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A Novel Approach For Solving Quadratic Riccati Differential Equations

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Abstract

In this paper, a numerical technique-differential transform method (DTM) is presented for the solution of Riccati differential equations. Three illustrative examples involving both constant and variable coefficients are solved. The DTM applied provides results that converge rapidly to the exact solution. To see the accuracy of this method, the results are compared with the exact solutions.

Keywords: DTM, Riccati Differential equation, Series solution

Introduction

The Riccati differential equationwas named after the Italian nobleman, Count Jacopo Francesco Riccati (1676-1754). The fundamental theories of Riccati differential equation are contained in the book of Reid[1] which has applications to random processes, optimal control, and diffusion problems as well as in stochastic realization theory, robust stabilization, network synthesis and more recently financial mathematics [2, 3]. Some numerical methods have been proposed to solve Riccati equations, these include: Adomian decomposition method [4,5], variational iteration method (VIM) [6], He's variational method [7] and Laplace transform and new homotopy perturbation method (LTNHPM) [8].

The differential transform method applied in this work was proposed by Zhou [9] to solve both linear and nonlinear initial value problems in electric circuit analysis. Differential transform method is an iterative technique for obtaining analytic Taylor series solution of differential equations. The solutions via the DTM converge faster to exact solutions, and less computational work is involved, unlike the traditional Taylor series method which involved more computations especially for higher order derivatives. The method (DTM) has been adopted in solving various physical problems [10-12].

Fundamentals of Differential Transform Method

An arbitrary function $f(x)$ in Taylor series about a point $x = 0$ is expressed as:

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\frac{d^k f}{dx^k} \right]_{x=0} \tag{1}$$

The differential transformation of $f(x)$ is defined as:

$$F(x) = \frac{1}{k!} \left[\frac{d^k y}{dx^k} \right]_{x=0} \tag{2}$$

Then, the inverse differential transform is

$$f(x) = \sum_{k=0}^{\infty} x^k F(k) \tag{3}$$

The following theorems can be deduced from (1), (2) and (3):

Theorem 1: If $y(x) = y_1(x) \pm y_2(x)$, then $Y(k) = Y_1(k) \pm Y_2(k)$

Theorem 2: If $y(x) = cy_1(x)$, then, where c is a constant.

Theorem 3: If $y(x) = 1$, then, $Y(k) = \delta(k)$

Theorem 4: If $y(x) = y_1(x) y_2(x)$, then $Y(k) = \sum_{k_1=0}^{k} Y_1(k_1) Y_2(k - k_1)$

Theorem 5: If $y(x) = x^n$, then $Y(k) = \partial(k-n)$ where $\delta(k-n) = \begin{matrix} 1 & k = n \\ 0 & k \neq n \end{matrix}$

Theorem 6: If $y(x) = e^{\beta x}$, then $Y(x) = \frac{(\beta)^k}{k!}$

Theorem 7: If $y(x) = \frac{dy_*(x)}{dx}$, then $Y(k) = (k+1) Y_*(k+1)$

Numerical Examples

In this section, we present some illustrative examples to test the proposed semi-numerical method.

Example 1

We first consider the Riccati equation below with constant coefficient:

$$y'(t) = y^2(t) - y(t) \quad [8] \tag{4}$$

$$y(0) = \frac{1}{2} \tag{5}$$

The theoretical solution for this problem is:

$$y = \frac{e^{-t}}{1 + e^{-t}} \tag{6}$$

As such, the differential Transformation of equation gives:

$$(k+1)Y(k+1)=\sum_{l=0}^{n}Y(l)Y(n-l)-Y(n) \tag{7}$$

and from the initial condition of equation (5) we obtain:

$$Y(0)=\frac{1}{2} \tag{8}$$

Substituting (8) into (7), we have the following results:

$$Y(1)=-\frac{1}{4},\quad Y(2)=0\,,Y(3)=\frac{1}{48},\quad Y(4)=0\,,\quad Y(5)=-\frac{1}{480}\,,\quad Y(6)=0\,,$$

$$Y(7)=\frac{17}{80640}\,,\quad Y(8)=0,\quad Y(9)=-\frac{31}{1451520}\,,Y(10)=0\,,Y(11)=\frac{691}{319334400}$$

Thus, the series solution using (3) is then given as:

$$y(t)=\frac{1}{2}-\frac{t}{4}+\frac{t^3}{48}-\frac{t^5}{480}+\frac{t^7}{80640}-\frac{31t^9}{1451520}+\frac{t^{11}}{319334400}+\cdots \tag{9}$$

Example 2

Considering the Riccati equation below

$$y'=e^t-e^{3t}+2e^{2t}y-e^ty^2 \quad [8] \tag{10}$$

subject to the initial condition

$$y(0)=1 \tag{11}$$

The theoretical solution of (10) subject to (11) is:

$$y=e^t \tag{12}$$

Taking the differential transform of the equations (10) and (11), we obtain:

$$Y(n+1)=\frac{1}{(n+1)}\left(\frac{1}{n!}-\frac{3^n}{n!}+2\sum_{l=0}^{n}\frac{2^l}{l!}Y(n-l)-\sum_{l=0}^{n}\sum_{s=0}^{l}\frac{1}{s!}Y(l-s)Y(n-l)\right) \tag{13}$$

$$y(0)=1 \tag{14}$$

Substituting (14) into(13)the following results are obtained:

$$Y(1)=1\,,\quad Y(2)=\frac{1}{2},\quad Y(3)=\frac{1}{6},\quad Y(4)=\frac{1}{24},\quad Y(5)=\frac{1}{120}\,,$$

$$Y(6)=\frac{1}{720}\,,Y(7)=\frac{1}{5040}\,,\quad Y(8)=\frac{1}{40320} \tag{15}$$

The closed form of the series solution via (3) can be written as:

$$y(t)=1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+\frac{t^6}{720}+\frac{t^7}{5040}+\cdots=e^t \tag{16}$$

Example 3

We finally consider the Riccati equation with variable coefficient

$$y'=16t^2-5+8ty+y^2\ [8] \tag{17}$$

with the initial condition:

$$y(0)=1 \tag{18}$$

The exact solution of the problem is:

$$y=1-4t \tag{19}$$

Taking the differential transformation of the (17) and (18), we have:

$$Y(n+1)=\frac{1}{(n+1)}\left(16\delta(n-2)-5\delta(n)+8\sum_{l=0}^{n}\delta(l-1)Y(n-1)+\sum_{r=0}^{n}Y(r)Y(n-r)\right) \tag{20}$$

$$Y(0)=1 \tag{21}$$

Substituting (21) in (20), we obtain the following results:

$$Y(1)=-4,\ Y(2)=Y(3)=Y(4)=Y(5)=Y(6)=Y(7)=\cdots=0 \tag{22}$$

Thus, combining (22) and (3) yield the result:

$$y(t)=1-4t \tag{23}$$

which is the exact solution.

Remark 3.1

We show in table **1**, fig **1** and fig **2;** a table showing the numerical values, the graphs of the approximate solution and the exact solution respectively of the problem in example **1**, for illustrative purpose.In examples 2 and 3, the cases where the method yields the exact solutions were presented, as such would give the same table and graph for each of the cases.

Table1: Computations showing comparison between the exact solution and numerical solution for example 1

t	DTM	EXACT	ABSOLUTE ERROR
0	0.500000000	0.500000000	0
0.1	0.475020813	0.475020813	0
0.2	0.450166003	0.450166003	0
0.3	0.425557483	0.425557483	5.55E-17
0.4	0.40131234	0.40131234	4.44E-16
0.5	0.377540669	0.377540669	1.67E-14
0.6	0.354343694	0.354343694	3.68E-13
0.7	0.331812228	0.331812228	4.99E-12
0.8	0.310025519	0.310025519	4.76E-11
0.9	0.289050498	0.289050497	3.47E-10
1	0.268941423	0.268941421	2.04E-09

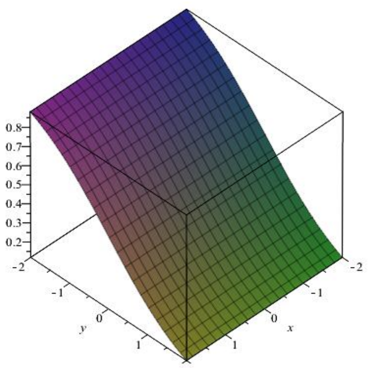

Figure 1: Graph of DTM for example 1

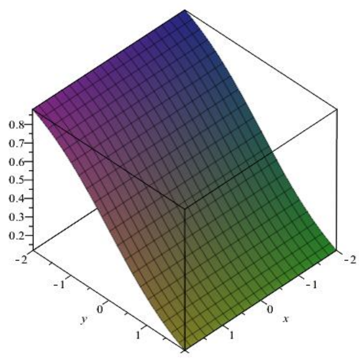

Figure 2: Graph of the exact solution for example 1

Conclusion
In this paper, a semi-analytical method (differential transformation technique) has been applied to solve Riccati differential equations with constant and variable coefficients. The size of computations involved and the rapidity of its convergence to the exact solution attest to the efficiency of this method in handling various types of physical problems.

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