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COUPLED FIXED POINT THEOREMS OF INTEGRAL TYPE MAPPINGS IN CONE METRIC SPACES

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Abstract. In this paper, we prove some coupled fixed point theorems in cone metric spaces. Furthermore, we introduce and prove the integral version of coupled fixed point theorems in cone metric spaces. Our results unify, extend and generalize the known results on coupled fixed point theorems in cone metric spaces.

1. Introduction

Huang and Zhang [6] generalized the concept of metric spaces by considering vector-valued metrics (cone metrics) with values in ordered real Banach spaces. Since then, several fixed point theorems have been proved in the context of cone metric spaces (for example, see [1, 3, 5, 6, 7, 12, 14, 15, 16, 17, 18]. The concept of coupled fixed point was recently introduced by T. G. Bhaskar and V. Lakshmikantham [3], Lakshmikantham and Ćirić [10], with Wei-Shih Du [16] later proved some results on coupled fixed points. Recently, F. Sabetghadam, H. P. Masiha and A. H. Sanatpour [14] introduced and proved some coupled fixed point theorems in the context of cone metric spaces. In this paper, we unify, extend and generalize the results in [14] using the idea of a nondecreasing function similar to the one introduced in [15].

Fixed point theorems of mappings of integral type in metric spaces are common in literature, for example, see [2]. Khojasteh et al. [17, 18] recently extended some fixed point results for mappings of integral type from metric spaces to cone metric spaces. In this paper, the authors introduced and prove some coupled fixed point theorems of integral type mappings in cone metric spaces and subsequently in metric spaces.

Definition 1.1. ([6]). A cone P is a subset of a real Banach space E such that

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(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$;
(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, the partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. The notation $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P. Also, we will use $x < y$ to indicate that $x \leq y$ and $x \neq y$.

The cone P is called normal if there exists a constant $M > 0$ such that for every $x, y \in E$ if $0 \leq x \leq y$ then $\|x\| \leq M\|y\|$. The least positive number satisfying this inequality is called the normal constant of P.

In this paper, we suppose that E is a real Banach space, $P \subseteq E$ is a cone with $intP \neq \emptyset$ and $\leq$ is partial ordering with respect to P. We also note that the relations $P + intP \subseteq intP$ and $\lambda intP \subseteq intP$ $(\lambda > 0)$ always hold true.

Definition 1.2. ([6]). Let X be a nonempty set and let E be a real Banach space equipped with the partial ordering $\leq$ with respect to the cone $P \subseteq E$. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 1.3. ([6]). Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ be a sequence in X. Then

(i) $\{x_n\}_{n \geq 1}$ converges to x, denoted by $\lim_{n \to \infty} x_n = x$, if for every $c \in E$ with $0 \ll c$ there exist a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$;
(ii) $\{x_n\}_{n \geq 1}$ is a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exists a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

A cone metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 1.4. ([17]). Let (X, d) be a cone metric space. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Recently, F. Sabetghadam, H. P. Masiha and A. H. Sanatpour [14] proved the existence of unique coupled fixed point for the following contractive conditions in a cone metric space:

$$d(F(x, y), F(u, v)) \leq kd(x, u) + ld(y, v), \tag{1.1}$$

where k, l are nonnegative constants with $k + l < 1$.

$$d(F(x, y), F(u, v)) \leq kd(F(x, y), x) + ld(F(u, v), u), \tag{1.2}$$

where k, l are nonnegative constants with $k + l < 1$.

$$d(F(x,y), F(u,v)) \leq kd(F(x,y), u) + ld(F(u,v), x), \tag{1.3}$$

where k, l are nonnegative constants with $k + l < 1$.

In this research paper, we unify, extend and generalize the contractive conditions (1.1), (1.2) and (1.3). Furthermore, we introduce and prove the integral version of coupled fixed point theorem, thereby generalizing the known results on coupled fixed point theorems.

Definition 1.5. ([15]). Let P be a cone and let $\{\omega_n\}$ be a sequence in P. One says that $\omega_n \xrightarrow{\ll} 0$ if for every $\epsilon \in P$ with $0 \ll \epsilon$ there exists $N > 0$ such that $\omega_n \ll \epsilon$ for all $n \geq N$.

Definition 1.6. For a non-decreasing mapping $T : P \to P$, we define the following conditions which will be used in the sequel:

(T_1) For every $\omega_n \in P$, $\omega_n \xrightarrow{\ll} \theta$ if and only if $T\omega_n \xrightarrow{\ll} \theta$;
(T_2) For every $\omega_1, \omega_2 \in P$, $T(\alpha\omega_1 + \beta\omega_2) \preceq \alpha T(\omega_1) + \beta T(\omega_2)$ for $\alpha,\ \beta \in\ [0,1)$.

Lemma 1.1. *If a mapping $T : P \to P$ satisfies (T_1), then, for all $\omega \in P$, $T(\omega) = \theta \Longleftrightarrow \omega = \theta$.*

Proof. Let $\omega \in P$ be such that $T(\omega) = \theta$, i.e. $T(\omega) = \theta \xrightarrow{\ll} \theta$. Following (T_1), $\omega \xrightarrow{\ll} \theta$. This implies that $\omega \ll \epsilon$ for all ϵ satisfying $\theta \ll \epsilon$. Hence $\omega = \theta$. Conversely, it is clear by (T_2) that $T(\theta) = \theta$. □

2. Main Results

Theorem 2.1. *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition:*

$$T(d(F(x,y), F(u,v))) \leq T(j), \tag{2.1}$$

for all $x, y, u, v \in X$, where

$$\begin{aligned} j &= c_1 d(x,u) + c_2 d(y,v) + c_3 d(F(x,y), x) \\ &+ c_4 d(F(u,v), u) + c_5 d(F(x,y), u) + c_6 d(F(u,v), x), \end{aligned}$$

$c_1, c_2, c_3, c_4, c_5, c_6$ are nonnegative constants with $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 < 1$ and $T : P \to P$ is a nondecreasing mapping satisfying $(T_1) - (T_2)$. Then F has a unique coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set

$$x_1 = F(x_0, y_0), \quad y_1 = F(y_0, x_0), \quad \ldots, \quad x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n),$$

we have:

$$
\begin{aligned}
T\left(d(x_n, x_{n+1})\right) &= T\left(d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))\right) \\
&\leq T\left(c_1 d(x_{n-1}, x_n) + c_2 d(y_{n-1}, y_n)\right. \\
&\quad + c_3 d(F(x_{n-1}, y_{n-1}), x_{n-1}) + c_4 d(F(x_n, y_n), x_n) \\
&\quad \left. + c_5 d(F(x_{n-1}, y_{n-1}), x_n) + c_6 d(F(x_n, y_n), x_{n-1})\right) \\
&\leq c_1 T\left(d(x_{n-1}, x_n)\right) + c_2 T\left(d(y_{n-1}, y_n)\right) \\
&\quad + c_3 T\left(d(F(x_{n-1}, y_{n-1}), x_{n-1})\right) + c_4 T\left(d(F(x_n, y_n), x_n)\right) \\
&\quad + c_5 T\left(d(F(x_{n-1}, y_{n-1}), x_n)\right) + c_6 T\left(d(F(x_n, y_n), x_{n-1})\right) \\
&\leq \left(\frac{c_1 + c_2}{2}\right) T\left(d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\right) \\
&\quad + \left(\frac{c_3 + c_4}{2}\right) T\left(d(x_{n-1}, x_n) + d(x_{n+1}, x_n)\right) \\
&\quad + \left(\frac{c_5 + c_6}{2}\right) T\left(d(x_{n+1}, x_{n-1})\right). \qquad (2.2)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
T\left(d(y_n, y_{n+1})\right) &= T\left(d\left(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\right)\right) \\
&\leq \left(\frac{c_1 + c_2}{2}\right) T\left(d(y_{n-1}, y_n) + d(x_{n-1}, x_n)\right) \\
&\quad + \left(\frac{c_3 + c_4}{2}\right) T\left(d(y_n, y_{n-1}) + d(y_{n+1}, y_n)\right) \\
&\quad + \left(\frac{c_5 + c_6}{2}\right) T\left(d(y_{n+1}, y_{n-1})\right). \qquad (2.3)
\end{aligned}
$$

Set $d_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1})$ and add (2.2) and (2.3) to obtain:

$$
\begin{aligned}
d_n \leq\; & (c_1 + c_2) T\left(d(x_n, x_{n-1}) + d(y_n, y_{n-1})\right) \\
& + \left(\frac{c_3 + c_4}{2}\right) T\left(d(x_n, x_{n-1}) + d(x_{n+1}, x_n) + d(y_n, y_{n-1}) + d(y_{n+1}, y_n)\right) \\
& + \left(\frac{c_5 + c_6}{2}\right) T\left(d(x_{n+1}, x_{n-1}) + d(y_{n+1}, y_{n-1})\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
(1 &- \frac{(c_3 + c_4)}{2} - \frac{(c_5 + c_6)}{2}) T\left(d(x_n, x_{n+1}) + d(y_n, y_{n+1})\right) \\
&\leq \left(c_1 + c_2 + \frac{c_3 + c_4}{2} + \frac{c_5 + c_6}{2}\right) T\left(d(x_n, x_{n-1}) + d(y_n, y_{n-1})\right) \\
&= \left(\frac{2c_1 + 2c_2 + c_3 + c_4 + c_5 + c_6}{2}\right) T\left(d(x_n, x_{n-1}) + d(y_n, y_{n-1})\right).
\end{aligned}
$$

Hence,

$$
T\left(d(x_n, x_{n+1}) + d(y_n, y_{n+1})\right) \leq \delta T\left(d(x_n, x_{n-1}) + d(y_n, y_{n-1})\right),
$$

i.e. $T(d_n) \leq \delta T(d_{n-1})$, where

$$\delta = \frac{2c_1 + 2c_2 + c_3 + c_4 + c_5 + c_6}{2 - c_3 - c_4 - c_5 - c_6} < 1.$$

For each $n \in \aleph$, we have

$$0 \leq T(d_n) \leq \delta T(d_{n-1}) \leq \delta^2 T(d_{n-2}) \leq \cdots \leq \delta^n T(d_0).$$

If $d_0 = 0$ then (x_0, y_0) is a coupled fixed point of F. Now, let $d_0 > 0$, for each $n \geq m$ we have:

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m),$$

$$d(y_n, y_m) \leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \cdots + d(y_{m+1}, y_m).$$

Therefore,

$$\begin{aligned} T\left(d(x_n, x_m) + d(y_n, y_m)\right) &\leq T\left(d_{n-1} + d_{n-2} + \cdots + d_m\right) \\ &\leq (\delta^{n-1} + \delta^{n-2} + \cdots + \delta^m) T(d_0) \\ &\leq \frac{\delta^m}{1-\delta} T(d_0), \end{aligned}$$

which implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X, and there exist $x^*, y^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. Thus

$$\begin{aligned} T\left(d(F(x^*, y^*), x^*)\right) &\leq T\left(d(F(x^*, y^*), x_{N+1}) + d(x_{N+1}, x^*)\right) \\ &= T\left(d(F(x^*, y^*), F(x_N, y_N)) + d(x_{N+1}, x^*)\right) \\ &\leq c_1 T\left(d(x^*, x_N)\right) + c_2 T\left(d(y^*, y_N)\right) + c_3 T\left(d(F(x^*, y^*), x^*)\right) \\ &\quad + c_4 T\left(d(x_{N+1}, x_N)\right) + c_5 T\left(d(F(x^*, y^*), x_N)\right) \\ &\quad + c_6 T\left(d(x_{N+1}, x^*)\right) + T\left(d(x_{N+1}, x^*)\right). \end{aligned}$$

This implies that

$$\begin{aligned} (1 - c_3 - c_5) T(d(F(x^*, y^*), x^*)) &\leq (c_1 + c_4 + c_5) T(d(x^*, x_N)) \\ &\quad + (c_4 + c_6) T(d(x_{N+1}, x^*)) \\ &\quad + c_2 T(d(y^*, y_N)) + T(d(x_{N+1}, x^*)). \end{aligned}$$

Hence,

$$\begin{aligned} T(d(F(x^*, y^*), x^*)) &\leq \left(\frac{c_1 + c_4 + c_5}{1 - c_3 - c_5}\right) T(d(x^*, x_N)) + \left(\frac{c_2}{1 - c_3 - c_5}\right) T(d(y^*, y_N)) \\ &\quad + \left(\frac{c_4 + c_6}{1 - c_3 - c_5}\right) T(d(x^*, x_{N+1})) + \left(\frac{1}{1 - c_3 - c_5}\right) T(d(x^*, x_{N+1})). \end{aligned}$$

Let $c \in E$ with $0 \ll c$, by T_1 and the convergence of $\{x_n\}$ and $\{y_n\}$ to x^*, we can choose

$$\begin{aligned}
T(d(x^*, x_N)) &\ll (\frac{1-c_3-c_5}{c_1+c_4+c_5})\frac{c}{4};\\
T(d(y^*, y_N)) &\ll (\frac{1-c_3-c_5}{c_2})\frac{c}{4};\\
T(d(x^*, x_{N+1})) &\ll (\frac{1-c_3-c_5}{c_4+c_6})\frac{c}{4}
\end{aligned}$$

and

$$T(d(x^*, x_{N+1})) \ll (\frac{1-c_3-c_5}{c_4+c_6})\frac{c}{4}.$$

Then,

$$T(d(F(x^*, y^*), x^*)) \ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c.$$

Consequently,

$$T(d(F(x^*, y^*), x^*)) \ll c.$$

Thus $d(F(x^*, y^*), x^*) = 0$ and hence $F(x^*, y^*) = x^*$.

Similarly, we have $F(y^*, x^*) = y^*$ meaning that (x^*, y^*) is a coupled fixed point of F.

Next, we show uniqueness. If (x', y') is another coupled fixed point of F, then

$$\begin{aligned}
T(d(x', x^*)) &= T(d(F(x', y'), F(x^*, y^*)))\\
&\leq c_1T(d(x', x^*)) + c_2T(d(y', y^*))\\
&\quad +c_3T(d(F(x', y'), x')) + c_4T(d(F(x^*, y^*), x^*))\\
&\quad +c_5T(d(F(x', y'), x^*)) + c_6T(d(F(x^*, y^*), x'))\\
&= c_1T(d(x', x^*)) + c_2T(d(y', y^*))\\
&\quad +c_3T(d((x', x')) + c_4T(d(x^*, x^*))\\
&\quad +c_5T(d(x', x^*)) + c_6T(d(x^*, x')).
\end{aligned}$$

Similarly,

$$\begin{aligned}
T(d(y', y^*)) &= T(d(F(y', x'), F(y^*, x^*)))\\
&\leq c_1T(d(y', y^*)) + c_2T(d(x', x^*))\\
&\quad +c_3T(d(y', y')) + c_4T(d(y^*, y^*))\\
&\quad +c_5T(d(y', y^*)) + c_6T(d(y^*, y')).
\end{aligned}$$

Hence,

$$T(d(x', x^*) + d(y', y^*)) \leq \delta T(d(x', x^*) + d(y', y^*)), \tag{2.4}$$

where $\delta = (c_1 + c_2 + c_5 + c_6) < 1$.

(2.4) implies that $d(x', x^*) + d(y', y^*) = 0$. Hence, we have $d(x', x^*) = d(y', y^*) = 0$, i.e $x' = x^*$ and $y' = y^*$. Thus, $(x', y') = (x^*, y^*)$ and the proof of the theorem is complete. □

If $T = I_d$ (Identity map), then Theorem 2.1 leads to the following Corollary:

Corollary 2.1. *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition:*

$$\begin{aligned} d(F(x,y), F(u,v)) &\leq c_1 d(x,u) + c_2 d(y,v) + c_3 d(F(x,y),x) + c_4 d(F(u,v),u) \\ &+ c_5 d(F(x,y),u) + c_6 d(F(u,v),x), \end{aligned} \tag{2.5}$$

for all $x, y, u, v \in X$, where $c_1, c_2, c_3, c_4, c_5, c_6$ are nonnegative constants with $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 < 1$. Then F has a unique coupled fixed point.

Remark 2.1.

(i) Theorem 2.1 is the coupled fixed point theorem version of the result of Olaleru [12].

(ii) In Corollary 2.1, if we set $c_3 = c_4 = c_5 = c_6 = 0$, then we obtain [14, Theorem 2.2]. If $c_1 = c_2 = c_5 = c_6 = 0$, then we obtain [14, Theorem 2.5] and if $c_1 = c_2 = c_3 = c_4 = 0$, then we obtain [14, Theorem 2.6].

The following is a combination of [14, Corollary 2.3], of [14, Corollary 2.7] and of [14, Corollary 2.8]:

Corollary 2.2. *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition:*

$$\begin{aligned} T(d(F(x,y), F(u,v))) &\leq (\frac{c_1}{6}) T((d(x,u) + d(y,v) + d(F(x,y),x) \\ &+ d(F(u,v),u) + d(F(x,y),u) + d(F(u,v),x))), \end{aligned}$$

for all $x, y, u, v \in X$, where $c_1 \in [0,1)$ is a nonnegative constant and $T : P \to P$ is a nondecreasing mapping satisfying $(T_1) - (T_2)$. Then F has a unique coupled fixed point.

Example 2.1. Let $E = \mathbb{R}^2$, $P = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\} \subseteq \mathbb{R}^2$, and $X = [0,1]$. Define $d : X \times X \to E$ with $d(x,y) = (|x-y|, |x-y|)$. Then (X, d) is a complete cone metric space. Consider the mapping $F : X \times X \to X$ with $F(x,y) = \frac{(x+y)}{2}$. Then F satisfies the contractive condition of Corollary 2.2 for $c_1 = 1$, i.e,

$$\begin{aligned} d(F(x,y), F(u,v)) &\leq \frac{1}{6}(d(x,u) + d(y,v) + d(F(x,y),x) \\ &+ d(F(u,v),u) + d(F(x,y),u) + d(F(u,v),x)). \end{aligned}$$

Hence, by Corollary 2.2, F has a unique coupled fixed point, which in this case is $(0,0)$. Observe that F does not have a unique coupled fixed point in [14] when $k = c_1 = 1$. (See [14, Corollary 2.3], contractive condition (2.14)).

3. Coupled fixed point of operators satisfying contractive condition of integral type

We start with some definitions, examples and properties as introduced in [17].

Definition 3.1. ([17]) Suppose that P is a normal cone in E. Let $a, b \in E$ and $a < b$. We define

$$[a,b] := \{x \in E : x = tb + (1-t)a, \text{ where } t \in [0,1]\}$$
$$[a,b) := \{x \in E : x = tb + (1-t)a, \text{ where } t \in [0,1)\}.$$

Definition 3.2. ([17]) The set $\{a = x_0, x_1, \ldots, x_n = b\}$ is called a partition for $[a,b]$ if and only if the sets $[x_{i-1}, x_i]$, $1 \le i \le n$, are pairwise disjoint and

$$[a,b] = \{\bigcup_{i=1}^{n} [x_{i-1}, x_i)\} \cup \{b\}.$$

Definition 3.3. ([17]) Suppose that P is a normal cone in E, $\phi : [a,b] \to P$ a map. ϕ is said to be *integrable on* $[a,b]$ *with respect to cone* P (or *cone integrable function*) if and only if for all partition Q of $[a,b]$

$$\lim_{n\to\infty} L_n^{Con}(\phi, Q) = S^{Con} = \lim_{n\to\infty} U_n^{Con}(\phi, Q)$$

where S^{Con} must be unique and:

$$L_n^{Con} = \sum_{i=0}^{n-1} \phi(x_i)\|x_i - x_{i+1}\| \text{ (Cone lower summation) and}$$
$$U_n^{Con} = \sum_{i=0}^{n-1} \phi(x_{i+1})\|x_i - x_{i+1}\| \text{ (Cone upper summation).}$$

We note

$$S^{Con} = \int_a^b \phi(x) d_P(x) = \int_a^b \phi d_P.$$

The set of all cone integrable functions $\phi : [a,b] \to P$ is denoted $L^1([a,b], P)$.

Definition 3.4. ([17]) The function $\phi : P \to E$ is called *subadditive cone integrable function* if and only if for all $a, b \in P$

$$\int_0^{a+b} \phi d_P \le \int_0^a \phi d_P + \int_0^b \phi d_P.$$

Example 3.1. ([17]) Let $E = X = \mathbb{R}$, $d(x,y) = |x-y|$, $P = [0, +\infty)$ and $\phi(t) = \frac{1}{t+1}$ for all $t > 0$. Then ϕ is a subbaditive cone integral function.

We are now in position to state the following Theorem:

Theorem 3.1. *Let (X,d) be a cone metric space and let and P a normal cone. Let $\phi : P \to P$ be a nonvanishing map and a subbaditive cone integrable on each $[a,b]$. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition*

$$\int_0^{d(F(x,y),F(u,v))} \phi(t) d_P(t) \le \int_0^{j(x,y,u,v)} \phi(t) d_P(t), \tag{3.1}$$

for all $x, y, u, v \in X$, *where*

$$\begin{aligned} j(x,y,u,v) &= c_1 d(x,u) + c_2 d(y,v) + c_3 d(F(x,y),x) \\ &+ c_4 d(F(u,v),u) + c_5 d(F(x,y),u) + c_6 d(F(u,v),x), \end{aligned}$$

$c_1, c_2, c_3, c_4, c_5, c_6$ *are nonnegative constants with* $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 < 1$. *Then* F *has a unique coupled fixed point.*

Proof. Theorem 3.1 is a Corollary of Theorem 2.1 when $T\left(j(x,y,u,v)\right) = \int_0^{j(x,y,u,v)} \phi d_P$. Under this case, T satisfies conditions $(T_1) - (T_2)$. (T_2) results from the subbaditivity of ϕ. The condition (T_1) results from the continuity of T and its inverse in θ. In fact, in a normal cone, if $\omega_n \xrightarrow{\ll} \omega$, then ω_n converges to ω. Now, since T is continuous in θ, for every sequence ω_n converging to θ, $T(\omega_n)$ converges to $T(\theta) = \theta$. Since T^{-1} is continuous, given any sequence $T(\omega_n)$ converging to θ, $T^{-1}(T(\omega_n)) = \omega_n$ converges to $T^{-1}(\theta) = \theta$; thus (T_1) is satisfied. □

Theorem 3.1 lead to the following Corollary:

Corollary 3.1. *Let* (X, d) *be a cone metric space and let and* P *a normal cone. Let* $\phi : P \to P$ *be a nonvanishing map and a subbaditive cone integrable on each* $[a, b]$. *Suppose that the mapping* $F : X \times X \to X$ *satisfies the following contractive condition*

$$\int_0^{d(F(x,y),F(u,v))} \phi(t) d_P(t) \leq \int_0^{j(x,y,u,v)} \phi(t) d_P(t), \tag{3.2}$$

for all $x, y, u, v \in X$, *where*

$$\begin{aligned} j(x,y,u,v) &= \frac{c_1}{6}\left(d(x,u) + d(y,v) + d(F(x,y),x)\right. \\ &\left. + d(F(u,v),u) + d(F(x,y),u) + d(F(u,v),x)\right)) \end{aligned}$$

and $c_1 \in [0,1)$ *is a nonnegative constant. Then* F *has a unique coupled fixed point.*

Remark 3.1.

(i) Theorem 3.1 is the Coupled fixed point of integral type of Theorem 2.1.
(ii) We can recover Corollary 2.1 from Theorem 3.1 if we set $\phi(t) = constant$.

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