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FIXED POINT THEOREMS FOR NONLINEAR EQUATIONS IN BANACH SPACES

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Abstract. We introduce a new class of nonlinear mappings, the class of ϕ-strongly quasi-accretive operators and approximate the unique common solution of a family of three of these operators in Banach spaces. Our results improves and generalizes the results of Xue and Fan [25], Yang *et al.* [26] and several others in literature.

Keywords: Three-step iterative scheme with errors, Banach spaces, ϕ-strongly quasi-accretive operators, common fixed point, strongly accretive.

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1. Introduction

Let E be a real Banach space, D a nonempty subset of E and $\phi : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ be a continuous strictly increasing function such that $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. We associate a ϕ-normalized duality mapping $J_\phi : E \to 2^{E^*}$ to the function ϕ defined by

$$J_\phi(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|\phi(\|x\|) \; and \; \|f^*\| = \phi(\|x\|)\}, \tag{1.1}$$

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where E^* denotes the dual space of E and $\langle ., . \rangle$ denotes the duality pairing.

We shall denote a single-valued duality mapping by j_ϕ. If $\phi(t) = t$, then J_ϕ reduces to the usual duality mapping J.

The following relationship exists between J_ϕ and J, which can easily be shown.

$$J_\phi(x) = \frac{\phi(\|x\|)}{\|x\|} J(x) \ \forall \ x \neq 0. \tag{1.2}$$

The following definitions was given in [9].

Let $T : D(T) \subset E \to E$ be a mapping with domain $D(T)$ and $F(T)$ be the nonempty set of fixed points of T.

Definition 1.1. [9]. T is said to be ϕ-nonexpansive if for all $x, y \in D(T)$, the following inequality holds:

$$\|Tx - Ty\| \leq \phi(\|x - y\|). \tag{1.3}$$

Definition 1.2. [9]. T is said to be ϕ-uniformly L-Lipschitzian if there exists $L > 0$ such that for all $x, y \in D(T)$

$$\|T^n x - T^n y\| \leq L.\phi(\|x - y\|). \tag{1.4}$$

Definition 1.3. [9]. T is said to be asymptotically ϕ-nonexpansive, if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \phi(\|x - y\|) \ \forall \ x, y \in D(T), \ n \geq 1. \tag{1.5}$$

Every ϕ-nonexpansive mapping is asymptotically ϕ-nonexpansive map. Every asymptotically ϕ-nonexpansive mapping is ϕ-uniformly L-Lipschitzian.

Definition 1.4. [9]. T is said to be asymptotically ϕ-pseudocontractive, if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j_\phi(x - y) \in J_\phi(x - y)$ such that

$$\langle T^n x - T^n y, j_\phi(x - y) \rangle \leq k_n (\phi(\|x - y\|))^2 \ \forall x, y \in D(T), \ n \geq 1. \tag{1.6}$$

Every asymptotically ϕ-nonexpansive mapping is asymptotically ϕ-pseudocontractive mapping.

Example 1.5. [9]. Let $E = \mathbb{R}$ have the usual norm and $D = [0, 2\pi]$. Define $T : D \to \mathbb{R}$ by

$$Tx = \frac{2x \cos x}{3}$$

for each $x \in D$. Define a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(x) = \ln(x+1)$ for each $x \in \mathbb{R}^+$ and take $j_\phi(x-y) = \ln(|x-y|+1)$.

It was shown by Kim and Lee [9] that T is asymptotically ϕ-pseudocontractive mapping.

Definition 1.6. [9]. T is said to be asymptotically ϕ-hemicontractive, if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j_\phi(x-y) \in J_\phi(x-y)$ such that for some $n_0 \in \mathbb{N}$

$$\langle T^n x - y, j_\phi(x-y)\rangle \leq k_n(\phi(\|x-y\|))^2 \ \forall\ x \in D(T),\ y \in F(T)\ n \geq n_0. \tag{1.7}$$

Every asymptotically ϕ-pseudocontractive mapping is asymptotically ϕ-hemicontractive mapping.

Definition 1.7. [20]. T is said to be *asymptotically pseudocontractive mapping in the intermediate sense* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\langle T^n x - T^n y, x-y\rangle - k_n\|x-y\|^2) \leq 0. \tag{1.8}$$

Put

$$\tau_n = \max\left\{0, \sup_{x,y\in C} (\langle T^n x - T^n y, x-y\rangle - k_n\|x-y\|^2)\right\}. \tag{1.9}$$

It follows that $\tau_n \to 0$ as $n \to \infty$. Hence, (1.8) is reduced to the following:

$$\langle T^n x - T^n y, x-y\rangle \leq k_n\|x-y\|^2 + \tau_n, \forall n \geq 1, x, y \in C. \tag{1.10}$$

In real Hilbert spaces, we observe that (1.10) is equivalent to

$$\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x-y\|^2 + \|(I-T^n)x - (I-T^n)y\|^2 + 2\tau_n, \forall n \geq 1, x, y \in C. \tag{1.11}$$

Qin *et al.* [20] recently introduced the class of asymptotically pseudocontractive mappings in the intermediate sense. We remark that if $\tau_n = 0 \ \forall n \geq 1$, then the class of asymptotically pseudocontractive mappings in the intermediate sense is reduced to the class of asymptotically pseudocontractive mappings. Olaleru and Okeke [19] proved some strong convergence results of Noor type iteration for a uniformly L-Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense without assuming any form of compactness.

Bruck *et al.* [2] in 1993 introduced the class of *asymptotically nonexpansive mappings in the intermediate sense* as follows.
The mapping $T : D \to D$ is said to be *asymptotically nonexpansive in the intermediate sense* provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in D} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{1.12}$$

Motivated by the facts above, we introduce the following class of nonlinear operators.

Definition 1.8. A mapping A is called *ϕ-strongly quasi-accretive* if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j_\phi(x - p) \in J_\phi(x - p)$ such that for some $n_0 \in \mathbb{N}$, $x \in D(A)$, $p \in N(A)$, then

$$\langle Ax - Ap, j_\phi(x - p)\rangle \geq k_n(\phi(\|x - p\|))^2. \tag{1.13}$$

The following definitions will be needed in this study.

Definition 1.9. [21]. A map $T : E \to E$ is called strongly accretive if there exists a constant $k > 0$ such that, for each $x, y \in E$, there is a $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y)\rangle \geq k\|x - y\|^2. \tag{1.14}$$

Definition 1.10. [21]. An operator T with domain $D(T)$ and range $R(T)$ in E is called *strongly pseudocontractive* if for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ and a constant $0 < k < 1$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq k\|x-y\|^2. \tag{1.15}$$

The class of strongly accretive operators is closely related to the class of strongly pseudocontractive operators. It is well known that T is strongly pseudocontractive if and only if $(I - T)$ is strongly accretive, where I denotes the identity operator. Browder [1] and Kato [8] indepedently introduced the concept of accretive operators in 1967. One of the early results in the theory of accretive operators credited to Browder states that the initial value problem

$$\frac{du(t)}{dt} + Tu(t) = 0,\ u(0) = u_0 \tag{1.16}$$

is solvable if T is locally Lipschitzian and accretive in an appropriate Banach space. These class of operators have been studied extensively by several authors (see [3, 4, 9, 10, 18, 21, 25]).

In 1953, Mann [11] introduced the Mann iterative scheme and used it to prove the convergence of the sequence to the fixed points for which the Banach principle is not applicable. Later in 1974, Ishikawa [7] introduced an iterative process to obtain the convergence of a Lipschitzian pseudocontractive operator when Mann iterative scheme failed to converge. In 2000 Noor [14] gave the following three-step iterative scheme (or Noor iteration) for solving nonlinear operator equations in uniformly smooth Banach spaces.

Let D be a nonempty convex subset of E and let $T : D \to D$ be a mapping. For a given $x_0 \in K$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative schemes

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n,\ n \geq 0 \end{cases} \tag{1.17}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0, 1]$ satisfying some conditions.

If $\gamma_n = 0$ and $\beta_n = 0$, for each $n \in \mathbb{Z}$, $n \geq 0$, then (1.17) reduces to the iterative scheme

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ n \in \mathbb{Z},\ n \geq 0, \tag{1.18}$$

which is called the one-step (or Mann iterative scheme), introduced by Mann [11].

For $\gamma_n = 0$, (1.17) reduces to:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \ n \geq 0 \end{cases} \tag{1.19}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two real sequences in $[0,1]$ satisfying some conditions. Equation (1.19) is called the two-step (or Ishikawa iterative process) introduced by Ishikawa [6].

In 1989, Glowinski and Le-Tallec [5] used a three-step iterative process to solve elasto-viscoplasticity, liquid crystal and eigenvalue problems. They established that three-step iterative scheme performs better than one-step (Mann) and two-step (Ishikawa) iterative schemes. Haubruge *et al.* [6] studied the convergence analysis of the three-step iterative processes of Glowinski and Le-Tallec [5] and used the three-step iteration to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iteration also lead to highly parallelized algorithms under certain conditions. Hence, we can conclude by observing that three-step iterative scheme play an important role in solving various problems in pure and applied sciences.

Rafiq [21] recently introduced the following modified three-step iterative scheme and used it to approximate the unique common fixed point of a family of strongly pseudocontractive operators.

Let $T_1, T_2, T_3 : D \to D$ be three given mappings. For a given $x_0 \in D$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative scheme

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n \\ y_n = (1-\beta_n)x_n + \beta_n T_2 z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n, \ n \geq 0, \end{cases} \tag{1.20}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ satisfying some conditions. Observe that iterative schemes (1.17), (1.18) and (1.19) are special cases of (1.20).

More recently, Suantai [22] introduced the following three-step iterative schemes. Let E be a normed space, D be a nonempty convex subset of E and $T : D \to D$ be a given mapping. Then for a given $x_1 \in D$, compute the sequence $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ by the iterative scheme

$$\begin{cases} z_n = a_n T^n x_n + (1 - a_n)x_n \\ y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n)x_n \\ x_{n+1} = \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n)x_n, \ n \geq 1, \end{cases} \tag{1.21}$$

where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in $[0, 1]$.

Yang *et al.* [26] in 2009 introduced the following three step iterative scheme. Let E be a normed space, D be a nonempty convex subset of E. Let $T_i : D \to D (i = 1, 2, 3)$ be given asymptotically nonexpansive mappings in the intermediate sense. Then for a given $x_1 \in D$ and $n \geq 1$, compute the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ defined by

$$\begin{cases} x_{n+1} = (1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})x_n + a_{n1}T_1^n y_n + b_{n1}T_1^n z_n + e_{n1}T_1^n x_n + c_{n1}u_n, \\ y_n = (1 - a_{n2} - b_{n2} - c_{n2})x_n + a_{n2}T_2^n z_n + b_{n2}T_2^n x_n + c_{n2}v_n, \\ z_n = (1 - a_{n3} - c_{n3})x_n + a_{n3}T_3^n x_n + c_{n3}w_n, \end{cases} \tag{1.22}$$

where $\{a_{ni}\}$, $\{c_{ni}\}$, $\{b_{n1}\}$, $\{b_{n2}\}$, $\{e_{n1}\}$, $\{a_{n3} + c_{n3}\}$, $\{a_{n2} + b_{n2} + c_{n2}\}$ and $\{a_{n1} + b_{n1} + c_{n1} + e_{n1}\}$ are appropriate sequences in [0,1] for $i = 1, 2, 3$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in D. The iterative schemes (1.22) are called the modified three-step iterations with errors. If $T_1 = T_2 = T_3 = T$ and $e_{n1} \equiv 0$, then (1.22) reduces to the modified Noor iterations with errors defined in [13].

$$\begin{cases} x_{n+1} = (1 - a_{n1} - b_{n1} - c_{n1})x_n + a_{n1}T^n y_n + b_{n1}T^n z_n + c_{n1}u_n, \\ y_n = (1 - a_{n2} - b_{n2} - c_{n2})x_n + a_{n2}T^n z_n + b_{n2}T^n x_n + c_{n2}v_n, \\ z_n = (1 - a_{n3} - c_{n3})x_n + a_{n3}T^n x_n + c_{n3}w_n, \end{cases} \tag{1.23}$$

where $\{a_{ni}\}$, $\{c_{ni}\}$, $\{b_{n1}\}$, $\{b_{n2}\}$ are appropriate sequences in [0,1] for $i = 1, 2, 3$ and $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C.

If $T_1 = T_2 = T_3 = T$ and $b_{n1} = b_{n2} = c_{n1} = c_{n2} = c_{n3} = e_{n1} \equiv 0$, then (1.22) reduces to the Noor iteration defined in [14]. If $b_{n1} = e_{n1} = c_{n1} = b_{n2} = c_{n2} = c_{n3} \equiv 0$, then (1.22) reduces to (1.20). This means that the modified Noor iterative scheme introduced by Rafiq [21] is a special case of the modified three-step iterations with errors introduced by Yang *et al.* [26].

Rafiq [21] in 2006 proved the following theorem

Theorem R. [21]. Let E be a real Banach space and D be a nonempty closed convex subset of E. Let T_1, T_2, T_3 be strongly pseudocontractive self maps of D with $T_1(D)$ bounded and T_1, T_3 be uniformly continuous. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n \\ y_n = (1-\beta_n)x_n + \beta_n T_2 z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n, \ n \geq 0, \end{cases}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ satisfying the conditions:

$\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.
If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2, T_3.

Xue and Fan [25] in 2008 obtained the following convergence results which in turn is a correction of Theorem R.

Theorem XF. [25]. Let E be a real Banach space and D be a nonempty closed convex subset of E. Let T_1, T_2 and T_3 be strongly pseudocontractive self maps of D with $T_1(D)$ bounded and T_1, T_2 and T_3 uniformly continuous. Let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.20), where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ which satisfy the conditions: $\alpha_n, \beta_n \to 0$ as $n \to \infty$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2 and T_3.

In this study, we approximate the common fixed points of a family of three asymptotically ϕ-hemicontractive mappings using the three step iterative scheme (1.22) introduced by Yang *et al.* [26]. Our results improves and generalizes the results of Kim and Lee [9], Xue and Fan [25], Yang *et al.* [26] and several others in literature.

The following lemmas will be needed in this study.

Lemma 1.1. [9]. Let $J_\phi : E \to 2^{E^*}$ be a ϕ-normalized duality mapping. Then for any $x, y \in E$, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\frac{\|x+y\|}{\phi(\|x+y\|)}\langle y, j_\phi(x+y)\rangle \ \forall\ j_\phi(x+y) \in J_\phi(x+y).$$

We remark that if ϕ is an identity, then we have the following inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \ \forall\ j(x+y) \in J(x+y).$$

Lemma 1.2. [23]. Let $\{\rho\}_{n=0}^\infty$ be a nonnegative sequence which satisfies the following inequality:

$$\rho_{n+1} \leq (1-\lambda_n)\rho_n + \sigma_n,\ n \geq 0,$$

where $\lambda_n \in (0,1)$, $n = 0, 1, 2, \cdots, \sum_{n=0}^\infty \lambda_n = \infty$ and $\sigma_n = o(\lambda_n)$. Then $\rho_n \to 0$ as $n \to \infty$.

3. Main results

Theorem 2.1. Let E be a real Banach space and D be a nonempty closed convex subset of E. Let T_1, T_2 and T_3 be asymptotically ϕ-hemicontractive self maps of D with $T_1(D)$ bounded and T_1, T_2 and T_3 uniformly continuous. Let $\{x_n\}_{n=0}^\infty$ be defined by (1.22), where $\{a_{ni}\}$, $\{c_{ni}\}$, $\{b_{n1}\}$, $\{b_{n2}\}$, $\{e_{n1}\}$, $\{a_{n3}+c_{n3}\}$, $\{a_{n2}+b_{n2}+c_{n2}\}$ and $\{a_{n1}+b_{n1}+c_{n1}+e_{n1}\}$ are appropriate sequences in [0,1] for $i = 1, 2, 3$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in D satisfying the conditions: $\{a_{n1}\}, \{a_{n2}\}, \{b_{n1}\}, \{b_{n2}\}, \{c_{n1}\}, \{c_{n2}\}, \{e_{n1}\} \to 0$ as $n \to \infty$ and $\sum_{n=0}^\infty a_{n1} = \infty$. If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^\infty$ converges strongly to the common fixed point of T_1, T_2 and T_3.

Proof. Since T_1, T_2, T_3 are asymptotically ϕ-hemicontractive mappings, there exists a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j_\phi(x-p) \in J_\phi(x-p)$ such that for some $n_0 \in \mathbb{N}$

$$\langle T_i^n x - p, j_\phi(x-p)\rangle \leq k_n(\phi(\|x-p\|))^2,\ \forall x \in D,\ p \in F(T),\ n \geq n_0, i = 1, 2, 3. \quad (2.1)$$

Let $p \in F(T_1) \cap F(T_2) \cap F(T_3)$ and

$$\begin{aligned} M_1 &= \|x_0 - p\| + \sup_{n\geq 0} \|T_1^n y_n - p\| + \sup_{n\geq 0} \|T_1^n z_n - p\| \\ &\quad + \sup_{n\geq 0} \|T_1^n x_n - p\| + \sup_{n\geq 0} \|u_n - p\|. \end{aligned} \tag{2.2}$$

Clearly, M_1 is finite. We now show that $\{x_n - p\}_{n\geq 0}$ is also bounded. Observe that $\|x_0 - p\| \leq M_1$. It follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})(x_n - p) + a_{n1}(T_1^n y_n - p) \\ &\quad + b_{n1}(T_1^n z_n - p) + e_{n1}(T_1^n x_n - p) + c_{n1}(u_n - p)\| \\ &\leq (1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})\|x_n - p\| + a_{n1}\|T_1^n y_n - p\| \\ &\quad + b_{n1}\|T_1^n z_n - p\| + e_{n1}\|T_1^n x_n - p\| + c_{n1}\|u_n - p\| \\ &\leq (1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})M_1 + a_{n1}M_1 + b_{n1}M_1 \\ &\quad + e_{n1}M_1 + c_{n1}M_1 \\ &= M_1, \end{aligned} \tag{2.3}$$

using the uniform continuity of T_3, we obtain that $\{T_3^n x_n\}$ is bounded. Denote

$$M_2 = \max\left\{M_1, \sup_{n\geq 0}\{\|T_3^n x_n - p\|\}, \sup_{n\geq 0}\{\|w_n - p\|\}\right\}, \tag{2.4}$$

then we have:

$$\begin{aligned} \|z_n - p\| &\leq (1 - a_{n3} - c_{n3})\|x_n - p\| + a_{n3}\|T_3^n x_n - p\| + c_{n3}\|w_n - p\| \\ &\leq (1 - a_{n3} - c_{n3})M_1 + a_{n3}M_2 + c_{n3}M_2 \\ &\leq (1 - a_{n3} - c_{n3})M_2 + a_{n3}M_2 + c_{n3}M_2 \\ &= M_2. \end{aligned} \tag{2.5}$$

Recall that T_2 is uniformly continuous, so that $\{T_2^n z_n\}$ is bounded. Let

$$M = \sup_{n\geq 0} \|T_2^n z_n - p\| + \sup_{n\geq 0} \|T_2^n x_n - p\| + \sup_{n\geq 0} \|v_n - p\| + M_2,$$

then M is finite. Since $\{x_n - p\}_{n\geq 0}$ is bounded and ϕ is a continuous strictly increasing function, $M^* := \sup_{n\geq 0} \phi(\|x_{n+1} - p\|)$ is also finite. Using Lemma 1.1, (1.24) and (2.1), then for $n \geq 0$ and $j_\phi(x_{n+1}-p) \in J(x_{n+1}-p)$, we have:

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-a_{n1}-b_{n1}-c_{n1}-e_{n1})(x_n-p)+a_{n1}(T_1^n y_n-p) \\
&\quad +b_{n1}(T_1^n z_n-p)+e_{n1}(T_1^n x_n-p)+c_{n1}(u_n-p)\|^2 \\
&\leq (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2 \\
&\quad +2\langle a_{n1}(T_1^n y_n-p)+b_{n1}(T_1^n z_n-p)+e_{n1}(T_1^n x_n-p) \\
&\quad +c_{n1}(u_n-p), \tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)} j_\phi(x_{n+1}-p)\rangle \\
&= (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2 \\
&\quad +2a_{n1}\left\langle T_1^n y_n-p, \tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)} j_\phi(x_{n+1}-p)\right\rangle \\
&\quad +2b_{n1}\left\langle T_1^n z_n-p, \tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)} j_\phi(x_{n+1}-p)\right\rangle \\
&\quad +2e_{n1}\left\langle T_1^n x_n-p, \tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)} j_\phi(x_{n+1}-p)\right\rangle \\
&\quad +2c_{n1}\left\langle u_n-p, \tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)} j_\phi(x_{n+1}-p)\right\rangle \\
&= (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2 \\
&\quad +2a_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n y_n-T_1^n x_{n+1}+T_1^n x_{n+1}-p, j_\phi(x_{n+1}-p)\rangle \\
&\quad +2b_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n z_n-T_1^n x_{n+1}+T_1^n x_{n+1}-p, j_\phi(x_{n+1}-p)\rangle \\
&\quad +2e_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n x_n-T_1^n x_{n+1}+T_1^n x_{n+1}-p, j_\phi(x_{n+1}-p)\rangle \\
&\quad +2c_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle u_n-p, j_\phi(x_{n+1}-p)\rangle \\
&= (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2 \\
&\quad +2a_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n y_n-T_1^n x_{n+1}, j_\phi(x_{n+1}-p)\rangle \\
&\quad +2a_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n x_{n+1}-p, j_\phi(x_{n+1}-p)\rangle \\
&\quad +2b_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n z_n-T_1^n x_{n+1}, j_\phi(x_{n+1}-p)\rangle \\
&\quad +2b_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n x_{n+1}-p, j_\phi(x_{n+1}-p)\rangle \\
&\quad +2e_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n x_n-T_1^n x_{n+1}, j_\phi(x_{n+1}-p)\rangle \\
&\quad +2e_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n x_{n+1}-p, j_\phi(x_{n+1}-p)\rangle \\
&\quad +2c_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle u_n-p, j_\phi(x_{n+1}-p)\rangle \\
&\leq (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2+2a_{n1}\|x_{n+1}-p\|\|T_1^n y_n-T_1^n x_{n+1}\| \\
&\quad +2a_{n1}k_n\|x_{n+1}-p\|\phi(\|x_{n+1}-p\|)+2b_{n1}\|x_{n+1}-p\|\|T_1^n z_n-T_1^n x_{n+1}\| \\
&\quad +2b_{n1}k_n\|x_{n+1}-p\|\phi(\|x_{n+1}-p\|)+2e_{n1}\|x_{n+1}-p\|\|T_1^n x_n-T_1^n x_{n+1}\| \\
&\quad +2e_{n1}k_n\|x_{n+1}-p\|\phi(\|x_{n+1}-p\|)+2c_{n1}\|x_{n+1}-p\|\|u_n-p\|
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2+2a_{n1}\|x_{n+1}-p\|\|T_1^ny_n-T_1^nx_{n+1}\| \\
&\quad +2a_{n1}k_nM^*\|x_{n+1}-p\|+2b_{n1}\|x_{n+1}-p\|\|T_1^nz_n-T_1^nx_{n+1}\| \\
&\quad +2b_{n1}k_nM^*\|x_{n+1}-p\|+2e_{n1}\|x_{n+1}-p\|\|T_1^nx_n-T_1^nx_{n+1}\| \\
&\quad +2e_{n1}k_nM^*\|x_{n+1}-p\|+2c_{n1}\|x_{n+1}-p\|\|u_n-p\| \\
&\leq (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2+2a_{n1}k_nM^*\|x_{n+1}-p\| \\
&\quad +2b_{n1}k_nM^*\|x_{n+1}-p\|+2e_{n1}k_nM^*\|x_{n+1}-p\| \\
&\quad +2M_1\{a_{n1}\|T_1^ny_n-T_1^nx_{n+1}\|+b_{n1}\|T_1^nz_n-T_1^nx_{n+1}\| \\
&\quad +e_{n1}\|T_1^nx_n-T_1^nx_{n+1}\|+c_{n1}\|u_n-p\|\} \\
&= (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2+2a_{n1}k_nM^*\|x_{n+1}-p\| \\
&\quad +2b_{n1}k_nM^*\|x_{n+1}-p\|+2e_{n1}k_nM^*\|x_{n+1}-p\|+2\delta_n, \qquad (2.6)
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_n &= M_1\{a_{n1}\|T_1^ny_n-T_1^nx_{n+1}\|+b_{n1}\|T_1^nz_n-T_1^nx_{n+1}\| \\
&\quad +e_{n1}\|T_1^nx_n-T_1^nx_{n+1}\|+c_{n1}\|u_n-p\|\}. \qquad (2.7)
\end{aligned}
$$

Using (1.24), we have

$$
\begin{aligned}
\|y_n-x_{n+1}\| &= \|y_n-x_n+x_n-x_{n+1}\| \\
&\leq \|y_n-x_n\|+\|x_n-x_{n+1}\| \\
&= \|(1-a_{n2}-b_{n2}-c_{n2})x_n+a_{n2}T_2^nz_n+b_{n2}T_2^nx_n+c_{n2}v_n-x_n\| \\
&\quad +\|x_n-\{(1-a_{n1}-b_{n1}-c_{n1}-e_{n1})x_n+a_{n1}T_1^ny_n+b_{n1}T_1^nz_n \\
&\quad +e_{n1}T_1^nx_n+c_{n1}u_n\}\| \\
&= \|-a_{n2}(x_n-T_2^nz_n)-b_{n2}(x_n-T_2^nx_n)-c_{n2}(x_n-v_n)\| \\
&\quad +\|a_{n1}(x_n-T_1^ny_n)+b_{n1}(x_n-T_1^nz_n)+c_{n1}(x_n-u_n) \\
&\quad +e_{n1}(x_n-T_1^nx_n)\| \\
&= \|-a_{n2}(x_n-p+p-T_2^nz_n)-b_{n2}(x_n-p+p-T_2^nx_n) \\
&\quad -c_{n2}(x_n-p+p-v_n)\|+\|a_{n1}(x_n-p+p-T_1^ny_n) \\
&\quad +b_{n1}(x_n-p+p-T_1^nz_n)+c_{n1}(x_n-p+p-u_n) \\
&\quad +e_{n1}(x_n-p+p-T_1^nx_n)\|
\end{aligned}
$$

$$\begin{aligned}
\|y_n - x_{n+1}\| &\leq a_{n2}\|x_n - p\| + a_{n2}\|p - T_2^n z_n\| + b_{n2}\|x_n - p\| + b_{n2}\|p - T_2^n x_n\| \\
&\quad + c_{n2}\|x_n - p\| + c_{n2}\|p - v_n\| + a_{n1}\|x_n - p\| + a_{n1}\|p - T_1^n y_n\| \\
&\quad + b_{n1}\|x_n - p\| + b_{n1}\|p - T_1^n z_n\| + c_{n1}\|x_n - p\| + c_{n1}\|p - u_n\| \\
&\quad + e_{n1}\|x_n - p\| + e_{n1}\|p - T_1^n x_n\| \\
&\leq 2Ma_{n2} + 2Mb_{n2} + 2Mc_{n2} + 2Ma_{n1} + 2Mb_{n1} + 2Mc_{n1} + 2Me_{n1} \\
&= 2M\left(a_{n2} + b_{n2} + c_{n2} + a_{n1} + b_{n1} + c_{n1} + e_{n1}\right). \qquad (2.8)
\end{aligned}$$

Using the condition that $\{a_{n1}\}, \{a_{n2}\}, \{b_{n1}\}, \{b_{n2}\}, \{c_{n1}\}, \{c_{n2}\}, \{e_{n1}\} \to 0$ as $n \to \infty$, we obtain from (2.8)

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \qquad (2.9)$$

Using the uniform continuity of T_1, we have

$$\lim_{n\to\infty} \|T_1^n y_n - T_1^n x_{n+1}\| = 0. \qquad (2.10)$$

Similarly, $\lim_{n\to\infty} \|T_1^n z_n - T_1^n x_{n+1}\| = \lim_{n\to\infty} \|T_1^n x_n - T_1^n x_{n+1}\| = 0$. Hence, we have that $\lim_{n\to\infty} \delta_n = 0$.

Furthermore, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})(x_n - p) + a_{n1}(T_1^n y_n - p) \\
&\quad + b_{n1}(T_1^n z_n - p) + e_{n1}(T_1^n x_n - p) + c_{n1}(u_n - p)\| \\
&\leq (1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})\|x_n - p\| + a_{n1}\|T_1^n y_n - p\| \\
&\quad + b_{n1}\|T_1^n z_n - p\| + e_{n1}\|T_1^n x_n - p\| + c_{n1}\|u_n - p\| \\
&\leq (1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})\|x_n - p\| \\
&\quad + (a_{n1} + b_{n1} + e_{n1} + c_{n1})M. \qquad (2.11)
\end{aligned}$$

Since $\{a_{n1}\}, \{a_{n2}\}, \{b_{n1}\}, \{b_{n2}\}, \{c_{n1}\}, \{c_{n2}\}, \{e_{n1}\} \to 0$ as $n \to \infty$, for every $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $(a_{n1} + b_{n1} + c_{n1} + e_{n1}) \leq \epsilon$ for all $n \geq k$. Let $\{t_n\} = \{a_{n1} + b_{n1} +$

$c_{n1}+e_{n1}\}$. Substituting (2.11) into (2.6), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-t_n)^2\|x_n-p\|^2+2M^*k_n(a_{n1}+b_{n1}+e_{n1})\|x_{n+1}-p\| \\
&\quad +2\delta_n \\
&\leq (1-t_n)^2\|x_n-p\|^2+2M^*k_n(a_{n1}+b_{n1}+e_{n1}) \\
&\quad \times\{(1-t_n)^2\|x_n-p\|+t_nM\}+2\delta_n \\
&\leq (1-t_n)^2\|x_n-p\|^2+2M^*t_nk_n\{(1-t_n)\|x_n-p\|+t_nM\} \\
&\quad +2\delta_n \\
&= (1-t_n)^2\|x_n-p\|^2+2M^*t_nk_n(1-t_n)\|x_n-p\| \\
&\quad +2MM^*t_n^2k_n+2\delta_n \\
&\leq (1-t_n)^2\|x_n-p\|^2+2M^*t_nk_n(1-t_n) \\
&\quad \times\{(1-t_{n-1})\|x_{n-1}-p\|+t_{n-1}M\}+2[MM^*t_n^2k_n+\delta_n] \\
&\leq (1-t_n)^2\|x_n-p\|^2+2M^*k_nt_n(1-t_n)(1-t_{n-1})\|x_{n-1}-p\| \\
&\quad +2[M^*k_nt_n(1-t_n)t_{n-1}M+MM^*t_n^2k_n+\delta_n] \\
&= (1-t_n)^2\|x_n-p\|^2+2M^*t_nk_n(1-t_n)(1-t_{n-1})\|x_{n-1}-p\| \\
&\quad +2\left[MM^*k_nt_n\{(1-t_n)t_{n-1}+t_n\}+\delta_n\right] \\
&\leq \cdots \\
&\leq (1-t_n)^2\|x_n-p\|^2+2t_nk_n\textstyle\prod_{j=k}^{n}(1-t_j)M^*\|x_k-p\| \\
&\quad +2\{t_n^2MM^*k_n \\
&\quad +t_nk_nMM^*\textstyle\sum_{j=k}^{n-1}\left(t_{n-1-j}\prod_{j=k}^{n-1}(1-t_{n-j})\right)+\delta_n\} \\
&\leq (1-t_n)^2\|x_n-p\|^2+2\{t_n^2k_n\textstyle\prod_{j=k}^{n}(1-t_j)MM^*+t_n^2MM^*k_n \\
&\quad +t_nk_nMM^*\textstyle\sum_{j=k}^{n-1}\left(t_{n-1-j}\prod_{j=k}^{n-1}(1-t_{n-j})\right)+\delta_n\} \\
&\leq (1-t_n)^2\|x_n-p\|^2+2\theta_n, \qquad (2.12)
\end{aligned}$$

where

$$\begin{aligned}
\theta_n &= \left[t_n\textstyle\prod_{j=k}^{n}(1-t_j)+t_n+\sum_{j=k}^{n-1}\left(t_{n-1-j}\prod_{j=k}^{n-1}(1-t_{n-j})\right)\right]t_nk_nMM^* \\
&\quad +\delta_n. \qquad (2.13)
\end{aligned}$$

Observe that $\{\theta_n\}_{n\geq 0}$ converges to 0 as $n \to \infty$. Clearly,
$\prod_{j=k}^{n}(1-t_j) \leq e^{-\sum_{j=k}^{n} t_j} \longrightarrow 0$ as $n \to \infty$ and

$$\sum_{j=k}^{n-1}\left\{t_{n-1-j}\prod_{j=k}^{n-1}(1-t_{n-j})\right\} \leq \sum_{j=k}^{n-1}\epsilon \to 0$$

as $\epsilon \to 0$. Let $\rho_n = \|x_n - p\|^2$, $\lambda_n = t_n$ and $\sigma_n = 2\theta_n$. Using the fact that $\lim_{n\to\infty}\theta_n = \lim_{n\to\infty}\delta_n = 0$ and Lemma 1.2, we have from (2.12) that

$$\lim_{n\to\infty}\|x_n - p\| = 0. \tag{2.14}$$

The proof of Theorem 2.1 is completed. □

Remark 2.2. Theorem 2.1 improves and generalizes the results of Yang *et al.* [26], Xue and Fan [25] which in turn is a correction of the results of Rafiq [21].

Theorem 2.3. Let E be a real Banach space, $T_1, T_2, T_3 : E \to E$ be uniformly continuous and *ϕ-strongly quasi-accretive* operators with $R(I - T_1)$ bounded, where I is the identity mapping on E. Let p denote the unique common solution to the equation $T_ix = f,\ (i = 1, 2, 3)$. For a given $f \in E$, define the operator $H_i : E \to E$ by $H_ix = f + x - T_ix,\ (i = 1, 2, 3)$. For any $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{cases} x_{n+1} = (1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})x_n + a_{n1}H_1y_n + b_{n1}H_1z_n + e_{n1}H_1x_n + c_{n1}u_n, \\ y_n = (1 - a_{n2} - b_{n2} - c_{n2})x_n + a_{n2}H_2z_n + b_{n2}H_2x_n + c_{n2}v_n, \\ z_n = (1 - a_{n3} - c_{n3})x_n + a_{n3}H_3x_n + c_{n3}w_n, \end{cases} \tag{2.15}$$

where $\{a_{ni}\}, \{c_{ni}\}, \{b_{n1}\}, \{b_{n2}\}, \{e_{n1}\}, \{a_{n3}+c_{n3}\}, \{a_{n2}+b_{n2}+c_{n2}\}$ and $\{a_{n1}+b_{n1}+c_{n1}+e_{n1}\}$ are appropriate sequences in [0,1] for $i = 1, 2, 3$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in D satisfying the conditions: $\{a_{n1}\}, \{a_{n2}\}, \{b_{n1}\}, \{b_{n2}\}, \{c_{n1}\}, \{c_{n2}\}, \{e_{n1}\} \to 0$ as $n \to \infty$ and $\sum_{n=0}^{\infty} a_{n1} = \infty$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique common solution to $T_ix = f\ (i = 1, 2, 3)$.

Proof. Clearly, if p is the unique common solution to the equation $T_ix = f\ (i = 1, 2, 3)$, it follows that p is the unique common fixed point of H_1, H_2 and H_3. Using the fact that T_1, T_2 and T_3 are all ϕ-strongly quasi-accretive in the intermediate sense operators, then H_1, H_2 and H_3 are all asymptotically ϕ-hemicontractive mappings. Since T_i $(i = 1, 2, 3)$ is uniformly continuous with $R(I - T_1)$ bounded, this implies that H_i $(i = 1, 2, 3)$ is uniformly continuous with $R(H_1)$ bounded. Hence, Theorem 2.3 follows from Theorem 2.1. □

Remark 2.4. Theorem 2.3 improves and extends Theorem 2.2 of Xue and Fan [25] which in turn is a correction of the results of Rafiq [21].

Example 2.5. Let $E = (-\infty, +\infty)$ with the usual norm and let $D = [0, +\infty)$. We define $T_1 : D \to D$ by $T_1 x := \frac{x}{2(1+x)}$ for each $x \in D$. Hence, $F(T_1) = \{0\}$, $R(T_1) = [0, \frac{1}{2})$ and T_1 is a uniformly continuous and asymptotically ϕ-hemicontractive mapping in the intermediate sense. Define $T_2 : D \to D$ by $T_2 x := \frac{x}{4}$ for all $x \in D$. Hence, $F(T_2) = \{0\}$ and T_2 is a uniformly continuous and strongly pseudocontractive mapping. Define $T_3 : D \to D$ by $T_3 x := \frac{\sin^4 x}{4}$ for each $x \in D$. Then $F(T_3) = \{0\}$ and T_3 is a uniformly continuous and asymptotically ϕ-hemicontractive mapping in the intermediate sense. Set $\{a_{ni}\} = \{c_{ni}\} = \frac{1}{n^4}$, $\{b_{n1}\} = \{e_{n1}\} = \{b_{n2}\} = \frac{1}{n^3}$, $\{k_n\} = 1$, for all $n \geq 0$ and $\phi(t) = \frac{t^2}{2}$ for each $t \in (-\infty, +\infty)$. Clearly, $F(T_1) \cap F(T_2) \cap F(T_3) = \{0\} = p \neq \emptyset$. For an arbitrary $x_0 \in D$, the sequence $\{x_n\}_{n=0}^{\infty} \subset D$ defined by (1.22) converges strongly to the common fixed point of T_1, T_2 and T_3 which is $\{0\}$, satisfying Theorem 2.1. This means that Theorem 2.1 is applicable.

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