

# Convergence Theorems on Asymptotically Generalized $\Phi$ -Hemicontractive Mappings in the Intermediate Sense

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## Abstract

In this study, we introduce two classes of nonlinear mappings, the class of asymptotically generalized  $\Phi$ -hemicontractive mappings in the intermediate sense and asymptotically generalized  $\Phi$ -pseudocontractive mappings in the intermediate sense; and prove the convergence of Mann type iterative scheme with errors to their fixed points. Our results generalize the results of Chang *et al.* [4], Chidume and Chidume [5] and Kim *et al.* [8] among others.

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## 1 Introduction

Let  $E$  be an arbitrary real normed linear space with dual  $E^*$ . We denote by  $J$  the normalized duality mapping from  $E$  into  $2^{E^*}$  defined by

$$J(x) := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad (1.1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the generalized duality pairing.

We give the following definitions which will be useful in this study

**Definition 1.1.** Let  $C$  be a nonempty subset of real normed linear space  $E$ . A mapping  $T : C \rightarrow E$  is said to be

(1) *strongly pseudocontractive mappings* [8] if for all  $x, y \in C$ , there exists a constant  $k \in (0, 1)$  and  $j(x - y) \in J(x - y)$  satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2, \quad (1.2)$$

(2)  *$\phi$ -strongly pseudocontractive mappings* [8] if for all  $x, y \in C$ , there exists a strictly increasing function  $\phi : [0, \infty) \rightarrow [0, \infty)$  with  $\phi(0) = 0$  and  $j(x - y) \in J(x - y)$  satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|. \quad (1.3)$$

The class of  $\phi$ -strongly pseudocontractive mappings includes the class of strongly pseudocontractive mappings by setting  $\phi(s) = ks$  for all  $s \in [0, \infty)$ . However, the converse is not true.

(3) *generalized  $\Phi$ -pseudocontractive mappings* [1, 5] if for all  $x, y \in C$ , there exists a strictly increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  with  $\Phi(0) = 0$  and  $j(x - y) \in J(x - y)$  satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|). \quad (1.4)$$

It is well known that the class of generalized  $\Phi$ -pseudocontractive mappings includes the class of  $\phi$ -strongly pseudocontractive mappings as a special case (if one sets  $\Phi(s) = s\phi(s)$  for all  $s \in [0, \infty)$ ).

(4) *generalized  $\Phi$ -hemi-contractive mappings* [5] if  $F(T) := \{x \in C : Tx = x\} \neq \emptyset$ , and there exists  $x^* \in F(T)$  and a strictly increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\Phi(0) = 0$  such that for all  $x \in C$ , there exists  $j(x - x^*) \in J(x - x^*)$  such that

$$\langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2 - \Phi(\|x - x^*\|). \quad (1.5)$$

Clearly, the class of generalized  $\Phi$ -hemi-contractive mappings includes the class of generalized  $\Phi$ -pseudocontractive mappings in which the fixed points set  $F(T) := \{x \in C : Tx = x\} \neq \emptyset$  is not empty.

(5) *generalized strongly successively  $\Phi$ -pseudocontractive mappings* [7] if for all  $x, y \in C$ , there exists  $j(x - y) \in J(x - y)$  and a strictly increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  with  $\Phi(0) = 0$  such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|). \tag{1.6}$$

Observe that if  $T^n = T$  for all  $n \in \mathbb{N}$  in (1.6), then we obtain (1.4).

(6) *asymptotically generalized  $\Phi$ -pseudocontractive mappings* [8] with sequence  $\{k_n\}$  if for each  $n \in \mathbb{N}$  and  $x, y \in C$ , there exists a constant  $k_n \geq 1$  with  $\lim_{n \rightarrow \infty} k_n = 1$  and a strictly increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  with  $\Phi(0) = 0$  and  $j(x - y) \in J(x - y)$  satisfying

$$\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2 - \Phi(\|x - y\|). \tag{1.7}$$

The class of asymptotically generalized  $\Phi$ -pseudocontractive maps was introduced by Kim *et al.* [8] in 2009 as a generalization of the class of generalized  $\Phi$ -pseudocontractive mappings. Observe that if  $k_n = 1$  for all  $n \in \mathbb{N}$  in (1.7), then we obtain (1.4).

(7) *asymptotically generalized  $\Phi$ -hemicontractive mappings* [8] with sequence  $\{k_n\}$  if  $F(T) \neq \emptyset$  and for each  $n \in \mathbb{N}$ ,  $x \in C$  and  $p \in F(T)$ , there exists a constant  $k_n \geq 1$  with  $\lim_{n \rightarrow \infty} k_n = 1$ , a strictly increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  with  $\Phi(0) = 0$  and  $j(x - p) \in J(x - p)$  satisfying

$$\langle T^n x - p, j(x - p) \rangle \leq k_n \|x - p\|^2 - \Phi(\|x - p\|). \tag{1.8}$$

Clearly, the class of asymptotically generalized  $\Phi$ -hemicontractive mappings is the most general among those defined by Huang [7], i.e the class of generalized  $\Phi$ -pseudocontractive maps and the class of generalized strongly successively  $\Phi$ -pseudocontractive maps.

Recently, Qin *et al.* [15] introduced the following class of nonlinear mappings.

**Definition 1.2.** [15]. A mapping  $T : C \rightarrow C$  is said to be *asymptotically pseudocontractive mapping in the intermediate sense* if

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2) \leq 0, \tag{1.9}$$

where  $\{k_n\}$  is a sequence in  $[1, \infty)$  such that  $k_n \rightarrow 1$  as  $n \rightarrow \infty$ . This is equivalent to

$$\langle T^n x - T^n y, x - y \rangle \leq k_n \|x - y\|^2 + \nu_n, \quad \forall n \geq 1, x, y \in C, \tag{1.10}$$

where

$$\nu_n = \max \left\{ 0, \sup_{x,y \in C} (\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2) \right\}. \quad (1.11)$$

Qin *et al.* [15] proved some weak convergence theorems for the class of asymptotically pseudocontractive mappings in the intermediate sense. They also established some strong convergence results without any compact assumption by considering the hybrid projection methods. Olaleru and Okeke [13] in 2012 proved a strong convergence of Noor type scheme for a uniformly  $L$ -Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense.

Motivated by the above facts, we now introduce the following classes of nonlinear mappings

**Definition 1.3.** Let  $C$  be a nonempty subset of a real normed linear space  $E$ . A mapping  $T : C \rightarrow C$  is said to be *asymptotically generalized  $\Phi$ -pseudocontractive mapping in the intermediate sense* with sequence  $\{k_n\}$  if for each  $n \in \mathbb{N}$  and  $x, y \in C$ , there exists a constant  $k_n \geq 1$  with  $\lim_{n \rightarrow \infty} k_n = 1$ , a strictly increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  with  $\Phi(0) = 0$  and  $j(x - y) \in J(x - y)$  satisfying

$$\limsup_{n \rightarrow \infty} \sup_{x,y \in C} (\langle T^n x - T^n y, j(x - y) \rangle - k_n \|x - y\|^2 + \Phi(\|x - y\|)) \leq 0. \quad (1.12)$$

Put

$$\tau_n = \max \left\{ 0, \sup_{x,y \in C} (\langle T^n x - T^n y, j(x - y) \rangle - k_n \|x - y\|^2 + \Phi(\|x - y\|)) \right\}. \quad (1.13)$$

It follows that  $\tau_n \rightarrow 0$  as  $n \rightarrow \infty$ . Hence (1.12) is reduced to the following

$$\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2 + \tau_n - \Phi(\|x - y\|). \quad (1.14)$$

We remark that if  $\tau_n = 0$  for all  $n \in \mathbb{N}$ , the class of asymptotically generalized  $\Phi$ -pseudocontractive mappings in the intermediate sense is reduced to the class of asymptotically generalized  $\Phi$ -pseudocontractive mappings introduced by Kim *et al.* [8] in 2009.

**Definition 1.4.** Let  $C$  be a nonempty subset of a real normed linear space  $E$ . A mapping  $T : C \rightarrow C$  is said to be *asymptotically generalized  $\Phi$ -hemicontractive mapping in the intermediate sense* with sequence  $\{k_n\}$  if  $F(T) := \{p \in C : p = Tp\} \neq \emptyset$  and for each  $n \in \mathbb{N}$ ,  $x \in C$  and  $p \in F(T)$ , there exists a constant  $k_n \geq 1$  with  $\lim_{n \rightarrow \infty} k_n = 1$  and a strictly increasing function

$\Phi : [0, \infty) \rightarrow [0, \infty)$  with  $\Phi(0) = 0$  and  $j(x - p) \in J(x - p)$  satisfying

$$\limsup_{n \rightarrow \infty} \sup_{x,p \in C \times F(T)} (\langle T^n x - p, j(x - p) \rangle - k_n \|x - p\|^2 + \Phi(\|x - p\|)) \leq 0. \tag{1.15}$$

Put

$$\tau_n = \max \left\{ 0, \sup_{x,p \in C \times F(T)} (\langle T^n x - p, j(x - p) \rangle - k_n \|x - p\|^2 + \Phi(\|x - p\|)) \right\}. \tag{1.16}$$

It follows that  $\tau_n \rightarrow 0$  as  $n \rightarrow \infty$ . Hence (1.15) is reduced to the following

$$\langle T^n x - p, j(x - p) \rangle \leq k_n \|x - p\|^2 + \tau_n - \Phi(\|x - p\|). \tag{1.17}$$

Clearly, the class of asymptotically generalized  $\Phi$ -hemicontractive mappings in the intermediate sense is the most general so far introduced in literature since it includes the class of asymptotically generalized  $\Phi$ -hemicontractive maps.

The following definitions will be needed in this study.

Let  $C$  be a nonempty subset of a normed linear space  $E$ . A mapping  $T : C \rightarrow E$  is said to be *Lipschitzian* if there exists a constant  $L > 0$  such that

$$\|Tx - Ty\| \leq L\|x - y\| \tag{1.18}$$

for all  $x, y \in C$  and *generalized Lipschitzian* [8] if there exists a constant  $L > 0$  such that

$$\|Tx - Ty\| \leq L(\|x - y\| + 1) \tag{1.19}$$

for all  $x, y \in C$ . A mapping  $T : C \rightarrow C$  is called *uniformly L-Lipschitzian* [8] if for each  $n \in \mathbb{N}$ , there exists a constant  $L > 0$  such that

$$\|T^n x - T^n y\| \leq L\|x - y\| \tag{1.20}$$

for all  $x, y \in C$ .

Clearly, every Lipschitzian mapping is a generalized Lipschitzian mapping. Every mapping with a bounded range is a generalized Lipschitzian mapping. The following example shows that the class of generalized Lipschitzian mappings properly contains the class of Lipschitzian mappings and that of mappings with bounded range.

**Example 1.5.** [3]. Let  $E = (-\infty, \infty)$  and  $T : E \rightarrow E$  be defined by

$$Tx = \begin{cases} x - 1 & \text{if } x \in (-\infty, -1), \\ x - \sqrt{1 - (x + 1)^2} & \text{if } x \in [-1, 0), \\ x + \sqrt{1 - (x - 1)^2} & \text{if } x \in [0, 1], \\ x + 1 & \text{if } x \in (1, \infty). \end{cases}$$

Then  $T$  is a generalized Lipschitzian mapping which is not Lipschitzian and whose range is not bounded.

Sahu [16] in 2005 introduced a new class of nonlinear mappings which is more general than the class of generalized Lipschitzian mappings and the class of uniformly  $L$ -Lipschitzian mappings.

**Definition 1.6.** [16]. Let  $C$  be a nonempty subset of a Banach space  $E$  and fix a sequence  $\{a_n\}$  in  $[0, \infty)$  with  $a_n \rightarrow 0$ .

(1) A mapping  $T : C \rightarrow C$  is said to be *nearly Lipschitzian* with respect to  $\{a_n\}$  if for each  $n \in \mathbb{N}$ , there exists a constant  $k_n > 0$  such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n) \quad (1.21)$$

for all  $x, y \in C$ .

The infimum of constants  $k_n$  in (1.21) is called *nearly Lipschitz constant* and is denoted by  $\eta(T^n)$ .

(2) A nearly Lipschitzian mapping  $T$  with sequence  $\{(a_n, \eta(T^n))\}$  is said to be *nearly uniformly  $L$ -Lipschitzian* if  $k_n = L$  for all  $n \in \mathbb{N}$ , i.e.

$$\|T^n x - T^n y\| \leq L(\|x - y\| + a_n) \quad (1.22)$$

and *nearly asymptotically nonexpansive* if  $k_n \geq 1$  for all  $n \in \mathbb{N}$  with  $\lim_{n \rightarrow \infty} k_n = 1$ .

(3) A mapping  $T : C \rightarrow E$  will be called *generalized  $(M, L)$ -Lipschitzian* if there exist two constants  $L, M > 0$  such that

$$\|Tx - Ty\| \leq L(\|x - y\| + M) \quad (1.23)$$

for all  $x, y \in C$ .

Observe that the class of generalized  $(M, L)$ -Lipschitzian mappings is a generalization of the class of Lipschitzian mappings. Clearly, the class of nearly uniformly  $L$ -Lipschitzian mappings properly contains the class of generalized  $(M, L)$ -Lipschitzian mappings and the class of uniformly  $L$ -Lipschitzian mappings. We remark that every nearly asymptotically nonexpansive mapping is nearly uniformly  $L$ -Lipschitzian.

It has been shown by Sahu [16] that a nearly uniformly  $L$ -Lipschitzian map is not necessarily continuous. Sahu [16] extended the results of Goebel and Kirk [6] to demicontinuous mappings and proved that if  $C$  is a nonempty closed convex bounded subset of a uniformly convex Banach space, then every demicontinuous nearly asymptotically nonexpansive self-mapping of  $C$  has a fixed point.

Chidume and Chidume [5] in 2005 obtained result for the class of generalized  $\Phi$ -hemi-contractive mappings while Kim *et al.* [8] in 2009 generalized the result to nearly uniformly  $L$ -Lipschitzian asymptotically generalized  $\Phi$ -hemicontractive mappings. They established a strong convergence result of the iterative sequence generated by these mappings in a general Banach space.

**Theorem KSN.** [8]. Let  $C$  be a nonempty convex subset of a real Banach space  $E$  and  $T : C \rightarrow C$  a nearly uniformly  $L$ -Lipschitzian mapping with sequence  $\{a_n\}$  and asymptotically generalized  $\Phi$ -hemicontractive mapping with sequence  $\{k_n\}$  and  $F(T) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence in  $[0, 1]$  satisfying the conditions:

- (i)  $\{\frac{a_n}{\alpha_n}\}$  is bounded,
- (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (iii)  $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$  and  $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$ .

Let  $\{x_n\}$  be the sequence in  $E$  generated from arbitrary  $x_1 \in C$  by

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n T^n x_n + \beta_n u_n, \quad n \in \mathbb{N}. \tag{1.24}$$

Then the sequence  $\{x_n\}$  in  $C$  defined by (1.24) converges strongly to a unique fixed point of  $T$ .

It is our purpose in this study to use the concept of nearly uniformly  $L$ -Lipschitzian (not necessarily continuous) mappings to prove a strong convergence result for the class of asymptotically generalized  $\Phi$ -hemicontractive mappings in the intermediate sense in a general Banach space. Our results are improvements and generalizations of Chidume and Chidume [5], Theorem KSN of Kim *et al.* [8] and Chang *et al.* [4] among others.

The following Lemmas will be useful in this study

**Lemma 1.1.** [2]. Let  $E$  be a Banach space. Then for each  $x, y \in E$ , there exists  $j(x + y) \in J(x + y)$  such that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.$$

**Lemma 1.2.** [14]. Let  $\{\delta_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  be three sequences of nonnegative numbers such that

$$\delta_{n+1} \leq (1 + \beta_n)\delta_n + \gamma_n$$

for all  $n \in \mathbb{N}$ . If  $\sum_{n=1}^{\infty} \beta_n < \infty$  and  $\sum_{n=1}^{\infty} \gamma_n < \infty$ , then  $\lim_{n \rightarrow \infty} \delta_n$  exists.

**Lemma 1.3.** [10]. Let  $\{\theta_n\}$  be a sequence of nonnegative real numbers and  $\{\lambda_n\}$  a real sequence in  $[0, 1]$  such that  $\sum_{n=1}^{\infty} \lambda_n = \infty$ . If there exists a strictly increasing function  $\phi : [0, \infty) \rightarrow [0, \infty)$  with  $\phi(0) = 0$  such that

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n$$

for all  $n \geq n_0$ , where  $n_0$  is some nonnegative integer and  $\{\sigma_n\}$  is a sequence of nonnegative numbers such that  $\sigma_n = o(\lambda_n)$ , then  $\lim_{n \rightarrow \infty} \theta_n = 0$ .

**Lemma 1.4.** [8]. Let  $\{\delta_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$  and  $\{\sigma_n\}$  be four sequences of nonnegative numbers such that

$$\delta_{n+1}^2 \leq (1 + \beta_n)\delta_n^2 + \gamma_n(\delta_n + \sigma_n)^2$$

for all  $n \in \mathbb{N}$ . If  $\sum_{n=1}^\infty \beta_n < \infty$ ,  $\sum_{n=1}^\infty \gamma_n < \infty$  and  $\{\sigma_n\}$  is bounded, then  $\lim_{n \rightarrow \infty} \delta_n$  exists.

## 2 Main Results

We prove the following lemma which will be needed in this study.

**Lemma 2.1.** Let  $\{\delta_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{\sigma_n\}$  and  $\{\rho_n\}$  be five sequences of nonnegative numbers such that

$$\delta_{n+1}^2 \leq (1 + \beta_n)\delta_n^2 + \gamma_n(\delta_n + \sigma_n)^2 + \rho_n^2 \tag{2.1}$$

for all  $n \in \mathbb{N}$ . If  $\sum_{n=1}^\infty \beta_n < \infty$ ,  $\sum_{n=1}^\infty \gamma_n < \infty$ ,  $\sum_{n=1}^\infty \rho_n < \infty$  and  $\{\sigma_n\}$  is bounded, then  $\lim_{n \rightarrow \infty} \delta_n$  exists.

**Proof.** Using (2.1), we obtain

$$\begin{aligned} \delta_{n+1}^2 &\leq (1 + \beta_n)\delta_n^2 + \gamma_n(\delta_n + \sigma_n)^2 + \rho_n^2 \\ &\leq (1 + \beta_n)\delta_n^2 + 2\gamma_n(\delta_n^2 + \sigma_n^2) + \rho_n^2 \\ &\leq (1 + \beta_n + 2\gamma_n)\delta_n^2 + 2\gamma_n\sigma_n^2 + \rho_n^2. \end{aligned} \tag{2.2}$$

Since  $\{\sigma_n\}$  is bounded and  $\sum_{n=1}^\infty \rho_n < \infty$ , then by Lemma 1.2, it follows that  $\lim_{n \rightarrow \infty} \delta_n$  exists.  $\square$

**Theorem 2.2.** Let  $C$  be a nonempty convex subset of a real Banach space  $E$  and  $T : C \rightarrow C$  a nearly uniformly  $L$ -Lipschitzian mapping with sequence  $\{a_n\}$  and asymptotically generalized  $\Phi$ -hemicontractive mapping in the intermediate sense with sequences  $\{\tau_n\}$  and  $\{k_n\}$  as defined in (1.17) and  $F(T) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence in  $[0, 1]$  satisfying the conditions:

- (i)  $\frac{1}{\alpha_n + \alpha_n L + \beta_n}$  is bounded, (ii)  $\sum_{n=1}^\infty \alpha_n = \infty$ ,  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,
- (iii)  $\sum_{n=1}^\infty \alpha_n^2 < \infty$ ,  $\sum_{n=1}^\infty \tau_n < \infty$  and  $\sum_{n=1}^\infty \alpha_n(k_n - 1) < \infty$ .

Let  $\{x_n\}$  be the sequence in  $E$  generated from arbitrary  $x_1 \in C$  by

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n T^n x_n + \beta_n u_n, \quad n \in \mathbb{N}, \tag{2.3}$$

where  $\{u_n\}$  is a bounded sequence in  $E$ . Then the sequence  $\{x_n\}$  in  $C$  defined by (2.3) converges strongly to the unique fixed point of  $T$ .



**Proof.** Fix  $p \in F(T)$  and using (1.17), (1.22), (2.3) we obtain

$$\begin{aligned}
 \|x_{n+1} - x_n\| &= \|-\alpha_n(x_n - T^n x_n) - \beta_n(x_n - u_n)\| \\
 &\leq \alpha_n \|T^n x_n - x_n\| + \beta_n \|u_n - x_n\| \\
 &\leq \alpha_n \{\|T^n x_n - p\| + \|x_n - p\|\} + \beta_n \{\|u_n - p\| + \|x_n - p\|\} \\
 &\leq \alpha_n \{L(\|x_n - p\| + a_n) + \|x_n - p\|\} + \beta_n \{\|x_n - p\| + \|u_n - p\|\} \\
 &= (\alpha_n L + \alpha_n + \beta_n) \|x_n - p\| + \beta_n \|u_n - p\| + \alpha_n a_n L \\
 &\leq (\alpha_n(1 + L) + \beta_n) \|x_n - p\| + \beta_n \|u_n - p\| + a_n L.
 \end{aligned} \tag{2.4}$$

Using Lemma 1.1, (1.17), (1.22), (2.3) and (2.4), we obtain

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &= \|(1 - \alpha_n - \beta_n)(x_n - p) + \alpha_n(T^n x_n - p) + \beta_n(u_n - p)\|^2 \\
 &\leq (1 - \alpha_n - \beta_n)^2 \|x_n - p\|^2 \\
 &\quad + 2\langle \alpha_n(T^n x_n - p) + \beta_n(u_n - p), j(x_{n+1} - p) \rangle \\
 &= (1 - \alpha_n - \beta_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle T^n x_n - p, j(x_{n+1} - p) \rangle \\
 &\quad + 2\beta_n \langle u_n - p, j(x_{n+1} - p) \rangle \\
 &\leq (1 - \alpha_n - \beta_n)^2 \|x_n - p\|^2 \\
 &\quad + 2\alpha_n \{ \langle T^n x_{n+1} - p, j(x_{n+1} - p) \rangle \\
 &\quad + \langle T^n x_n - T^n x_{n+1}, j(x_{n+1} - p) \rangle \} + 2\beta_n \langle u_n - p, j(x_{n+1} - p) \rangle \\
 &\leq (1 - \alpha_n - \beta_n)^2 \|x_n - p\|^2 + 2\alpha_n \{ k_n \|x_{n+1} - p\|^2 + \tau_n \\
 &\quad - \Phi(\|x_{n+1} - p\|) + L(\|x_{n+1} - x_n\| + a_n) \|x_{n+1} - p\| \} \\
 &\quad + 2\beta_n \|u_n - p\| \times \|x_{n+1} - p\| \\
 &\leq (1 - \alpha_n - \beta_n)^2 \|x_n - p\|^2 + 2\alpha_n \{ k_n \|x_{n+1} - p\|^2 + \tau_n \\
 &\quad - \Phi(\|x_{n+1} - p\|) + L[(\alpha_n(1 + L) + \beta_n) \|x_n - p\| + \beta_n \|u_n - p\| \\
 &\quad + (1 + L)a_n] \|x_{n+1} - p\| \} + 2\beta_n \|u_n - p\| \times \|x_{n+1} - p\| \\
 &= (1 - \alpha_n - \beta_n)^2 \|x_n - p\|^2 + 2\alpha_n k_n \|x_{n+1} - p\|^2 + 2\alpha_n \tau_n \\
 &\quad - 2\alpha_n \Phi(\|x_{n+1} - p\|) + 2\alpha_n L \{ (\alpha_n + \alpha_n L + \beta_n) \|x_n - p\| \\
 &\quad + \beta_n(1 + \frac{1}{\alpha_n L}) \|u_n - p\| + (1 + L)a_n \} \|x_{n+1} - p\| \\
 &\leq (1 - \alpha_n - \beta_n)^2 \|x_n - p\|^2 + 2\alpha_n k_n \|x_{n+1} - p\|^2 + 2\alpha_n \tau_n \\
 &\quad - 2\alpha_n \Phi(\|x_{n+1} - p\|) + 2\alpha_n L \{ (\alpha_n + \alpha_n L + \beta_n) \|x_n - p\| \\
 &\quad + \beta_n(1 + \frac{1}{\alpha_n L}) \|u_n - p\| + (1 + L)a_n \}^2 + \|x_{n+1} - p\|^2 \}.
 \end{aligned} \tag{2.5}$$

Set  $A_n := 2\alpha_n(k_n - 1) + 2\alpha_n L(\alpha_n + \alpha_n L + \beta_n)$  and  $B_n := 1 - 2\alpha_n k_n - 2\alpha_n L$ .

From (2.5), we obtain

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq \frac{(1 - \alpha_n - \beta_n)^2}{B_n} \|x_n - p\|^2 + \frac{2\alpha_n \tau_n}{B_n} - \frac{2\alpha_n}{B_n} \Phi(\|x_{n+1} - p\|) \\
 &\quad + \frac{2\alpha_n L}{B_n} [(\alpha_n + \alpha_n L + \beta_n) \|x_n - p\| + \beta_n(1 + \frac{1}{\alpha_n L}) \|u_n - p\| \\
 &\quad + (1 + L)a_n]^2 \\
 &\leq (1 + \frac{A_n}{B_n}) \|x_n - p\|^2 + \frac{2\alpha_n \tau_n}{B_n} - \frac{2\alpha_n}{B_n} \Phi(\|x_{n+1} - p\|) \\
 &\quad + \frac{2\alpha_n L}{B_n} [(\alpha_n + \alpha_n L + \beta_n) \|x_n - p\| + \beta_n(1 + \frac{1}{\alpha_n L}) \|u_n - p\| \\
 &\quad + (1 + L)a_n]^2.
 \end{aligned} \tag{2.6}$$

But  $B_n := 1 - 2\alpha_n k_n - 2\alpha_n L \rightarrow 1$ , since  $\lim_{n \rightarrow \infty} \alpha_n = 0$  there exists a number  $n_0 \in \mathbb{N}$  such that  $\frac{1}{2} < B_n \leq 1$  for all  $n \geq n_0$ . From (2.6), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1 + 2A_n)\|x_n - p\|^2 + 4\alpha_n\tau_n - 2\alpha_n\Phi(\|x_{n+1} - p\|) \\ &\quad + 2\alpha_nL[(\alpha_n + \alpha_nL + \beta_n)\|x_n - p\| + \beta_n(1 + \frac{1}{\alpha_nL})\|u_n - p\| \\ &\quad + (1 + L)a_n]^2 \\ &\leq (1 + 2A_n)\|x_n - p\|^2 + 4\alpha_n\tau_n + 2\alpha_nL[(\alpha_n + \alpha_nL + \beta_n)\|x_n - p\| \\ &\quad + \beta_n(1 + \frac{1}{\alpha_nL})\|u_n - p\| + (1 + L)a_n]^2 \\ &\leq 1 + 2A_n\|x_n - p\|^2 + 4\alpha_n\tau_n + 2\alpha_nL(\alpha_n + \alpha_nL + \beta_n)^2 \times \\ &\quad \left[ \|x_n - p\| + \frac{1}{\alpha_n + \alpha_nL + \beta_n} \left\{ \beta_n(1 + \frac{1}{\alpha_nL})\|u_n - p\| + (1 + L)a_n \right\} \right]^2 \end{aligned} \tag{2.7}$$

Using the conditions  $\sum_{n=1}^\infty \alpha_n(k_n - 1) < \infty$  and  $\sum_{n=1}^\infty \alpha_n^2 < \infty$ , we have  $\sum_{n=1}^\infty A_n < \infty$ . Since  $\{u_n\}$  is a bounded sequence in  $E$  and  $\frac{1}{\alpha_n + \alpha_nL + \beta_n}$  is bounded, from (2.7) and Lemma 2.1 we have that  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists. Hence  $\{x_n\}$  is bounded. Now, we set  $M_1 := \sup\{\|x_n - p\| : n \in \mathbb{N}\}$ ,  $M_2 := \sup\{\beta_n(1 + \frac{1}{\alpha_nL}) : n \in \mathbb{N}\}$ ,  $M_3 := \sup\{\|u_n - p\| : n \in \mathbb{N}\}$ ,  $M_4 := \sup\{4\alpha_n\tau_n : n \in \mathbb{N}\}$  and  $M_5 := \sup\{(1 + L)a_n : n \in \mathbb{N}\}$ . Then from (2.6), we obtain

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + M_4 - 2\alpha_n\Phi(\|x_{n+1} - p\|) + 2\alpha_nL(\alpha_n + \alpha_nL + \beta_n)^2 \times \{[(\alpha_n(1 + L) + \beta_n)M_1 + M_2M_3 + M_5]^2\} + 2A_nM_1^2. \tag{2.8}$$

Taking  $\theta_n = \|x_n - p\|^2$ ,  $\lambda_n = 2\alpha_n$  and  $\sigma_n = 2\alpha_nL(\alpha_n + \alpha_nL + \beta_n)^2\{[(\alpha_n(1 + L) + \beta_n)M_1 + M_2M_3 + M_5]^2\} + 2A_nM_1^2 + M_4$ , (2.8) reduces to

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n\phi(\theta_{n+1}) + \sigma_n.$$

Hence from Lemma 1.3, we have that  $\|x_n - p\| \rightarrow 0$ . The proof of Theorem 2.2 is completed.  $\square$

**Corollary 2.3.** Let  $C$  be a nonempty convex subset of a real Banach space  $E$  and  $T : C \rightarrow C$  a nearly uniformly  $L$ -Lipschitzian mapping with sequence  $\{a_n\}$  and asymptotically generalized  $\Phi$ -hemicontractive with sequence  $\{k_n\}$  as defined in (1.5) and  $F(T) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence in  $[0, 1]$  satisfying the conditions:

- (i)  $\frac{1}{\alpha_n + \alpha_nL + \beta_n}$  is bounded, (ii)  $\sum_{n=1}^\infty \alpha_n = \infty$ ,
- (iii)  $\sum_{n=1}^\infty \alpha_n^2 < \infty$  and  $\sum_{n=1}^\infty \alpha_n(k_n - 1) < \infty$ .

Let  $\{x_n\}$  be the sequence in  $E$  generated from arbitrary  $x_1 \in C$  by

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_nT^n x_n + \beta_nu_n, \quad n \in \mathbb{N}, \tag{2.9}$$

where  $\{u_n\}$  is a bounded sequence in  $E$ . Then the sequence  $\{x_n\}$  in  $C$  defined by (2.9) converges strongly to a unique fixed point of  $T$ .

**Remark 2.4.** We remark that Theorem 2.2 and of course, Corollary 2.3 improves and generalizes the results of Chang *et al.* [4], Ofoedu [11], Liu *et*

al. [9], Olaleru and Mogbademu [12] and includes the results of Chidume and Chidume [5] and Theorem KSN as special cases since the class of *asymptotically generalized  $\Phi$ -hemicontractive mappings in the intermediate sense* introduced in this paper is more general than those defined by those authors.

**Example 2.5.** Let  $E = \mathbb{R}$ ,  $C = [0, 1]$  and  $T : C \rightarrow C$  a mapping defined by

$$Tx = \begin{cases} \frac{x}{2} & \text{if } x \in [0, 1), \\ 0 & \text{if } x = 1. \end{cases} \quad (2.10)$$

Clearly,  $T$  is not a continuous mapping and the unique fixed point of  $T$  is  $x = 0$ . It was shown by Sahu and Beg [17] that  $T$  is not Lipschitzian, but it is nearly  $\frac{1}{2}$ -Lipschitzian with sequence  $\{\frac{1}{2^n}\}$ . We can easily show that  $T$  is an *asymptotically generalized  $\Phi$ -hemicontractive mapping in the intermediate sense* with sequences  $\{k_n = 1\}$ ,  $\tau_n = \frac{1}{n^2}$  and  $\Phi(t) = \frac{t^2}{2}$ ,  $t \in [0, \infty)$  as defined in (1.17).

Put  $\alpha_n = \frac{1}{n}$ ,  $\beta_n = \frac{1}{n^2}$ . We see that the assumptions (i), (ii) and (iii) of Theorem 2.2 are satisfied.

If  $x_1 = 1$ , then  $x_n = 0$  for each  $n \geq 2$ . Hence, the sequence  $\{x_n\}$  converges to 0. Moreover, if  $x_1 \in [0, 1)$ , then using (2.3), we obtain

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n \frac{x_n}{2^n} + \beta_n u_n \\ &= (1 - (1 - \frac{1}{2^n})\alpha_n)x_n + \beta_n u_n, \quad n \in \mathbb{N}. \end{aligned} \quad (2.11)$$

Since  $\sum_{n=1}^{\infty} (1 - \frac{1}{2^n})\alpha_n = \infty$  and  $\{u_n\}$  is bounded, from (2.11), we have that  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ .

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