasing function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ with $\Phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\limsup_{n\longrightarrow\infty} \sup_{x,y\in C} \left(\langle T^n x - T^n y, j(x-y)\rangle - k_n\|x-y\|^2 + \Phi(\|x-y\|)\right) \leq 0. \tag{1.9}$$

Put

$$\tau_n = \max\left\{0, \sup_{x,y\in C} \left(\langle T^n x - T^n y, j(x-y)\rangle - k_n\|x-y\|^2 + \Phi(\|x-y\|)\right)\right\}. \tag{1.10}$$

It follows that $\tau_n \longrightarrow 0$ as $n \longrightarrow \infty$. Hence (1.9) is reduced to the following

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n\|x-y\|^2 + \tau_n - \Phi(\|x-y\|). \tag{1.11}$$

We remark that if $\tau_n = 0$ for all $n \in \mathbb{N}$, the class of asymptotically generalized Φ-pseudocontractive mapping in the intermediate sense is reduced to the class of asymptotically generalized Φ-pseudocontractive.

Example 1.4. Let $E = \mathbb{R}^1$ and $C = [c, \infty)$, where $c > 0$ is any given constant. Define the mapping $T : C \longrightarrow 2^E$ by

$$Tx = \begin{cases} [0, c], \text{ if } x = c, \\ \frac{k(x-c)^2}{1+(x-c)}, \text{ if } x > c, \end{cases}$$

where $k \in (0, 1)$.

Clearly, T has a unique fixed point $p = c \in C$. Define $\Phi : [0, \infty) \longrightarrow [0, \infty)$ by $\Phi(t) = \frac{t^2}{(1+t)}$. Clearly, Φ is strictly increasing and $\Phi(0) = 0$. Now, for each $x \in C$, we have

$$\begin{aligned} \langle T^n x - T^n p, j(x-p)\rangle &= \tfrac{k(x-c)^3}{1+(x-c)} \\ &= k^n(|x-c|^2 - \tfrac{|x-c|^2}{1+|x-c|}) \\ &\leq k^n|x-p|^2 - \Phi(|x-p|) + k^n. \end{aligned}$$

Hence, T is asymptotically generalized Φ-pseudocontractive mapping in the intermediate sense.

Let C be a nonempty of a normed linear space E. A mapping $T : C \longrightarrow E$ is said to be *Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x-y\| \tag{1.12}$$

for all $x, y \in C$ and *generalized Lipschitzian* [12] if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(\|x-y\| + 1) \tag{1.13}$$

for all $x, y \in C$. A mapping $T : C \longrightarrow C$ is called *uniformly L-Lipschitzian* [12] if for each $n \in \mathbb{N}$, there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x-y\| \tag{1.14}$$

for all $x, y \in C$.

Clearly, every Lipschitzian mapping is a generalized Lipschitzian mapping. Every mapping with a bounded range is a generalized Lipschitzian mapping. The following example shows that the class of generalized Lipschitzian mappings properly contains the class of Lipschitzian mappings and that of mappings with bounded range.

Example 1.5. [4]. Let $E = (-\infty, \infty)$ and $T : E \longrightarrow E$ be defined by $Tx =$

$$\begin{cases} x - 1 & \text{if } x \in (-\infty, -1), \\ x - \sqrt{1-(x+1)^2} & \text{if } x \in [-1, 0), \\ x + \sqrt{1-(x-1)^2} & \text{if } x \in [0, 1], \\ x + 1 & \text{if } x \in (1, \infty). \end{cases}$$

Then T is a generalized Lipschitzian mapping which is not Lipschitzian and whose range is not bounded.

Sahu [20] introduced a new class of nonlinear mappings which is more general than the class of generalized Lipschitzian mappings and the class of uniformly L-Lipschitzian mappings.

Definition 1.6. [20]. Let C be a nonempty subset of a Banach space E and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \longrightarrow 0$.

(1) A mapping $T : C \longrightarrow C$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n) \tag{1.15}$$

for all $x, y \in C$.

The infimum of constants k_n in (1.15) is called *nearly Lipschitz constant* and is denoted by $\eta(T^n)$.

(2) A nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be *nearly uniformly L-Lipschitzian* if $k_n = L$ for all $n \in \mathbb{N}$, i.e.

$$\|T^n x - T^n y\| \leq L(\|x - y\| + a_n) \tag{1.16}$$

and *nearly asymptotically nonexpansive* if $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\longrightarrow\infty} k_n = 1$.

(3) A mapping $T : C \longrightarrow E$ will be called *generalized (M, L)-Lipschitzian* if there exist two constants $L, M > 0$ such that

$$\|Tx - Ty\| \leq L(\|x - y\| + M) \tag{1.17}$$

for all $x, y \in C$.

Observe that the class of generalized (M, L)-Lipschitzian mappings is a generalization of the class of Lipschitzian mappings. Clearly, the class of nearly uniformly L-Lipschitzian mappings properly contains the class of generalized (M, L)-Lipschitzian mappings and the class of uniformly L-Lipschitzian mappings. We remark that every nearly asymptotically nonexpansive mapping is nearly uniformly L-Lipschitzian.

It has been shown by Sahu [20] that the class of nearly uniformly L-Lipschitzian is not necessarily continuous. Sahu [20] extended the results of Goebel and Kirk [8] to demicontinuous mappings and proved that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space, then every demicontinuous nearly asymptotically nonexpansive self-mapping of C has a fixed point.

It is our purpose in this study to use the concept of nearly uniformly L- Lipschitzian (not necessarily continuous) mappings to prove a strong convergence result for the class of asymptotically generalized Φ-pseudocontractive mappings in the intermediate sense in a general Banach space. Our results is an improvement of several other results in literature.

The following Lemmas will be useful in this study

Lemma 1.1. [3]. Let E be a Banach space. Then for each $x, y \in E$, there exists $j(x + y) \in J(x + y)$ such that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle.$$

Lemma 1.2. [18]. Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that

$$\delta_{n+1} \leq (1 + \beta_n)\delta_n + \gamma_n$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then $\lim_{n\longrightarrow\infty} \delta_n$ exists.

Lemma 1.3. [14]. Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\}$ a real sequence in $[0,1]$ such that $\sum_{n=1}^{\infty}\lambda_n = \infty$. If there exists a strictly increasing function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ such that

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n$$

for all $n \geq n_0$, where n_0 is some nonnegative integer and $\{\sigma_n\}$ is a sequence of nonnegative numbers such that $\sigma_n = o(\lambda_n)$, then $\lim_{n\longrightarrow\infty}\theta_n = 0$.

Lemma 1.4. [12]. Let $\{\delta_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\sigma_n\}$ be four sequences of nonnegative numbers such that

$$\delta_{n+1}^2 \leq (1+\beta_n)\delta_n^2 + \gamma_n(\delta_n + \sigma_n)^2$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty}\beta_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\{\sigma_n\}$ is bounded, then $\lim_{n\longrightarrow\infty}\delta_n$ exists.

2. Main Results

We prove the following Lemma which will be needed in this study.

Lemma 2.1. Let $\{\delta_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$ and $\{\rho_n\}$ be five sequences of nonnegative numbers such that

$$\delta_{n+1}^2 \leq (1+\beta_n)\delta_n^2 + \gamma_n(\delta_n + \sigma_n)^2 + \rho_n^2 \tag{2.1}$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty}\beta_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\rho_n < \infty$ and $\{\sigma_n\}$ is bounded, then $\lim_{n\longrightarrow\infty}\delta_n$ exists.

Proof. Using (2.1), we obtain

$$\begin{aligned}
\delta_{n+1}^2 &\leq (1+\beta_n)\delta_n^2 + \gamma_n(\delta_n + \sigma_n)^2 + \rho_n^2 \\
&\leq (1+\beta_n)\delta_n^2 + 2\gamma_n(\delta_n^2 + \sigma_n^2) + \rho_n^2 \\
&\leq (1+\beta_n + 2\gamma_n)\delta_n^2 + 2\gamma_n\sigma_n^2 + \rho_n^2.
\end{aligned} \tag{2.2}$$

Since $\{\sigma_n\}$ is bounded and $\sum_{n=1}^{\infty}\rho_n < \infty$, then by Lemma 1.2, it follows that $\lim_{n\longrightarrow\infty}\delta_n$ exists. The proof of Lemma 2.1 is completed. □

Theorem 2.2. Let C be a nonempty convex subset of a real Banach space E and $T : C \longrightarrow C$ a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$ and *asymptotically generalized* Φ*-pseudocontractive mapping in the intermediate sense* with sequences $\{\tau_n\}$ and $\{k_n\}$ as defined in (1.11) and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the conditions:

(i) $\{\frac{a_n}{\alpha_n}\}$ is bounded,

(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $\sum_{n=1}^{\infty}\alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty}\alpha_n(k_n - 1) < \infty$.

Let $\{x_n\}$ be the sequence in E generated from arbitrary $x_1 \in C$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,\ n \in \mathbb{N}. \tag{2.3}$$

Then the sequence $\{x_n\}$ in C defined by (2.3) converges strongly to a unique fixed point of T.

Proof. Fix $p \in F(T)$, using (1.16) and (2.3) and set

$A_n := 2\alpha_n(k_n - 1) + \alpha_n^2[1 + L(1+L)]$

and

$B_n := 1 - 2\alpha_n k_n - \alpha_n^2 L(1+L)$.

$$\begin{aligned} \|x_{n+1} - x_n\| &= \alpha_n \|T^n x_n - x_n\| \\ &\leq \alpha_n(\|T^n x_n - p\| + \|x_n - p\|) \\ &\leq \alpha_n(L\|x_n - p\| + a_n) + \|x_n - p\|) \\ &\leq \alpha_n(1+L)\|x_n - p\| + a_n L. \end{aligned} \tag{2.4}$$

Using (1.11), (1.16), (2.3), (2.4) and Lemma 1.1, we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\|^2 \\ &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\langle T^n x_n - p, j(x_{n+1} - p)\rangle \\ &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\{\langle T^n x_{n+1} - p, j(x_{n+1} - p)\rangle \\ &\quad + \langle T^n x_n - T^n x_{n+1}, j(x_{n+1} - p)\rangle\} \\ &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\{k_n\|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\ &\quad + \|T^n x_n - T^n x_{n+1}\| \times \|x_{n+1} - p\|\} \\ &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\{k_n\|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\ &\quad + L\left(\|x_{n+1} - x_n\| + a_n\right)\|x_{n+1} - p\|\} \\ &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\{k_n\|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\ &\quad + L\left(\alpha_n(1+L)\|x_n - p\| + a_n L + a_n\right)\|x_{n+1} - p\|\} \\ &= (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\{k_n\|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\ &\quad + \alpha_n L(1+L)\left(\|x_n - p\| + \tfrac{a_n}{\alpha_n}\right)\|x_{n+1} - p\|\} \\ &\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\left\{k_n\|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|)\right\} \\ &\quad + \alpha_n^2 L(1+L)\left\{\left(\|x_n - p\| + \tfrac{a_n}{\alpha_n}\right)^2 + \|x_{n+1} - p\|^2\right\}. \end{aligned} \tag{2.5}$$

From (2.5), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \left(\tfrac{(1-\alpha_n)^2}{1-2\alpha_n k_n - k_n^2 L(1+L)}\right)\|x_n - p\|^2 + \tfrac{2\alpha_n\tau_n}{1-2\alpha_n k_n - \alpha_n^2 L(1+L)} \\ &\quad - \tfrac{2\alpha_n\Phi(\|x_{n+1}-p\|)}{1-2\alpha_n k_n - \alpha_n^2 L(1+L)} + \tfrac{\alpha_n^2 L(1+L)}{1-2\alpha_n k_n - \alpha_n^2 L(1+L)}\left(\|x_n - p\| + \tfrac{a_n}{\alpha_n}\right)^2 \\ &= \left(\tfrac{(1-\alpha_n)^2}{B_n}\right)\|x_n - p\|^2 + \tfrac{2\alpha_n\tau_n}{B_n} - \tfrac{2\alpha_n}{B_n}\Phi(\|x_{n+1} - p\|) \\ &\quad + \tfrac{\alpha_n^2 L(1+L)}{B_n}\left(\|x_n - p\| + \tfrac{a_n}{\alpha_n}\right)^2. \end{aligned} \tag{2.6}$$

From (2.6), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \left(1 + \tfrac{A_n}{B_n}\right)\|x_n - p\|^2 + 2\tfrac{\alpha_n\tau_n}{B_n} - 2\tfrac{\alpha_n}{B_n}\Phi(\|x_{n+1} - p\|) \\ &\quad + \tfrac{2\alpha_n^2 L(1+L)}{B_n}\left(\|x_n - p\| + \tfrac{a_n}{\alpha_n}\right)^2. \end{aligned} \tag{2.7}$$

But $B_n = 1 - 2\alpha_n k_n - \alpha_n^2 L(1+L) \longrightarrow 1$, there exists a number $n_0 \in \mathbb{N}$ such that $\frac{1}{2} < B_n \leq 1$ for each $n \geq n_0$. From (2.7), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1 + 2A_n)\|x_n - p\|^2 + 4\alpha_n\tau_n - 2\alpha_n\Phi(\|x_{n+1} - p\|) \\ &\quad + 4\alpha_n^2 L(1+L)\left(\|x_n - p\| + \tfrac{a_n}{\alpha_n}\right)^2. \end{aligned} \tag{2.8}$$

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1 + 2A_n)\|x_n - p\|^2 + 4\alpha_n\tau_n \\ &\quad + 4\alpha_n^2 L(1+L)\left(\|x_n - p\| + \tfrac{a_n}{\alpha_n}\right)^2. \end{aligned} \tag{2.9}$$

From the conditions $\sum_{n=1}^{\infty}\alpha_n(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\alpha_n^2 < \infty$, it follows that $\sum_{n=1}^{\infty} A_n < \infty$. Since $\{\frac{a_n}{\alpha_n}\}$ is bounded, we have from (2.8) and Lemma 2.1 that $\lim_{n\longrightarrow\infty}\|x_n - p\|$ exists. Hence, $\{x_n\}$ is bounded. Now, we set $M_1 := \sup\{\|x_n - p\| :$

$n \in \mathbb{N}\}$, $M_2 := \sup\{\frac{a_n}{\alpha_n} : n \in \mathbb{N}\}$ and $M_3 := \sup\{\alpha_n \tau_n : n \in \mathbb{N}\}$. Then from (2.8),we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 + 4M_3 - 2\alpha_n \Phi(\|x_{n+1} - p\|) \\ &\quad + 4\alpha_n^2 L(1+L)(M_1 + M_2)^2 + 2A_n M_1^2. \end{aligned} \tag{2.10}$$

Taking $\theta_n = \|x_n - p\|$, $\lambda_n = 2\alpha_n$ and $\sigma_n = 4\alpha_n^2 L(1+L)(M_1+M_2)^2 + 2A_n M_1^2 + 4M_3$, (2.10) reduces to

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n.$$

Hence from Lemma 1.3, it follows that $\|x_n - p\| \longrightarrow 0$. The proof of Theorem 2.2 is completed. □

Corollary 2.3. Let C be a nonempty convex subset of a real Banach space E and $T : C \longrightarrow C$ a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$ and asymptotically generalized Φ-pseudocontractive mapping with sequence $\{k_n\}$ as defined in (1.4) and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the conditions:

(i) $\{\frac{a_n}{\alpha_n}\}$ is bounded,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.

Let $\{x_n\}$ be the sequence in E generated from arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n,\ n \in \mathbb{N}. \tag{2.11}$$

Then the sequence $\{x_n\}$ in C defined by (2.11) converges strongly to a unique fixed point of T.

Remark 2.4. The results of Theorem 2.2 shows that the class of asymptotically generalized Φ-pseudocontractive mappings in the intermediate sense includes the class of asymptotically generalized Φ-pseudocontractive mappings introduced by Kim *et al.* [12]. Furthermore, Our results extended the works of Qin *et al.* [19] and Zegeye *et al.* [22] from Hilbert spaces to the general Banach spaces.

Example 2.5. Let $E = \mathbb{R}$ and $C = [0, 1]$. For all $x \in C$, we define $T : C \longrightarrow C$ by

$$Tx = \begin{cases} (3 - \sqrt{x})^2 & \text{if } x \in [0, 1) \\ 0 & \text{if } x = 1 \end{cases}$$

It is easy to see that T is *asymptotically generalized* Φ*-pseudocontractive mapping in the intermediate sense* with sequence $\{k_n = 1\}$, $\Phi(t) = \frac{t^2}{3}$, $t \in [0, \infty)$ and $\{\tau\} = \frac{1}{n^2}$.

Put $\alpha_n = \frac{1}{n}$. We can see that the conditions (i), (ii) and (iii) of Theorem 2.2 are satisfied.

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