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Modified Noor Iterations with Errors for Generalized Strongly Φ-Pseudocontractive Maps in Banach Spaces

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Abstract : In this paper, we prove some strong convergence results for a family of three generalized strongly Φ-pseudocontractive (accretive) mappings in Banach spaces. Our results are generalizations and improvements of convergence results obtained by several authors in literature. In particular, they generalize and improve the results of Olaleru and Mogbademu [17], Xue and Fan [22] which is in turn a correction of Rafiq [18].

Keywords : noor iterative scheme with errors; banach spaces; generalized strongly Φ-pseudocontractive operators; unique common fixed point; strongly Φ-accretive operators.
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1 Introduction

In this study, we assume that E is a real Banach space and D is a nonempty closed convex subset of E. We denote by J the normalized duality from E to 2^{E^*}

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defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\} \tag{1.1}$$

where E^* denotes the dual space of E and $\langle .,. \rangle$ denotes the generalized duality pairing. We shall also denote the single-valued duality mapping by j.

Definition 1.1 [18]. A map $T : E \to E$ is called strongly accretive if there exists a constant $k > 0$ such that, for each $x, y \in E$, there is a $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2. \tag{1.2}$$

Definition 1.2 [18]. An operator T with domain $D(T)$ and range $R(T)$ in E is called *strongly pseudocontractive* if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ and a constant $0 < k < 1$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq k\|x-y\|^2. \tag{1.3}$$

The class of strongly accretive operators is closely related to the class of strongly pseudocontractive operators. It is well known that T is strongly pseudocontractive if and only if $(I-T)$ is strongly accretive, where I denotes the identity operator. Browder [1] and Kato [7] indepedently introduced the concept of accretive operators in 1967. One of the early results in the theory of accretive operators credited to Browder states that the initial value problem

$$\frac{du(t)}{dt} + Tu(t) = 0, \quad u(0) = u_0 \tag{1.4}$$

is solvable if T is locally Lipschitzian and accretive on E.
These class of operators have been studied extensively by several authors (see [2, 3, 8, 9, 10, 14 - 15, 17, 18, 22]).

Definition 1.3 [24]. A mapping T is called strongly ϕ-pseudocontractive if for all $x, y \in E$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|.$$

Definition 1.4 [24]. A mapping T is called generalized strongly Φ- pseudocontractive if for all $x, y \in E$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|).$$

Clearly, every strongly ϕ-pseudocontractive operator is a generalized strongly Φ-pseudocontractive operator with $\Phi : [0,\infty) \to [0,\infty)$ defined by $\Phi(s) = \phi(s)s$, and every strongly pseudocontractive operator is strongly ϕ-pseudocontractive operator with ϕ defined by $\phi(s) = ks$ for all $k \in (0,1)$ while the converses need not

be true.

Definition 1.5. [18]. A mapping $T : E \to E$ is called Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in D(T). \tag{1.5}$$

In 1953, Mann [9]introduced the Mann iterative scheme and used it to prove the convergence of the sequence to the fixed points for which the Banach principle is not applicable. Ishikawa [6] in 1974, introduced an iterative process to obtain the convergence of a Lipschitzian pseudocontractive operator when Mann iterative scheme failed to converge. Noor [13,14] in 2000 gave the following three-step iterative scheme for solving nonlinear operator equations in uniformly smooth Banach spaces.
Let D be a nonempty convex subset of E and let $T : D \to D$ be a mapping. For a given $x_0 \in D$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative schemes

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, & n \in \mathbb{Z},\ n \geq 0, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, & n \in \mathbb{Z},\ n \geq 0, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, & n \in \mathbb{Z},\ n \geq 0, \end{cases} \tag{1.6}$$

which is called the three-step iterative process, where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0, 1]$ satisfying some certain conditions.

If $\gamma_n = 0$ and $\beta_n = 0$, for each $n \in \mathbb{Z}$, $n \geq 0$, then (1.6) reduces to:

for a given $x_0 \in D$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative scheme

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \in \mathbb{Z},\ n \geq 0, \tag{1.7}$$

which is called the one-step Mann iterative scheme, introduced by Mann [9].

For $\gamma_n = 0$, (1.6) reduces to:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, & n \in \mathbb{Z},\ n \geq 0, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, & n \in \mathbb{Z},\ n \geq 0, \end{cases} \tag{1.8}$$

which is called the two-step Ishikawa iterative process introduced by Ishikawa [6], where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two real sequences in $[0, 1]$ satisfying some certain conditions.

In 1989, Glowinski and Le Tallec [4] used a three-step iterative process to solve elastoviscoplasticity, liquid crystal and eigenvalue problems. They established that three-step iterative schemes performs better than one-step (Mann) and two-step (Ishikawa) iterative schemes. Haubruge *et al.* [5] studied the convergence analysis of the three-step iterative processes of Glowinski and Le Tallec [4] and used these three-step iterations to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of

convex functions. They also proved that three-step iterations also lead to highly parallelized algorithms under certain conditions. Hence, we can conclude by observing that three-step iterative schemes play an important role in solving various problems in pure and applied sciences.

In 2006, Rafiq [18] introduced the following modified three-step iterative schemes and used it to approximate the unique common fixed point of a family of strongly pseudocontractive operators.

Let $T_1, T_2, T_3 : D \to D$ be three given mappings. For a given $x_0 \in D$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative scheme

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_2 z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n, \quad n \geq 0, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ satisfying some certain conditions. Equation (1.9) is called the modified three-step iterative process. Observe that algorithms (1.6)-(1.8) are special cases of (1.9).

Suantai [19] introduced the following three-step iterative schemes.
Let E be a normed space, D be a nonempty convex subset of E, and $T : D \to D$ be a given mapping. Then for a given $x_1 \in D$, compute the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{cases} z_n = a_n T^n x_n + (1-a_n)x_n, \\ y_n = b_n T^n z_n + c_n T^n x_n + (1-b_n-c_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n + \beta_n T^n z_n + (1-\alpha_n-\beta_n)x_n, \quad n \geq 1, \end{cases} \tag{1.10}$$

where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are appropriate sequences in $[0,1]$.

Motivated by the facts above, we now introduce the following modified three-step iterative scheme with errors which we shall use in this paper to approximate the unique common fixed point of a family of strongly pseudocontractive maps.

Let E be a real Banach space, D be a nonempty convex subset of E, and $T_i : D \to D$, $(i = 1, 2, 3)$ be a family of three mappings. Then for a given $x_0, u_0, v_0, w_0 \in D$, compute the sequence $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ by the iterative scheme

$$\begin{cases} x_{n+1} = (1-\alpha_n-\beta_n-e_n)x_n + \alpha_n T_1 y_n + \beta_n T_1 z_n + e_n u_n, \quad n \geq 0, \\ y_n = (1-a_n-b_n-e'_n)x_n + a_n T_2 z_n + b_n T_2 x_n + e'_n v_n, \\ z_n = (1-c_n-e''_n)x_n + c_n T_3 x_n + e''_n w_n, \end{cases} \tag{1.11}$$

where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{e_n\}_{n=0}^{\infty}$, $\{e'_n\}_{n=0}^{\infty}$, $\{e''_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying certain conditions and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are

bounded sequences in D.

Observe that (1.6)-(1.10) and the modified three step iteration process with errors introduced by Mogbademu and Olaleru [10] are special cases of (1.11). In this paper, we shall use algorithm (1.11) to approximate the unique common fixed point of a family of three generalized strongly Φ-pseudocontractive operators in Banach spaces. Hence, our results are generalizations and improvements of the results of Olaleru and Mogbademu [10,17], Xue and Fan [22] which in turn is a correction of Rafiq [18].

Rafiq [17] proved the following theorem

Theorem R [18]. Let E be a real Banach space and D be a nonempty closed convex subset of E. Let T_1, T_2, T_3 be strongly pseudocontractive self maps of D with $T_1(D)$ bounded and T_1, T_3 be uniformly continuous. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_2 z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ satisfying the conditions:

$\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.
If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2, T_3.

Xue and Fan [22] obtained the following convergence results which is in turn a correction of the results of Rafiq [18].

Theorem XF [22]. Let E be a real Banach space and D be a nonempty closed convex subset of E. Let T_1, T_2 and T_3 be strongly pseudocontractive self maps of D with $T_1(D)$ bounded and T_1, T_2 and T_3 uniformly continuous. Let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.9), where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ which satisfy the conditions: $\alpha_n, \beta_n \to 0$ as $n \to \infty$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2 and T_3.

In this study, we shall prove convergence theorems using our newly introduced iterative scheme (1.11). Our results are generalizations and improvements of the results of Ćirić and Ume [3], Olaleru and Mogbademu [17], Xue and Fan [22] which in turn is a correction of Rafiq [18].

The following lemma will be useful in this study.

Lemma 1.1 [18]. Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall\ j(x+y) \in J(x+y). \tag{1.12}$$

Lemma 1.2 [11]. Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying
$\sum_{n=0}^{\infty} b_n = +\infty$ and $\lim_{n\to\infty} b_n = 0$.
Suppose that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying $a_{n+1}^2 < a_n^2 + o(b_n) - b_n\Phi(a_{n+1})$ for all $n \geq N_0$. where $\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, then $\lim_{n\to\infty} a_n = 0$.

2 Main Results

Theorem 2.1 Let E be a real Banach space and D be a nonempty closed convex subset of E. Let T_1, T_2 and T_3 be generalized strongly Φ-pseudocontractive self maps of D with $T_1(D)$ bounded and T_1, T_2 and T_3 uniformly continuous. Let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.11), where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{e_n\}_{n=0}^{\infty}$, $\{e_n'\}_{n=0}^{\infty}$, $\{e_n''\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying the conditions: $a_n, b_n, c_n, e_n', e_n'', \alpha_n, \beta_n,\ e_n \longrightarrow 0$ as $n \to \infty$, $\alpha_n + \beta_n + e_n < 1$, $a_n + b_n + e_n' < 1$, $c_n + e_n'' < 1$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$ are bounded sequences in D. If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2 and T_3.

Proof. Since T_1, T_2, T_3 are generalized strongly Φ-pseudocontractive, there exists $j(x-y) \in J(x-y)$ and a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle T_i x - T_i y, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|), \quad i = 1,2,3. \tag{2.1}$$

Assume that $p \in F(T_1) \cap F(T_2) \cap F(T_3)$, using the fact that T_i is generalized strongly Φ-pseudocontractive for each $i = 1,2,3$ we obtain $F(T_1) \cap F(T_2) \cap F(T_3) = p \neq \emptyset$. Since T_1 has a bounded range, we let

$$M_1 = \|x_0 - p\| + \sup_{n\geq 0} \|T_1 y_n - p\| + \sup_{n\geq 0} \|T_1 z_n - p\| + \|u_n - p\|. \tag{2.2}$$

We shall prove by induction that $\|x_n - p\| \leq M_1$ holds for all $n \in \mathbb{N}$. We observe from (2.2) that $\|x_0 - p\| \leq M_1$. Assume that $\|x_n - p\| \leq M_1$ holds for all $n \in \mathbb{N}$. We will prove that $\|x_{n+1} - p\| \leq M_1$. Using (1.11), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n-\beta_n-e_n)(x_n - p) + \alpha_n(T_1 y_n - p) + \\
&\quad \beta_n(T_1 z_n - p) + e_n(u_n - p)\| \\
&\leq (1-\alpha_n-\beta_n-e_n)\|x_n - p\| + \alpha_n\|T_1 y_n - p\| + \\
&\quad \beta_n\|T_1 z_n - p\| + e_n\|u_n - p\| \\
&\leq (1-\alpha_n-\beta_n-e_n)M_1 + \alpha_n M_1 + \beta_n M_1 + e_n M_1 \\
&= M_1.
\end{aligned} \tag{2.3}$$

Using the uniform continuity of T_3, we obtain that $\{T_3 x_n\}_{n=0}^{\infty}$ is bounded. We now set

$$M_2 = \max\left\{M_1, \sup_{n\geq 0}\{\|T_3 x_n - p\|\}, \sup_{n\geq 0}\{\|w_n - p\|\}\right\}, \tag{2.4}$$

hence

$$\begin{aligned}
\|z_n - p\| &= \|(1 - c_n - e_n'')(x_n - p) + c_n(T_3x_n - p) + e_n''(w_n - p)\| \\
&\leq (1 - c_n - e_n'')\|x_n - p\| + c_n\|T_3x_n - p\| + e_n''\|w_n - p\| \\
&\leq (1 - c_n - e_n'')M_1 + c_nM_2 + e_n''M_2 \\
&\leq (1 - c_n - e_n'')M_2 + c_nM_2 + e_n''M_2 \\
&= M_2. \qquad (2.5)
\end{aligned}$$

By the uniform continuity of T_2, we obtain $\{T_2z_n\}_{n=0}^{\infty}$ and $\{T_2x_n\}_{n=0}^{\infty}$ are bounded. Set

$$M = \sup_{n\geq 0}\|T_2z_n - p\| + \sup_{n\geq 0}\|x_n - p\| + \sup_{n\geq 0}\|v_n - p\| + M_2. \qquad (2.6)$$

Using Lemma 1.1 and (1.11), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1 - \alpha_n - \beta_n - e_n)(x_n - p) + \alpha_n(T_1y_n - p) \\
&\quad +\beta_n(T_1z_n - p) + e_n(u_n - p)\|^2 \\
&\leq (1 - \alpha_n - \beta_n - e_n)^2\|x_n - p\|^2 \\
&\quad +2\langle\alpha_n(T_1y_n - p) + \beta_n(T_1z_n - p) + e_n(u_n - p), j(x_{n+1} - p)\rangle \\
&= (1 - \alpha_n - \beta_n - e_n)^2\|x_n - p\|^2 + 2\alpha_n\langle T_1y_n - p, j(x_{n+1} - p)\rangle \\
&\quad +2\beta_n\langle T_1z_n - p, j(x_{n+1} - p)\rangle + 2e_n\langle u_n - p, j(x_{n+1} - p)\rangle \\
&\leq (1 - \alpha_n - \beta_n - e_n)^2\|x_n - p\|^2 \\
&\quad +2\alpha_n\langle T_1x_{n+1} - T_1p, j(x_{n+1} - p)\rangle \\
&\quad +2\alpha_n\langle T_1y_n - T_1x_{n+1}, j(x_{n+1} - p)\rangle \\
&\quad +2\beta_n\langle T_1x_{n+1} - T_1p, j(x_{n+1} - p)\rangle \\
&\quad +2\beta_n\langle T_1z_n - T_1x_{n+1}, j(x_{n+1} - p)\rangle + 2e_n\langle u_n - p, j(x_{n+1} - p)\rangle \\
&\leq (1 - \alpha_n - \beta_n - e_n)^2\|x_n - p\|^2 \\
&\quad +2\alpha_n(\|x_{n+1} - p\|^2 - \Phi(\|x_{n+1} - p\|)) \\
&\quad +2\alpha_n\|T_1y_n - T_1x_{n+1}\|.\|x_{n+1} - p\| \\
&\quad +2\beta_n(\|x_{n+1} - p\|^2 - \Phi(\|x_{n+1} - p)) \\
&\quad +2\beta_n\|T_1z_n - T_1x_{n+1}\|.\|x_{n+1} - p\| + 2e_nM \\
&\leq (1 - \alpha_n - \beta_n - e_n)^2\|x_n - p\|^2 \\
&\quad +(2\alpha_n + 2\beta_n)(\|x_{n+1} - p\|^2 - \Phi(\|x_{n+1} - p\|)) \\
&\quad +2\alpha_n\delta_nM_1 + 2\beta_n\tau_nM_1 + 2e_nM \\
&\leq (1 - \alpha_n - \beta_n - e_n)^2\|x_n - p\|^2 \\
&\quad +(2\alpha_n + 2\beta_n)\left(\|x_{n+1} - p\|^2 - \Phi(\|x_{n+1} - p\|)\right) \\
&\quad +2M(\alpha_n\delta_n + \beta_n\tau_n + e_n), \qquad (2.7)
\end{aligned}$$

where $\delta_n = \|T_1y_n - T_1x_{n+1}\| \longrightarrow 0$ as $n \to \infty$ and $\tau_n = \|T_1z_n - T_1x_{n+1}\| \longrightarrow 0$

as $n \to \infty$. But

$$\begin{aligned}
\|y_n - x_{n+1}\| &= \|(1-a_n-b_n-e'_n)x_n + a_nT_2z_n + b_nT_2x_n + e'_nv_n - \\
&\quad (1-\alpha_n-\beta_n-e_n)x_n - \alpha_nT_1y_n - \beta_nT_1z_n - e_nu_n\| \\
&= \|a_n(T_2z_n - x_n) + b_n(T_2x_n - x_n) + e'_n(v_n - x_n) + \\
&\quad \alpha_n(x_n - T_1y_n) + \beta_n(x_n - T_1z_n) + e_n(x_n - u_n)\| \\
&\leq a_n\|T_2z_n - x_n\| + b_n\|T_2x_n - x_n\| + e'_n\|v_n - x_n\| + \\
&\quad \alpha_n\|x_n - T_1y_n\| + \beta_n\|x_n - T_1z_n\| + e_n\|x_n - u_n\| \\
&\leq a_nM + b_nM + e'_nM + \alpha_nM_1 + \beta_nM_1 + e_nM_1 \\
&= M(a_n + b_n + e'_n) + M_1(\alpha_n + \beta_n + e_n) \\
&\leq 2M(a_n + b_n + e'_n + \alpha_n + \beta_n + e_n) \longrightarrow 0, \qquad (2.8)
\end{aligned}$$

as $n \to \infty$.

$$\begin{aligned}
\|z_n - x_{n+1}\| &= \|(1-c_n-e''_n)x_n + c_nT_3x_n + e''_nw_n - (1-\alpha_n-\beta_n-e_n)x_n \\
&\quad -\alpha_nT_1y_n - \beta_nT_1z_n - e_nu_n\| \\
&= \|c_n(T_3x_n - x_n) + e''_n(w_n - x_n) + \alpha_n(x_n - T_1y_n) \\
&\quad +\beta_n(x_n - T_1z_n) + e_n(x_n - u_n)\| \\
&\leq c_n\|T_3x_n - x_n\| + e''_n\|w_n - x_n\| + \alpha_n\|x_n - T_1y_n\| \\
&\quad +\beta_n\|x_n - T_1z_n\| + e_n\|x_n - u_n\| \\
&\leq c_nM_2 + e''_nM + \alpha_nM_1 + \beta_nM_1 + e_nM \\
&\leq M(c_n + e''_n + \alpha_n + \beta_n + e_n) \longrightarrow 0,
\end{aligned}$$

as $n \to \infty$. This implies that $\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$, since $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, $\lim_{n\to\infty} e'_n = 0$, $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$, $\lim_{n\to\infty} e_n = 0$. Using the uniform continuity of T_1, we obtain $\delta_n = \|T_1y_n - T_1x_{n+1}\| \longrightarrow 0$ as $n \to \infty$ and $\tau_n = \|T_1z_n - T_1x_{n+1}\| \longrightarrow 0$ as $n \to \infty$. Hence, there exists a positive integer N such that

$$\frac{1}{2} < 1 - 2\alpha_n - 2\beta_n < 1$$

for all $n > N$. Hence, from (2.7), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \tfrac{(1-\alpha_n-\beta_n-e_n)^2}{1-2\alpha_n-2\beta_n}\|x_n - p\|^2 - \tfrac{2(\alpha_n+\beta_n)}{1-2\alpha_n-2\beta_n}\Phi(\|x_{n+1} - p\|) \\
&\quad + \tfrac{2M(\alpha_n\delta_n + \beta_n\tau_n + e_n)}{1-2\alpha_n-2\beta_n} \\
&\leq \|x_n - p\|^2 - \tfrac{2(\alpha_n+\beta_n)}{1-2\alpha_n-2\beta_n}\Phi(\|x_{n+1} - p\|) \\
&\quad + \tfrac{2M(\alpha_n\delta_n + \beta_n\tau_n + e_n)}{1-2\alpha_n-2\beta_n}. \qquad (2.9)
\end{aligned}$$

Next, set $b_n = \frac{2(\alpha_n+\beta_n)}{1-2\alpha_n-2\beta_n}$ and observe that $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} b_n = +\infty$ since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = +\infty$. Hence, we observe that (2.9) becomes $a_{n+1}^2 \leq a_n^2 - b_n\Phi(a_{n+1}) + o(b_n)$ for all $n \geq N$, satisfying Lemma 1.2. This implies that $a_n \longrightarrow 0$ as $n \to \infty$. This means that $\lim_{n\to\infty}\|x_n - p\| = 0$. The proof of Theorem 2.1 is completed. □

Corollary 2.2 Let E be a real Banach space, D a nonempty closed and convex subset of E. Let T_1, T_2, T_3 be self maps of D with $T_1(D)$ bounded such that

$F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ and T_1, T_2 and T_3 uniformly continuous. Suppose T_1, T_2, T_3 are strongly pseudocontractive mappings. For $x_0, u_0, v_0, w_0 \in D$, the three step iteration with errors $\{x_n\}_{n=0}^{\infty}$ defined as follows

$$\begin{cases} x_{n+1} = a_n x_n + b_n T_1 y_n + c_n u_n \\ y_n = a'_n x_n + b'_n T_2 z_n + c'_n v_n \\ z_n = a''_n x_n + b''_n T_3 x_n + c''_n w_n \quad n \geq 0, \end{cases} \tag{2.10}$$

where $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in D. $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$, $\{c'_n\}_{n=0}^{\infty}$, $\{a''_n\}_{n=0}^{\infty}$, $\{b''_n\}_{n=0}^{\infty}$ and $\{c''_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$ satisfying the following conditions:

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$

(ii) $b_n, b'_n, c_n, c'_n \longrightarrow 0$ as $n \to \infty$.

(iii) $\sum_{n=1}^{\infty} b_n = \infty$

(iv) $\lim_{n\to\infty} \frac{c_n}{b_n} = 0$,

converges strongly to the unique common fixed point of T_1, T_2 and T_3.

Corollary 2.3 Let E be a real Banach space, D a nonempty closed and convex subset of E. Let $T_1, T_2, T_3 : D \to D$ be uniformly continuous and generalized Φ-pseudocontractive mappings such that $T_1(D)$ is bounded. Let $\{x_n\}$ be a sequence defined by (1.9) where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three sequences in $[0, 1]$ satisfying (i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0$, (ii) $\lim_{n=1}^{\infty} \alpha_n = \infty$. If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique common fixed point of T_1, T_2 and T_3.

Remark 2.4 Corollary 2.2 is Theorem 2.1 of Mogbademu and Olaleru [10] and Corollary 2.3 is Theorem 2.1 of Olaleru and Mogbademu [17]. Observe that Theorem 2.1 improves and generalizes the results of [10] since the class of strongly pseudocontractive maps is a subclass of the class of generalized strongly Φ - pseudocontractive maps. Clearly, our newly introduced iterative scheme (1.11) is more general than iterative scheme (1.9) used by Olaleru and Mogbademu [17]. Theorem 2.1 is also an improvement and a generalization of Theorem 2.1 of Xue and Fan [22] which in turn is a correction of Rafiq [18].

Theorem 2.5 Let E be a real Banach space, $T_1, T_2, T_3 : E \to E$ be uniformly continuous and generalized strongly Φ-accretive operators with $R(I - T_1)$ bounded, where I is the identity mapping on E. Let p denote the unique common solution to the equation $T_i x = f$, $(i = 1, 2, 3)$. For a given $f \in E$, define the operator $H_i : E \to E$ by $H_i x = f + x - T_i x$, $(i = 1, 2, 3)$. For any $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \beta_n - e_n)x_n + \alpha_n H_1 y_n + \beta_n H_1 z_n + e_n u_n, \quad n \geq 0, \\ y_n = (1 - a_n - b_n - e'_n)x_n + a_n H_2 z_n + b_n H_2 x_n + e'_n v_n, \\ z_n = (1 - c_n - e''_n)x_n + c_n H_3 x_n + e''_n w_n, \end{cases} \tag{2.11}$$

where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{e_n\}_{n=0}^{\infty}$, $\{e'_n\}_{n=0}^{\infty}$, $\{e''_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$ satisfying the conditions: $a_n, b_n, c_n, e'_n, e''_n, \alpha_n, \beta_n, e_n \longrightarrow$

0 as $n \to \infty$, $\alpha_n + \beta_n + e_n < 1$, $a_n + b_n + e'_n < 1$, $c_n + e''_n < 1$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$ are bounded sequences in E. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique common solution to $T_i x = f$, $(i = 1, 2, 3)$.

Proof. Clearly, if p is the unique common solution to the equation $T_i x = f$, $(i = 1, 2, 3)$, it follows that p is the unique common fixed point of H_1, H_2 and H_3. Using the fact that T_1, T_2 and T_3 are all generalized srtongly Φ-accretive operators, then H_1, H_2 and H_3 are all generalized strongly Φ-pseudocontractive with Φ a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ and $\Phi(0) = 0$. Since T_i $(i = 1, 2, 3)$ is uniformly continuous with $R(I - T_1)$ bounded, this implies that H_i $(i = 1, 2, 3)$ is uniformly continuous with $R(H_1)$ bounded. Hence, Theorem 2.5 follows from Theorem 2.1. □

Corollary 2.6 Let E be a real Banach space and $T_1, T_2, T_3 : D \to D$ be uniformly continuous and generalized Φ-accretive operator such that the equation $T_i x = f$, $1 \leq i \leq 3$, has common solution with the range of $(I - T_1)$ bounded. For a given $f \in E$, defined by the operator $H_i : E \to E$ by $H_i x = f + x - T_i x$, $(i = 1, 2, 3)$ and for $x_0 \in E$, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n H_1 y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n H_2 z_n, \\ z_n = (1 - \gamma_n)x_n + \gamma_n H_3 x_n, \quad n \geq 0, \end{cases} \tag{2.12}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma\}_{n=0}^{\infty}$ are three sequences in $[0, 1]$ satisfying
(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.
Then the sequence $\{x_n\}$ converges to a common solution of $T_i x = f$, $(i = 1, 2, 3)$.

Remark 2.7 Corollary 2.6 is Theorem 2.3 of Olaleru and Mogbademu [17]. Theorem 2.5 improves and extends Theorem 2.3 of Olaleru and Mogbademu [17] and Theorem 2.2 of Xue and Fan [22] which in turn is a correction of Rafiq [18].

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