

# Existence of Fixed Points of Asymptotically Generalized $\Phi$ -Hemicontractive Mappings in the Intermediate Sense

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## Abstract

Let  $E$  be a real Banach space,  $C$  be a nonempty closed convex subset of  $E$  and  $T : C \rightarrow C$  be a continuous generalized  $\Phi$ -pseudocontractive mapping, Xiang [Chang He Xiang, Fixed point theorem for generalized  $\Phi$ -pseudocontractive mappings, *Nonlinear Analysis* 70 (2009) 2277-2279] proved that  $T$  has a unique fixed point in  $C$ . It is our purpose in this study to extend the results of Xiang [11] to the class of asymptotically generalized  $\Phi$ -hemicontractive mappings in the intermediate sense, recently introduced by Okeke, Olaleru and Akewe [6].

**Mathematics Subject Classification:** 47H10, 47H09

**Keywords:** asymptotically generalized  $\Phi$ -hemicontractive mappings in the intermediate sense, Banach spaces, fixed point

## 1 Introduction and Preliminaries

Let  $E$  be an arbitrary real normed linear space with dual space  $E^*$  and  $C$  be a nonempty subset of  $E$ . We denote by  $J$  the normalized duality mapping from  $E$  to  $2^{E^*}$  defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E, \quad (1.1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the generalized duality pairing.

The following definitions will be needed in this study.

**Definition 1.1.** [11]. A mapping  $T : C \rightarrow E$  is called *strongly pseudocontractive* if there exists a constant  $k \in (0, 1)$  such that, for all  $x, y \in C$ , there exists  $j(x - y) \in J(x - y)$  satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq (1 - k)\|x - y\|^2. \quad (1.2)$$

$T$  is called  *$\phi$ -strongly pseudocontractive* if there exists a strictly increasing function  $\phi : [0, \infty) \rightarrow [0, \infty)$  with  $\phi(0) = 0$  such that, for all  $x, y \in C$ , there exists  $j(x - y) \in J(x - y)$  satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|. \quad (1.3)$$

$T$  is called *generalized  $\Phi$ -pseudocontractive* [2] if there exists a strictly increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  with  $\Phi(0) = 0$  such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|). \quad (1.4)$$

The class of generalized  $\Phi$ -pseudocontractive mappings is also called uniformly pseudocontractive mappings (see [2]). It is well known that these kinds of mappings play crucial roles in nonlinear functional analysis.

It has been proved (see [8]) that the class of  $\phi$ -strongly pseudocontractive mappings properly contains the class of strongly pseudocontractive mappings. By taking  $\Phi(s) = s\phi(s)$ , where  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a strictly increasing function with  $\phi(0) = 0$ , Clearly, the class of generalized  $\Phi$ -pseudocontractive mappings properly contains the class of  $\phi$ -strongly pseudocontractive mappings.

Bruck *et al.* [1] in 1993 introduced the class of *asymptotically nonexpansive mappings in the intermediate sense* as follows.

The mapping  $T : C \rightarrow C$  is said to be *asymptotically nonexpansive in the intermediate sense* provided  $T$  is uniformly continuous and

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \quad (1.5)$$

Recently, Qin *et al.* [10] introduced the following class of nonlinear mappings.

**Definition 1.2.** [10]. A mapping  $T : C \rightarrow C$  is said to be *asymptotically pseudocontractive mapping in the intermediate sense* if

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2) \leq 0, \quad (1.6)$$

where  $\{k_n\}$  is a sequence in  $[1, \infty)$  such that  $k_n \rightarrow 1$  as  $n \rightarrow \infty$ . This is equivalent to

$$\langle T^n x - T^n y, x - y \rangle \leq k_n \|x - y\|^2 + \nu_n, \quad \forall n \geq 1, x, y \in C, \quad (1.7)$$

where

$$\nu_n = \max \left\{ 0, \sup_{x, y \in C} (\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2) \right\}. \quad (1.8)$$

Qin *et al.* [10] proved some weak convergence theorems for the class of asymptotically pseudocontractive mappings in the intermediate sense. They also established some strong convergence results without any compact assumption by considering the hybrid projection methods. Olaleru and Okeke [7] in 2012 proved a strong convergence of Noor type scheme for a uniformly  $L$ -Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense.

The following classes of nonlinear mappings was introduced by Okeke *et al.* [6] as a generalization of those introduced by Kim *et al.* [4].

**Definition 1.3.** [6]. A mapping  $T : C \rightarrow C$  is called *asymptotically generalized  $\Phi$ -pseudocontractive mapping in the intermediate sense* if there exists a strictly increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  with  $\Phi(0) = 0$  and a sequence  $\{k_n\} \subset [1, \infty)$  such that  $k_n \rightarrow 1$  as  $n \rightarrow \infty$  satisfying

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\langle T^n x - T^n y, j(x - y) \rangle - k_n \|x - y\|^2 + \Phi(\|x - y\|)) \leq 0, \quad (1.9)$$

for all  $x, y \in C$  and for some  $j(x - y) \in J(x - y)$ . Put

$$\xi_n = \max \left\{ 0, \sup_{x, y \in C} (\langle T^n x - T^n y, j(x - y) \rangle - k_n \|x - y\|^2 + \Phi(\|x - y\|)) \right\}, \quad (1.10)$$

we observe that  $\xi_n \rightarrow 0$  as  $n \rightarrow \infty$ . Hence (1.9) reduces to

$$\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2 + \xi_n - \Phi(\|x - y\|). \quad (1.11)$$

Clearly, the class of asymptotically generalized  $\Phi$ -pseudocontractive mappings in the intermediate sense is a generalization of the class of asymptotically generalized  $\Phi$ -pseudocontractive mappings, introduced by Kim *et al.* [4].

**Definition 1.4.** [6]. A mapping  $T : C \rightarrow C$  is called *asymptotically generalized  $\Phi$ -hemicontractive mapping in the intermediate sense* if there exists a strictly increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  with  $\Phi(0) = 0$  and a sequence  $\{k_n\} \subset [1, \infty)$  such that  $k_n \rightarrow 1$  as  $n \rightarrow \infty$  satisfying

$$\limsup_{n \rightarrow \infty} \sup_{x \in C, p \in F(T)} (\langle T^n x - T^n p, j(x - p) \rangle - k_n \|x - p\|^2 + \Phi(\|x - p\|)) \leq 0, \quad (1.12)$$

for all  $x \in C$ ,  $p \in F(T) := \{x \in C : Tx = x\} \neq \emptyset$  and for some  $j(x - p) \in J(x - p)$ . Put

$$\tau_n = \max \left\{ 0, \sup_{x \in C, p \in F(T)} (\langle T^n x - T^n p, j(x - p) \rangle - k_n \|x - p\|^2 + \Phi(\|x - p\|)) \right\}, \quad (1.13)$$

we observe that  $\tau_n \rightarrow 0$  as  $n \rightarrow \infty$ . Hence (1.12) reduces to

$$\langle T^n x - T^n p, j(x - p) \rangle \leq k_n \|x - p\|^2 + \tau_n - \Phi(\|x - p\|). \quad (1.14)$$

The class of *asymptotically generalized  $\Phi$ -hemicontractive mapping in the intermediate sense* was introduced by Okeke *et al.* [6]. Clearly, the class of *asymptotically generalized  $\Phi$ -hemicontractive mappings in the intermediate sense* is the most general among those defined above.

Xiang [11] in 2009 obtained the following existence results for the class of generalized  $\Phi$ -pseudocontractive mappings.

**Theorem X.** [11]. Let  $E$  be a real Banach space,  $C$  be a nonempty closed convex subset of  $E$ , and  $T : C \rightarrow C$  be a continuous generalized  $\Phi$ -pseudocontractive mapping. Then  $T$  has a unique fixed point in  $C$ .

It is our purpose in this study to extend the results of Xiang [11] to the class of asymptotically generalized  $\Phi$ -hemicontractive mappings in the intermediate sense. Our results extends and generalizes the results of Xiang [11] among others.

The following lemmas will be needed in this study.

**Lemma 1.1.** [2]. Let  $E$  be a real normed linear space. Then for all  $x, y \in E$ , we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y).$$

**Lemma 1.2.** [5]. Let  $\psi : [0, \infty) \rightarrow [0, \infty)$  be a strictly increasing function with  $\psi(0) = 0$  and let  $\{\theta_n\}$ ,  $\{\sigma_n\}$  and  $\{\nu_n\}$  be nonnegative real sequences such that  $\sigma_n = o(\nu_n)$ ,  $\sum_{n \geq 0} \nu_n = \infty$ ,  $\lim_{n \rightarrow \infty} \nu_n = 0$ . Suppose that

$$\theta_{n+1}^2 \leq \theta_n^2 - \nu_n \psi(\theta_{n+1}) + \sigma_n, \quad n \geq 0.$$

Then  $\theta_n \rightarrow 0$  as  $n \rightarrow \infty$ .

## 2 Main Results

**Theorem 2.1.** Let  $E$  be a real Banach space,  $C$  be a nonempty closed convex subset of  $E$ , and  $T : C \rightarrow C$  be a continuous asymptotically generalized  $\Phi$ -hemicontractive mapping in the intermediate sense. Then  $T$  has a unique fixed point in  $C$ .

**Proof.** For each  $u \in C$ , the mapping  $S : C \rightarrow C$  defined by  $Sx = \frac{1}{2}u + \frac{1}{2}T^n x$  for each  $x \in C$  is a continuous strongly pseudocontractive mapping. By Corollary 2 of [3], we know that  $S$  has a unique fixed point in  $C$ . Meaning that given  $x_0 \in C$ , the sequence  $\{x_n\}$  defined by  $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}T^n x_{n+1}$  ( $\forall n \geq 0$ ) is well defined.

For each  $n \geq 1$ , we have

$$x_{n+1} = x_n - x_{n+1} + T^n x_{n+1}, \quad x_n = x_{n-1} - x_n + T^n x_n. \tag{2.1}$$

Using Lemma 1.1 and (1.14), it follows that there exists  $j(x_{n+1} - x_n) \in J(x_{n+1} - x_n)$  such that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_n - x_{n-1}) - (x_{n+1} - x_n) + (T^n x_{n+1} - T^n x_n)\|^2 \\ &\leq \|x_n - x_{n-1}\|^2 - 2\langle x_{n+1} - x_n, j(x_{n+1} - x_n) \rangle \\ &\quad + 2\langle T^n x_{n+1} - T^n x_n, j(x_{n+1} - x_n) \rangle \\ &\leq \|x_n - x_{n-1}\|^2 - 2\|x_{n+1} - x_n\|^2 \\ &\quad + 2\{k_n \|x_{n+1} - x_n\|^2 + \xi_n - \Phi(\|x_{n+1} - x_n\|)\} \\ &= \|x_n - x_{n-1}\|^2 - 2\|x_{n+1} - x_n\|^2 + 2k_n \|x_{n+1} - x_n\|^2 \\ &\quad + 2\xi_n - 2\Phi(\|x_{n+1} - x_n\|). \end{aligned} \tag{2.2}$$

From (2.2), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &\leq \frac{1}{3-2k_n} \|x_n - x_{n-1}\|^2 - \frac{2}{3-2k_n} \Phi(\|x_{n+1} - x_n\|) + \frac{2\xi_n}{3-2k_n} \\ &\leq \frac{1}{3-2k_n} \|x_n - x_{n-1}\|^2 - \frac{2\xi_n}{3-2k_n} \Phi(\|x_{n+1} - x_n\|) + \frac{2\xi_n}{3-2k_n} \end{aligned} \tag{2.3}$$

where  $\Phi : [0, \infty) \rightarrow [0, \infty)$  is a strictly increasing function with  $\Phi(0) = 0$  and  $\lim_{n \rightarrow \infty} k_n = 1$ . Let  $\theta_n = \|x_n - x_{n-1}\|$  ( $\forall n \geq 1$ ),  $\nu_n = \frac{2\xi_n}{3-2k_n}$ ,  $\sigma_n = \frac{2\xi_n}{3-2k_n}$  and  $\psi(s) = \Phi(\sqrt{s})$ . Then  $\theta_{n+1}^2 \leq \theta_n^2 - \nu_n \psi(\theta_{n+1}) + \sigma_n$  for all  $n \geq 1$ . By Lemma 1.2, we obtain  $\lim_{n \rightarrow \infty} \|x_n - x_{n-1}\|^2 = \lim_{n \rightarrow \infty} \theta_n^2 = 0$ . Hence,

$$\lim_{n \rightarrow \infty} \|x_n - x_{n-1}\| = 0. \quad (2.4)$$

Observe that  $x_n - x_{n-1} = T^n x_n - x_n$  for each  $n \geq 1$ , we obtain

$$\lim_{n \rightarrow \infty} \|T^n x_n - x_n\| = 0. \quad (2.5)$$

For each  $\epsilon > 0$ , we take  $\delta = \frac{\Phi(\epsilon)}{2\epsilon} > 0$ , it follows from (2.4) and (2.5) that there exists a natural number  $N$  such that  $\|x_{n+1} - x_n\| < \epsilon$  for every  $n \geq N$  and  $\|(T^n x_m - x_m) - (T^n x_n - x_n)\| < \delta$  for each  $m > n$ . Next, we prove by induction that

$$\|x_m - x_n\| < \epsilon, \quad \forall m > n \geq N. \quad (2.6)$$

For each natural number  $n \geq N$ , if we take  $m = n + 1$ , then we observe that (2.6) holds for some  $m \geq n + 1$ . Then

$$\|x_{m+1} - x_n\| \leq \|x_{m+1} - x_m\| + \|x_m - x_n\| < 2\epsilon. \quad (2.7)$$

Using (1.11), we obtain

$$\langle T^n x_{m+1} - T^n x_n, j(x_{m+1} - x_n) \rangle \leq k_n \|x_{m+1} - x_n\|^2 + \xi_n - \Phi(\|x_{m+1} - x_n\|). \quad (2.8)$$

From (2.8), we obtain

$$\begin{aligned} \Phi(\|x_{m+1} - x_n\|) &\leq k_n \|x_{m+1} - x_n\|^2 + \xi_n - \langle T^n x_{m+1} - T^n x_n, j(x_{m+1} - x_n) \rangle \\ &\leq k_n \langle (x_{m+1} - T^n x_{m+1}) - (x_n - T^n x_n), j(x_{m+1} - x_n) \rangle + \xi_n \\ &\leq k_n \|(x_{m+1} - T^n x_{m+1}) - (x_n - T^n x_n)\| \|x_{m+1} - x_n\| + \xi_n \\ &< \delta \cdot 2\epsilon + \xi_n \\ &= \Phi(\epsilon) + \xi_n. \end{aligned} \quad (2.9)$$

Since  $\Phi$  is a strictly increasing function and  $\xi_n \rightarrow 0$  as  $n \rightarrow \infty$ , we have that  $\|x_{m+1} - x_n\| < \epsilon$ , meaning that (2.6) holds for  $m + 1$ . By induction, (2.6) holds for all  $m > n \geq N$ , which implies that  $\{x_n\} \subset C$  is a Cauchy sequence. But  $E$  is a Banach space and  $C$  is closed, hence  $\{x_n\}$  converges to some  $p \in C$ . Since  $T : C \rightarrow C$  is continuous, we conclude that  $Tp = p$  using (2.5). From (1.14), we see that the fixed point of  $T$  is unique. The proof of Theorem 2.1 is completed.  $\square$

**Remark 2.2.** Theorem 2.1 improves, extends and generalizes Theorem 2.1 of Xiang [11] and the references therein.

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**Received: June 24, 2013**