Existence of Fixed Points of Some Classes of Nonlinear Mappings in Spaces with Weak Uniform Normal Structure

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Abstract

In this paper, we prove some fixed point results for some classes of nonlinear mappings recently introduced by Okeke and Olaleru [5]. Our results improves several other known results in literature, including the results of Sahu et al. [8] and Sahu [7].

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1 Introduction and Preliminaries

Let $C$ be a nonempty subset of a Banach space $X$ and $S : C \to C$ a Lipschitzian mapping, we use the symbol $\sigma(S)$ to denote the exact Lipschitz constant of $S,$
i.e.,

\[
\sigma(S) = \inf\{k \in [0, \infty] : \|Sx - S\| \leq k\|x - y\| \text{ for all } x, y \in C\}. \quad (1.1)
\]

A mapping \( T : C \to C \) is said to be

(a) **nonexpansive** if \( \sigma(T) = 1 \),

(b) **asymptotically nonexpansive** if \( \sigma(T^n) \geq 1 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \sigma(T^n) = 1 \),

(c) **uniformly \( L\)-Lipschitzian** if \( \sigma(T^n) = L \) for all \( n \in \mathbb{N} \) and for some \( L \in (0, \infty) \).

Sahu [7] recently introduced the following classes of nonlinear mappings as intermediate classes between the class of asymptotically nonexpansive mappings and that of mappings of asymptotically nonexpansive type (see, Goebel and Kirk [3], Kirk [4]).

**Definition 1.1** [7] Let \( C \) be a nonempty subset of a Banach space \( E \) and fix a sequence \( \{a_n\} \) in \([0, \infty)\) with \( a_n \to 0 \). A mapping \( T : C \to C \) will be called **nearly Lipschitzian** with respect to \( \{a_n\} \) if for each \( n \in \mathbb{N} \), there exists a constant \( k_n \geq 0 \) such that

\[
\|T^nx - T^ny\| \leq k_n(\|x - y\| + a_n) \quad \forall \ x, y \in C. \quad (1.2)
\]

The infimum of constants \( k_n \) for which (2.18) holds will be denoted by \( \eta(T^n) \) and called **nearly Lipschitz constant**.

\[
\eta(T^n) = \sup\left\{\frac{\|T^nx - T^ny\|}{\|x - y\| + a_n} : x, y \in C, x \neq y\right\}. \quad (1.3)
\]

A nearly Lipschitzian mapping \( T \) with sequence \( \{(a_n, \eta(T^n))\} \) is said to be

(i) **nearly contraction** if \( \eta(T^n) < 1 \) for all \( n \in \mathbb{N} \),

(ii) **nearly nonexpansive** if \( \eta(T^n) \leq 1 \) for all \( n \in \mathbb{N} \),

(iii) **nearly asymptotically nonexpansive** if \( \eta(T^n) \geq 1 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \eta(T^n) \leq 1 \),

(iv) **nearly uniformly \( k\)-Lipschitzian** if \( \eta(T^n) \leq k \) for all \( n \in \mathbb{N} \),

(v) **nearly uniformly \( k\)-contraction** if \( \eta(T^n) \leq k < 1 \) for all \( n \in \mathbb{N} \).

Inspired by the facts above, Okeke and Olaleru [5] introduced the following classes of nonlinear mappings.

**Definition 1.2** Let \( C \) be a nonempty subset of a Banach space \( E, \phi : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+ \) be a continuous strictly increasing function such that \( \phi(0) = 0 \), \( \lim_{t \to \infty} \phi(t) = \infty \) and fix a sequence \( \{a_n\} \) in \([0, \infty)\) with \( a_n \to 0 \). A mapping \( T : C \to C \) will be called **\( \phi\)-nearly Lipschitzian** with respect to \( \{a_n\} \) if for each \( n \in \mathbb{N} \), there exists a constant \( k_n \geq 0 \) such that

\[
\|T^nx - T^ny\| \leq k_n \phi(\|x - y\| + a_n) \quad \forall \ x, y \in C. \quad (1.4)
\]
The infimum of constants $k_n$ for which (1.6) holds will be denoted by $\eta(T^n)$ and called $\phi$-nearly Lipschitz constant. Notice that

$$\eta(T^n) = \sup \left\{ \frac{||T^n x - T^n y||}{\phi(||x - y|| + a_n)} : x, y \in C, x \neq y \right\}. \quad (1.5)$$

A $\phi$-nearly Lipschitzian mapping $T$ with sequence $\{(a_n, \eta(T^n))\}$ is said to be

(i) $\phi$-nearly contraction if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,

(ii) $\phi$-nearly nonexpansive if $\eta(T^n) \leq 1$ for all $n \in \mathbb{N}$,

(iii) $\phi$-nearly asymptotically nonexpansive if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \eta(T^n) \leq 1$,

(iv) $\phi$-nearly uniformly $k$-Lipschitzian if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,

(v) $\phi$-nearly uniformly $k$-contraction if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Observe that if $\phi$ is identity in Definition 1.2, then we obtain the concepts introduced by Sahu [7] (see Definition 1.1 above).

Our purpose in this paper is to prove some fixed point results for the classes of nonlinear mappings defined by Okeke and Olaleru [5], as given in Definition 1.2 above.

The following definitions and lemma will be needed in this study.

**Definition 1.3** [7] Let $C$ be a nonempty subset of a Banach space $E$ and $T : C \to C$ a mapping. $T$ is said to be *demicontinuous* if whenever a sequence $\{x_n\}$ in $C$ converges strongly to $x \in C$, then $\{Tx_n\}$ converges weakly to $Tx$.

**Definition 1.4** [2] The normal structure coefficient $N(E)$ of a Banach space $E$ is defined by

$$N(E) = \inf \left\{ \frac{\text{diam}(C)}{r_C(C)} : C \text{ is nonempty bounded convex subset of } E \text{ with } \text{diam} \ C > 0 \right\},$$

where $r_C(C) = \inf_{x \in C} \{\sup_{y \in C} ||x - y||\}$ is the *Chebyshev radius* of $C$ relative to itself and $\text{diam} (C) = \sup_{x,y \in C} ||x - y||$ is diameter of $C$. The space $E$ is said to have the *uniform normal structure* if $N(E) > 1$. A weakly convergent sequence coefficient of $E$ is defined by

$$WCS(E) = \sup \{ k : k \limsup_{n \to \infty} ||x_n|| < \text{diam}_a(\{x_n\}) \text{ for all } \{x_n\} \text{ in } E \text{ with } x_n \rightharpoonup 0 \}.$$  

The space $E$ is said to have the *weak uniform normal structure* if $WCS(E) > 1$.

**Definition 1.5** [1] Let $C$ be a nonempty subset of a Banach space $E$. A nonempty closed convex subset $D$ of $C$ is said to satisfy property $(\omega)$ with respect to a mapping $T : C \to C$ if

$$\omega_T(x) \subset D \text{ for every } x \in D,$$  

(1.6)
where $\omega_T(x)$ denotes the set of all weak subsequential limits of $\{T^n x : n \in \mathbb{N}\}$. Moreover, $T$ is said to satisfy the $(\omega)$-fixed point property if $T$ has a fixed point in every nonempty closed convex subset $D$ of $C$ which satisfies property $(\omega)$.

**Lemma 1.6** [8] Let $C$ be a nonempty closed convex subset of a Banach space and $T : C \to C$ a mapping such that $T^n u \to v$ as $n \to \infty$ for some $u, v \in C$. Suppose that $T$ is demicontinuous at $v$. Then $v$ is a fixed point of $T$ in $C$.

## 2 Main Results

**Theorem 2.1** Let $E$ be a Banach space with weak uniform normal structure, $C$ a nonempty weakly compact convex subset of $E$ and $T : C \to C$ a $\phi$-nearly Lipschitzian mapping with sequence $\{\{a_n, \eta(T^n)\}\}$ such that $\limsup_{n \to \infty} \eta(T^n) < \sqrt{WCS(E)}$. Also suppose that there exists a nonempty closed convex subset $M$ of $C$ which satisfies property $(\omega)$ with respect to $T$. Then

(a) for an arbitrary $x_0 \in M$, there exists an iterative sequence $\{x_m\}$ in $M$ defined by

$$x_m = w - \lim_{n \to \infty} T^n x_{m-1} \quad \forall m \in \mathbb{N}, \quad (2.1)$$

(b) if $T$ is asymptotically regular on $C$, then there exists an element $v \in M$ such that $\{x_m\}$ converges strongly to $v \in M$. Further, if $T$ is demicontinuous at $v$, then

$$v \in F(T).$$

**Proof.** (a) We can easily construct a nonempty closed convex separable subset $C_0$ of $C$ which is invariant under each $T^n$ (i.e. $T^n(C_0) \subset C_0$ for $n = 1, 2, \cdots$), we suppose that $C$ itself is separable.

Due to the separability of $C_0$, we can select a subsequence $\{T^n x\}$ such that $\{T^n x\}$ is weakly convergent for each $x \in C$. For every $x_0 \in M \subset C$, we consider a sequence $\{T^n x_0\}$ in $C$. Suppose that $w - \lim_{n \to \infty} T^n x_0 = x_1 \in C$. Using property $(\omega)$ we have that $x_1 \in M$. By induction, we can construct a sequence $\{x_m\}$ in $M$ defined by (2.1).

(b) Suppose that $T$ is asymptotically regular on $C$. The weak asymptotic regularity of $T$ ensures that $x_m = w - \lim_{n \to \infty} T^{n+r} x_{m-1}$ for each $r \in \mathbb{N}$. We are to show that $\{x_m\}$ converges strongly to a fixed point $T$. We set $L := \limsup_{n \to \infty} \eta(T^n)$, $D_m := \limsup_{n \to \infty} \|x_m - T^n x_m\|$ and $R_m := \limsup_{n \to \infty} \|x_{m+1} - T^n x_m\|$ for all $m = 0, 1, 2, \cdots$. Using the property of $WCS(E)$, we obtain

$$R_m = \limsup_{n \to \infty} \|x_{m+1} - T^n x_m\| \leq \frac{1}{WCS(E)}D\{\{T^n x_m\}\}. \quad (2.2)$$
Using the asymptotic regularity of $T$ and the $w$-$l.s.c.$ of the norm $\|\cdot\|$, we obtain

$$D[\{T^nx_m\}] = \limsup_{n \to \infty} (\limsup_{r \to \infty} \|T^n x_m - T^r x_m\|)$$

$$\leq \limsup_{n \to \infty} (\limsup_{r \to \infty} (\|T^n x_m - T^{n+r} x_m\|$$

$$+ \|T^{n+r} x_m - T^r x_m\|))$$

$$\leq \limsup_{n \to \infty} (\limsup_{r \to \infty} (\eta(T^n) \phi(\|x_m - T^r x_m\| + a_n)))$$

$$\leq L \limsup_{r \to \infty} (\phi(\limsup_{s \to \infty} (\|T^s x_{m-1} - T^r x_{m-1}\|$$

$$+ \|T^{r+s} x_{m-1} - T^r x_{m-1}\|)))$$

$$\leq L \limsup_{s \to \infty} (\phi(\limsup_{r \to \infty} (\|T^s x_{m-1} - T^r x_{m-1}\|$$

$$+ \eta(T^r) (\|T^s x_{m-1} - x_m\| + a_r))))$$

$$\leq L^2 \limsup_{s \to \infty} (\phi(\|T^s x_{m-1} - x_m\|)) = L^2 \times \phi(R_{m-1}). \quad (2.3)$$

We set $\lambda := \frac{L^2}{WCS(E)} < 1$. Using (2.2), we have

$$\phi(R_m) \leq \lambda \times \phi(R_{m-1}) \leq \lambda^2 \times \phi(R_{m-2}) \leq \cdots \leq \lambda^m \times \phi(R_0) \to 0 \quad (2.4)$$

as $m \to \infty$. For each $m \in \mathbb{N}$, we obtain

$$\|x_{m+1} - x_m\| \leq \limsup_{n \to \infty} (\|x_{m+1} - T^n x_m\| + \|T^n x_m - x_m\|)$$

$$\leq R_m + \limsup_{n \to \infty} (\limsup_{r \to \infty} (\|T^n x_m - T^r x_{m-1}\|$$

$$+ \|T^{n+r} x_{m-1} - T^r x_{m-1}\|))$$

$$\leq \limsup_{r \to \infty} (\phi(\limsup_{s \to \infty} (\eta(T^n) \times$$

$$\|x_m - T^r x_{m-1}\| + a_n)))$$

$$\leq (\lambda + L) \phi(R_{m-1})$$

$$\cdots$$

$$\leq (\lambda + L) \lambda^{m-1} \times \phi(R_0). \quad (2.5)$$

We see that $\{x_m\}$ is a Cauchy sequence in $M$ and hence there exists an element $v \in M$ such that $\lim_{m \to \infty} x_m = v$. Clearly,

$$\|v - T^n v\| \leq \|v - x_{m+1}\| + \|x_{m+1} - T^n x_m\| + \|T^n x_m - T^m v\|$$

$$\leq \|v - x_{m+1}\| + \|x_{m+1} - T^n x_m\| + \eta(T^n) \times$$

$$\phi(\|x_m - v\| + a_n). \quad (2.6)$$

Taking limit superior as $n \to \infty$ on both sides, we obtain

$$\limsup_{n \to \infty} \|v - T^n v\| \leq \|v - x_{m+1}\| + \phi(R_m) + L \|x_m - v\| \to 0,$$

as $m \to \infty$. Hence, we have that $T^n v \to v$ as $n \to \infty$. Furthermore, we assume that $T$ is demicontinuous at $v$. Therefore, using Lemma 1.6, we obtain $v \in F(T)$. □

**Remark 2.2** The results of Theorem 2.1 improves and generalizes several other known results in literature, including the results of Sahu et al. [8] and Sahu [7].
References


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