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Existence results for a fourth order multipoint boundary value problem at resonance

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S.A. Iyase

Department of Mathematics, Covenant University, Canaanland P.M.B. 1023, Ota, Ogun State, Nigeria

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Abstract

In this paper we present some existence results for a fourth order multipoint boundary value problem at resonance. Our main tools are based on the coincidence degree theory of Mawhin.

Keywords: Fourth order; Multipoint boundary value problem; Resonance; Coincidence degree

1. Introduction

In this paper, we shall discuss the solvability of the multipoint boundary value problem

$$x^{(iv)}(t) = f(t, x(t), x'(t), x''(t), x'''(t)) \tag{1.1}$$

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i) \qquad x'(0) = x''(0) = 0, \qquad x(1) = x(\eta) \tag{1.2}$$

where $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function $\alpha_i (1 \le i \le m-2) \in \mathbb{R}$, $0 < \xi_1 \le \xi_2 \le \cdots < \xi_{m-2} < 1$ and $\eta \in (0, 1)$.

Multipoint boundary value problems of ordinary differential equations arise in a variety of different areas of Applied Mathematics, Physics and Engineering. For example Bridges of small sizes are often designed with two supported points, which leads to a standard two-point boundary condition and bridges of Large sizes are sometimes contrived with multipoint supports which corresponds to a multipoint boundary condition.

Boundary value problem (1.1)–(1.2) is called a problem at resonance if $Lx = x^{(iv)}(t) = 0$ has non-trivial solutions under the boundary conditions (1.2) that is, when $\dim \ker L \ge 1$. On the interval $[0, 1]$ second order and third order boundary value problems at resonance have been studied by many authors (see [1–4]) and references therein.

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E-mail address: driyase2011@yahoo.com.

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Although the existing literature on solutions of multipoint boundary value problems is quite large, to the best of our knowledge there are few papers that have investigated the existence of solutions of fourth order multipoint boundary value problems at resonance. Our motivation for this paper is derived from these previous results.

In what follows, we shall use the classical spaces $C^k[0,1], k = 1,2,3$. For $x \in C^3[0,1]$ we use the norm $|x|_\infty = \max_{t\in[0,1]}|x(t)|$. We denote the norm in $L^1[0,1]$ by $|\ |_1$ and on $L^2[0,1]$ by $|\ |_2$. We will use the Sobolev Q2 spaces $W^{4,1}(0,1)$ which may be defined by

$$W^{4,1}(0,1) = \{x : [0,1] \longrightarrow \mathbb{R} : x, x', x'', x'''\}$$

are absolutely continuous on $[0,1]$ with $x^{(iv)} \in L^1[0,1]$.

2. Preliminaries

Consider the linear equation

$$Lx = x^{(iv)}(t) = 0 \tag{2.1}$$

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \qquad x'(0) = x''(0) = 0, \qquad x(1) = x(\eta). \tag{2.2}$$

If we consider a solution of the form

$$x(t) = \sum_{i=0}^{3} a_i t^i, \quad a_i \in \mathbb{R}. \tag{2.3}$$

Then this solution exists if and only if

$$a_3(1-\eta^3) = 0, \quad \eta \in (0,1). \tag{2.4}$$

In this case (2.1)–(2.2) has non-trivial solutions.

Hence if $Lx = y$ then L is not invertible. Therefore, the problem is said to be at resonance. We shall prove existence results for the boundary value problem (1.1)–(1.2) under the condition (2.4).

We shall apply the continuation Theorem of Mawhin [5] to get our results. We present some preliminaries needed to understand this continuation Theorem.

Let X and Z be real Banach spaces and $L : domL \subset X \longrightarrow Z$ be a linear operator which is Fredholm of index zero and $P : X \longrightarrow X$, $Q : Z \longrightarrow Z$ be continuous projections such that

$$ImP = \ker L, \ker Q = ImL \quad \text{and} \quad X = \ker L \oplus \ker P$$

$Z = ImL \oplus ImQ$. It follows that $L|_{domL\cap\ker P} \longrightarrow ImL$ is invertible and we write the inverse of this map by K_p. Let Ω be an open bounded subset of X such that $domL \cap \Omega \neq \Phi$ and let $N : \bar{\Omega} \longrightarrow Z$ be an L-compact mapping, that is, the maps $QN(\bar{\Omega})$ is bounded and $K_p(I-Q)N : \bar{\Omega} \longrightarrow X$ is compact. In order to obtain our existence results we shall use the following fixed point Theorem of Mawhin.

Theorem 2.1 (*See [5]*). *Let L be a Fredholm operator of index zero and let N be L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied*

(i) $Lx \neq \lambda Nx$ *for every* $(x,\lambda) \in [(domL\backslash\ker L) \cap \partial\Omega \times (0,1)]$
(ii) $Nx \notin ImL$ *for every* $x \in \ker L \cap \partial\Omega$
(iii) $\deg(JQN|_{\ker L\cap\partial\Omega}; \Omega \cap \ker L, 0) \neq 0$ *where* $Q : Z \to Z$ *is a continuous projection as above and* $J : ImQ \to \ker L$ *is an isomorphism. Then the equation* $Lx = Nx$ *has at least one solution in* $domL \cap \bar{\Omega}$.

We shall prove existence results for the boundary value problem (1.1)-(1.2) when

$$\sum_{i=1}^{m-2} \alpha_i \xi_i^3 = 0 \quad \text{and} \quad \sum_{i=1}^{m-2} \alpha_i = 1.$$

Let $X = C^3[0,1]$, $Z = L^1[0,1]$. Let $L : domL \subset X \longrightarrow Z$ be defined by

$$Lx = x^{(iv)}$$

where

$$domL = \left\{ x \in W^{4,1}(0,1),\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i),\ x'(0) = x''(0) = 0,\ x(1) = x(\eta) \right\}.$$

We define $N : X \longrightarrow Z$ by setting $N = f(t, x(t), x'(t), x''(t), x'''(t))$. Then the boundary value problem (1.1)–(1.2) can be put in the form

$$Lx = Nx. \tag{2.5}$$

In what follows we shall use the following lemmas.

Lemma 2.1. *If* $\sum_{i=1}^{m-2} \alpha_i = 1$ *then there exists* $l \in \{0, 1, 2, \ldots, m-4\}$ *such that* $\sum_{i=1}^{m-2} \alpha_i \xi_i^{l+4} \neq 0$.

Proof. Follows the same procedure as in [1]. $\square$

Lemma 2.2. *If* $\sum_{i=1}^{m-2} \alpha_i = 1$, $\sum_{i=1}^{m-2} \alpha_i \xi_i^3 = 0$ *then*

(A) $ImL = \left\{ y \in Z : \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s \int_0^{\tau_2} \int_0^{\tau_1} y(v)dvd\tau_1 d\tau_2 ds = 0 \right\}$
(B) $L : domL \subset X \longrightarrow Z$ *is a Fredholm operator of index zero.*

Proof. We will show that the problem

$$x^{(iv)}(t) = y \quad \text{for } y \in Z \tag{2.6}$$

has a solution $x(t)$ satisfying

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \qquad x'(0) = x''(0) = 0, \qquad x(1) = x(\eta) \tag{2.7}$$

if and only if

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s \int_0^{\tau_2} \int_0^{\tau_1} y(v)dvd\tau_1 d\tau_2 ds = 0. \tag{2.8}$$

Suppose (2.6) has a solution $x(t)$ satisfying (2.7) then from (2.6) we have

$$x(t) = x(0) + x'(0)t + \frac{t^2}{2}x''(0) + \frac{t^3}{6}x'''(0) + \int_0^t \int_0^s \int_0^{\tau_2} \int_0^{\tau_1} y(v)dvd\tau_1 d\tau_2 ds.$$

Using $\sum_{i=1}^{m-2} \alpha_i = 1$, $\sum_{i=1}^{m-2} \alpha_i \xi_i^3 = 0$ we obtain

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s \int_0^{\tau_2} \int_0^{\tau_1} y(v)dvd\tau_1 d\tau_2 ds = 0, \quad \text{for } y \in Z.$$

Now suppose

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s \int_0^{\tau_2} \int_0^{\tau_1} y(v)dvd\tau_1 d\tau_2 ds = 0.$$

Let

$$x(t) = c - \frac{t^3}{1-\eta^3} \int_\eta^1 \int_0^s \int_0^{\tau_2} \int_0^{\tau_1} y(v)dvd\tau_1 d\tau_2 ds + \int_0^t \int_0^s \int_0^{\tau_2} \int_0^{\tau_1} y(v)dvd\tau_1 d\tau_2 ds$$

where c is an arbitrary constant. Then $x(t)$ is a solution of (2.6) with

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s \int_0^{\tau_2} \int_0^{\tau_1} y(v)dvd\tau_1 d\tau_2 ds = 0.$$

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For $y \in Z$, we define the projection $Q : Z \longrightarrow Z$ by

$$(Qy)(t) = \frac{A}{\sum_{i=1}^{m-2} \alpha_i \xi_i^{l+4}} t^l \left(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau_2} \int_0^{\tau_1} y(v) dv d\tau_1 d\tau_2 ds \right)$$

where

$$A = (l+1)(l+2)(l+3)(l+4). \tag{2.9}$$

Let $y_1 = y - Qy$, that is $y_1 \in \ker Q$. Then by direct calculations we have

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau_2} \int_0^{\tau_1} y_1(v) dv d\tau_1 d\tau_2 ds$$

$$= \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau_2} \int_0^{\tau_1} y(v) dv d\tau_1 d\tau_2 ds \left(1 - \frac{A}{\sum_{i=1}^{m-2} \alpha_i \xi_i^{l+4}} \int_0^{\xi_i} \int_0^{s} \int_0^{\tau_2} \int_0^{\tau_1} v^l dv d\tau_1 d\tau_2 ds \right) = 0.$$

So, $y_1 \in ImL$. Hence $Z = ImL + ImQ$. Since $ImL \cap ImQ = \{0\}$ we obtain

$$Z = ImL \oplus ImQ.$$

Now $\ker L = \{x \in domL : x = c,\ c \in \mathbb{R}\}$.

Hence,

$$\dim \ker L = \dim ImQ = 1.$$

Hence L is a Fredholm operator of index zero. $\square$

Let $P : X \to X$ be defined by

$$Px(t) = x(0), \quad t \in [0, 1].$$

Lemma 2.3. *If* $\sum_{i=1}^{m-2} \alpha_i = 1$, $\sum_{i=1}^{m-2} \alpha_i \xi_i^3 = 0$. *Then the generalized inverse* $Kp : ImL \longrightarrow domL \cap \ker P$ *can be written as*

$$K_p y(t) = \frac{-t^3}{1-\eta^3} \int_\eta^1 \int_0^s \int_0^{\tau_1} \int_0^{\tau_2} y(v) dv d\tau_1 d\tau_2 ds + \int_0^t \int_0^s \int_0^{\tau_1} \int_0^{\tau_2} y(v) dv d\tau_1 d\tau_2 ds.$$

Proof. For any $y \in ImL$, we have

$$(LK_p)y(t) = (K_p y(t))^{(iv)} = y(t)$$

and for $x \in domL \cap \ker P$, one has

$$(K_p L)x(t) = K_p(x^{iv}) = \frac{-t^3}{1-\eta^3} \int_\eta^1 \int_0^s \int_0^{\tau_2} \int_0^{\tau_1} x^{(iv)}(v) dv d\tau_1 d\tau_2 ds$$

$$+ \int_0^t \int_0^s \int_0^{\tau_2} \int_0^{\tau_1} x^{(iv)}(v) dv d\tau_1 d\tau_2 ds$$

$$= \frac{-t^3}{1-\eta^3} \left[x(1) - x(\eta) - (1-\eta^3)x'(0) - \frac{(1-\eta^2)}{2} x''(0) - \frac{(1-\eta^3)}{6} x'''(0) \right]$$

$$+ x(t) - x(0) - tx'(0) - \frac{t^2}{2} x''(0) - t^3 x'''(0).$$

Since $x \in domL \cap \ker P$, $Px(t) = x(0) = 0$. Also $x'(0) = x''(0) = 0$.

Thus,

$$(K_pL)x(t) = x(t).$$

Hence,

$$K_p = (L|_{domL\cap \ker P})^{-1}. \quad \square$$

3. Main results

Theorem 3.1. *Let $\sum_{i=1}^{m-2} \alpha_i = 1$, $\sum_{i=1}^{m-2} \alpha_i \xi_i^3 = 0$ and let $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function and suppose that f has the decomposition*

$$f(t, x, y, w, z) = g(t, x, y, w, z) + h(t, x, y, w, z)$$

(H$_1$) *Assume there exists $M_1 > 0$ such that for all $x \in domL\backslash \ker L$ if $x(t) > M_1$, $t \in [0, 1]$ then*

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau_2} \int_0^{\tau_1} [f(v, x(v), x'(v), x''(v), x'''(v))]dvd\tau_2 d\tau_2 ds \neq 0 \tag{3.1}$$

(H$_2$)

$$zg(t, x, y, w, z) \leq 0 \quad \text{for all } (t, x, y, w, z) \in [0, 1] \times \mathbb{R}^4 \tag{3.2}$$

(a)

$$|h(t, x, y, w, z)| \leq M\{|x|^r + |y| + |w| + |z|^\theta\} \quad \text{for } 0 < r,\ \theta < 1 \tag{3.3}$$

(b)

$$z[f(t, x, y, w, z)] \leq (|z|^2 + 1)[D(t, x, y, w) + m(t)] \tag{3.4}$$

where $D(t, x, y, w)$ is bounded on bounded sets and $m(t) \in L^1[0, 1]$.

(H$_3$) *There exists $N^* > 0$ such that for all $c \in \mathbb{R}$, $|c| > N^*$ then either*

$$\frac{cA}{\sum_{i=1}^{m-2} \alpha_i \xi_i^{l+4}} \cdot t^l \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau_2} \int_0^{\tau_1} [f(v, c, 0, 0, 0)]dvd\tau_1 d\tau_2 ds < 0 \tag{3.5}$$

or

$$\frac{cA}{\sum_{i=1}^{m-2} \alpha_i \xi_i^{l+4}} \cdot t^l \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau_2} \int_0^{\tau_1} [f(v, c, 0, 0, 0)]dvd\tau_1 d\tau_2 ds > 0 \tag{3.6}$$

Then (1.1)–(1.2) has at least one solution in $C^3[0, 1]$ provided

$$M < \frac{B\pi^2}{4(B^2 + \pi^2 + 4)}$$

where $B = \sqrt{4 + \pi^2}$.

Proof. Set

$$\Omega_1 = \{x \in domK\backslash_{\ker L} : Lx = \lambda Nx, \quad \lambda \in [0, 1]\}$$

for $x \in \Omega_1$. Since $Lx = \lambda Nx$, then $\lambda \neq 0$, $Nx \in ImL = \ker Q$ hence

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau_2} \int_0^{\tau_1} \{f(v, x(v), x'(v), x''(v), x'''(v))\}dvd\tau_1 d\tau_2 ds = 0.$$

Thus by (H_1) there exists $t_0 \in [0,1]$ such that $|x(t_0)| \leq M_1$. Therefore,

$$|x(t)| \leq |x(t_0)| + \int_{t_0}^{t} |x'(s)|ds, \quad t \in [0,1]$$

$$|x|_\infty \leq M_1 + |x'|_\infty. \tag{3.7}$$

We note that for $x(1) = x(\eta)$ there exists $t_1 \in (\eta, 1)$ such that $x'(t_1) = 0$ and from $x'(t_1) = x'(0) = 0$ there exists $t_2 \in (0, t_1)$ such that $x''(t_2) = 0$ and $x''(0) = x''(t_2)$ there exists $t_3 \in (0, t_2)$ such that $x'''(t_3) = 0$. Hence for $x \in \Omega_1$ we have

$$\int_{t_3}^{t} x'''(s)x^{(iv)}(s)ds = \lambda \int_{t_3}^{t} x'''(s)g(s,x,x',x'',x''')ds + \lambda \int_{t_3}^{t} x'''(s)h(s,x,x',x'',x''')ds$$

$$\frac{1}{2}|x'''|_2^2 \leq \int_0^1 |x'''|\,|h(t,x,x',x'',x''')|dt.$$

Using the Cauchy inequality

$$|ab| \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon} \quad \text{for } \varepsilon > 0$$

we have

$$\int_0^1 |x'''|\,|h(t,x,x',x'',x''')|dt \leq \frac{\varepsilon}{2}\int_0^1 |x'''|^2 dt + \frac{1}{2\varepsilon}\int_0^1 |h(t,x,x',x'',x''')|dt.$$

From condition (H_{2a}) we obtain the estimate

$$|h(t,x,y,w,z)|^2 \leq 4M^2\{|x|^{2r} + |y|^2 + |w|^2 + |z|^{2\theta}\}.$$

Therefore,

$$\frac{1}{2}|x'''|_2^2 - \frac{\varepsilon}{2}|x'''|_2^2 \leq \frac{2M^2}{\varepsilon}|x|_2^{2r} + \frac{2M^2}{\varepsilon}|x'''|_2^{2\theta} + \frac{2M^2}{\varepsilon}|x'|_2^2 + \frac{2M^2}{\varepsilon}|x''|_2^2.$$

From Holder's inequality we get

$$\left(\frac{1}{2} - \frac{\varepsilon}{2} - \frac{32M^2}{\varepsilon\pi^4} - \frac{8M^2}{\varepsilon\pi^2}\right)|x'''|_2^2 \leq \frac{2M^2}{\varepsilon}\left(|x|_2^{2r} + |x'''|_2^{2\theta}\right)_1.$$

Since $0 \leq \theta, r < 1$ we infer the existence of a constant M_2 such that

$$|x'''|_2^2 < M_2 \tag{3.8}$$

provided

$$\frac{1}{2} > \frac{\varepsilon}{2} + \frac{32M^2}{\varepsilon\pi^4} + \frac{8M^2}{\varepsilon\pi^2}. \tag{3.9}$$

The choice $\varepsilon = \frac{4M\sqrt{4+\pi^2}}{\pi}$ minimizes the right hand side of (3.9) with a minimum value $\frac{2M(B+\pi^2+4)}{B\pi^2}$ where $B = \sqrt{4+\pi^2}$.

Hence (3.8) holds provided

$$M < \frac{B\pi^2}{4(B^2+\pi^2+4)}.$$

From (3.8) and $x'(0) = x''(0) = 0$ we get

$$|x'|_\infty < M_3 \quad \text{for some } M_3 > 0 \tag{3.10}$$

$$|x''|_\infty < M_4, \quad M_4 > 0 \tag{3.11}$$

and from (3.7) we derive

$$|x|_\infty \le M_1 + |x'|_\infty < M_1 + M_3 = M_5. \tag{3.12}$$

Now using condition (H_{2b}) of Theorem 3.1 we get

$$\frac{x'''x^{iv}}{|x'''|^2+1} \le D(t,x,x',x'') + m(t)$$

and hence

$$\log_e |x'''| \le \int_{t_3}^{t} \frac{x'''(s)x^{(iv)}(s)}{|x'''|^2+1} ds = \left[\frac{1}{2}\log_e(|x'''(s)|^2+1)\right]_{t_3}^{t} \le D + |m|_1 \tag{3.13}$$

where the constant D depends on M_3, M_4 and M_5. Since $x'''(t_3) = 0$ we infer from (3.13) that

$$|x'''|_\infty < e^{N_0} = M_6 \tag{3.14}$$

where $N_0 = D + |m|_1$.

Hence

$$\|x\| = \max\{|x|_\infty,\ |x'|_\infty, |x''|_\infty,\ |x'''|_\infty \le \max\{M_4, M_3, M_5, M_6\}\}.$$

Therefore, Ω_1 is bounded.

Let

$$\Omega_2 = \{x \in \ker L : Nx \in ImL\}.$$

For $x \in \Omega_2$, we have $x = c \in \mathbb{R}$, thus

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i}\int_0^{s}\int_0^{\tau_2}\int_0^{\tau_1} [f(v, x(v), 0, 0, 0)]dv d\tau_1 d\tau_2 ds = 0. \tag{3.15}$$

Then we have by H_3 and (3.15) that

$$\|x\| = c \le N^*$$

which shows that Ω_2 is bounded.

We define the isomorphism $J : ImQ \longrightarrow \ker L$ by

$$J(c) = c, \quad c \in \mathbb{R}.$$

If (3.5) holds we set

$$\Omega_3 = \{x \in \ker L : -\lambda x + (1-\lambda)JQNx = 0\},\ \lambda \in [0,1]$$

For $c_0 \in \Omega_3$, we obtain

$$\lambda c_0 = \frac{(1-\lambda)A}{\sum_{i=0}^{m-2} \alpha_i \xi_i^{l+4}} \cdot t^l \left(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i}\int_0^{s}\int_0^{\tau_2}\int_0^{\tau_1} [f(v, c_0, 0, 0, 0)]dv d\tau_1 d\tau_2 ds\right)$$

if $\lambda = 1$ then $c_0 = 0$ and if $|c_0| > N^*$ then from (3.5) we have

$$\lambda c_0^2 = \frac{(1-\lambda)c_0 A}{\sum_{i=0}^{m-2} \alpha_i \xi_i^{l+4}} \cdot t^l \left(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i}\int_0^{s}\int_0^{\tau_2}\int_0^{\tau_1} [f(v, c_0, 0, 0, 0)]dv d\tau_1 d\tau_2 ds\right) < 0$$

which contradicts $\lambda c_0^2 \ge 0$. Thus Ω_3 is bounded.

If (3.6) holds, then let

$$\Omega_3 = \{x \in \ker L : \lambda x + (1-\lambda)JQNx = 0,\ \lambda \in [0,1]\}.$$

Following the above argument we can show that Ω_3 is bounded.

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Let Ω be a bounded open subset of X such that $\cup_{i=1}^{3} \Omega_i \subset \Omega$. By the Arzela–Ascoli Theorem we can show that $K_p(I-Q)N : \Omega \longrightarrow X$ is compact [5]. So N is L-compact. Thus we have shown that

(i) $Lx \neq \lambda Nx$ for every $(x,\lambda) \in [domL \setminus_{\ker L} \cap \partial\Omega \times (0,1)]$
(ii) $Nx \notin ImL$ for every $x \in \ker L \cap \partial\Omega$.

Finally we shall prove that (iii) of Theorem 2.1 is satisfied.

Define

$$H(x,\lambda) = \pm\lambda x + (1-\lambda)QNx,$$

we have

$$H(x,1) = \pm x, \qquad H(x,0) = QNx.$$

Thus $H(x,\lambda)$ is a homotopy from the identity $\pm I$ to QN and is such that $H(x,\lambda) \neq 0$ for every $x \in \partial\Omega \cap \ker L$. Therefore

$$\begin{aligned} \deg(JQN|_{\ker L\cap\partial\Omega},\ \Omega\cap\ker L, 0) &= \deg(H(\cdot,0), \Omega\cap\ker L, 0) \\ &= \deg(H(\cdot,1), \Omega\cap\ker L, 0) \\ &= \deg(\pm I, \Omega\cap\ker L, 0) \neq 0. \end{aligned}$$

Then by Theorem 2.1 $Lx = Nx$ has at least one solution in $domL \cap \bar{\Omega}$. In other words (1.1)–(1.2) has at least one solution in $C^3[0,1]$. □

Uncited references

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[6], [7], [8], [9], [10], [11], [12] and [13].

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