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On the Exponentiated Weighted Exponential Distribution and Its Basic Statistical Properties

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A B S T R A C T

We explore a generalization of the Weighted Exponential (WE) distribution using the exponentiated class/family of distributions. The proposed model is named Exponentiated Weighted Exponential distribution and it serves as an alternative to both the Weighted Exponential distribution and the Exponential distribution. Some of the basic statistical properties of the proposed model are studied and provided. The method of maximum likelihood estimation (MLE) was proposed in estimating the parameters of the model.

Key words: *Distribution, Exponential, Exponentiated Weighted Exponential, Generalization, Weighted Exponential*.

Introduction

Extreme Value distributions have been found to be useful in the fields of engineering, insurance and modern science. According to Ramadan (2013), mostly because of the crisis in currency, crashes in stock markets and defaults in large credit, a number of researches have been carried out to analyze the extreme variations that financial markets are subject to. Besides, details on the usefulness of extreme value theory in risk management have been discussed by several notable authors and the tail behavior of financial series has also been rigorously discussed (See Andjelic et al (2010); Cooley et al (2007); and Ramadan (2013) for details).

Gupta and Kundu (2009) developed a two-parameter Weighted Exponential (WE) distribution as a lifetime model and it has been widely used engineering and medicine. In addition, the WE distribution has been discovered to be a competitor to the Weibull, Gamma and Generalized Exponential distributions.

The probability density function (pdf) and the cumulative density function (cdf) of the WE distribution are given respectively by;

$$g(x)=\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left[1-e^{-\lambda\alpha x}\right] \quad ; \quad x>0,\alpha>0,\lambda>0 \qquad (1)$$

$$G(x)=\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right] \quad ; \quad x>0,\alpha>0,\lambda>0 \qquad (2)$$

where;
α is the shape parameter
λ is the scale parameter

The aim of this article is to generalize the WE distribution by inducing it by an additional shape parameter so as to be able to withstand data sets with strong asymmetry. To this end, we shall make use of the Exponentiated family of distributions which has been used by a number of notable authors to generalize several known theoretical distributions in the literature; The Exponentiated Weibull distribution (Mudholkar et al; 1995), Exponentiated Gamma distribution (Nadarajah and Gupta; 2007), Exponentiated Exponential distribution (Gupta and Kundu; 2001), Exponentiated Weibull distribution (Badmus and Bamiduro; 2014) and many more are notable examples.

The rest of this article is organized as follows; Section 2 discusses the proposed Exponentiated Weighted Exponential distribution, Section 3 discusses some basic statistical properties of the distribution and the estimation of model parameters using the method of maximum likelihood estimation followed by a concluding remark.

The Exponentiated Weighted Exponential Distribution

The Exponentiated family of distribution is derived by raising the cdf of an arbitrary parent distribution by a shape parameter say; $\theta > 0$. Its pdf is given by;

$$f(x) = \theta g(x) G(x)^{\theta - 1} \tag{3}$$

Its corresponding pdf is given by;

$$F(x) = G(x)^{\theta} \tag{4}$$

With this understanding, we insert Equations (1) and (2) into Equation (3) to give the pdf of the ExpWE as;

$$f(x) = \theta \left(\frac{\alpha + 1}{\alpha} \lambda e^{-\lambda x} \left[1 - e^{-\lambda \alpha x} \right] \right) \left\{ \frac{\alpha + 1}{\alpha} \left[1 - e^{-\lambda x} - \frac{1}{1 + \alpha} \left(1 - e^{-\lambda x (1 + \alpha)} \right) \right] \right\}^{\theta - 1} \tag{5}$$

The corresponding cdf of the ExpWE distribution is given by;

$$F(x) = \left\{ \frac{\alpha + 1}{\alpha} \left[1 - e^{-\lambda x} - \frac{1}{1 + \alpha} \left(1 - e^{-\lambda x (1 + \alpha)} \right) \right] \right\}^{\theta} \tag{6}$$

For $x > 0, \alpha > 0, \lambda > 0, \theta > 0$

where;

α and θ are the shape parameters

λ is the scale parameter

Special Case

For $\theta = 1$ in Equation (5), the ExpWE distribution reduces to give the Weighted Exponential distribution.

The author studied the plots for the ExpWE distribution at various parameter values, but for brevity, we only report and present plots at $\alpha = 2, \lambda = 1.5, \theta = 1.5$

The plot for the pdf and cdf of the ExpWE distribution are given in Figures 1 and 2 respectively.

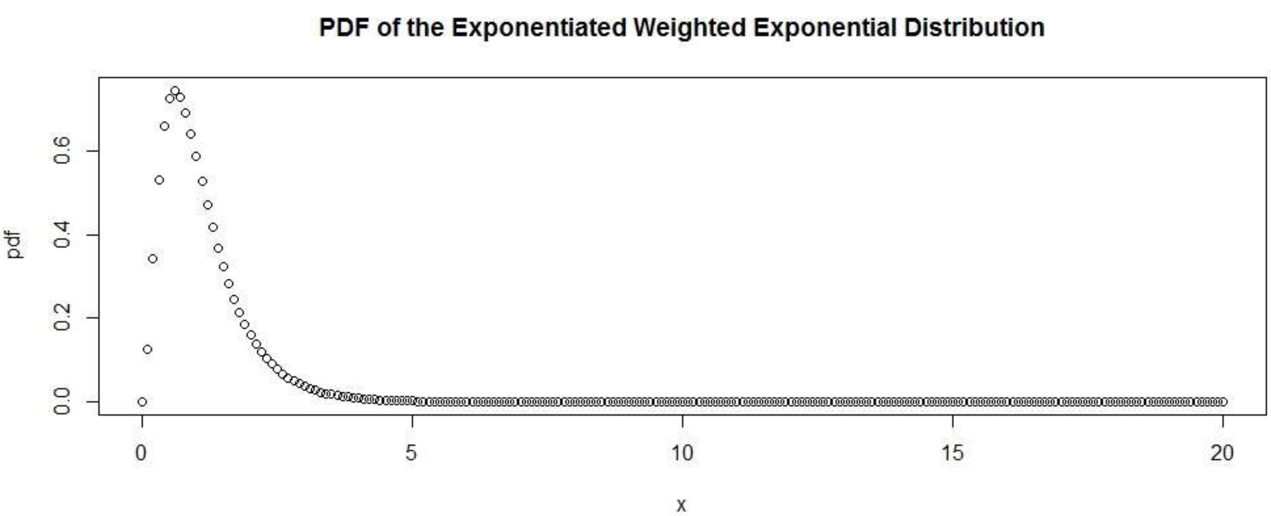

Figure 1. Plot for the pdf of the ExpWE distribution at $\alpha = 2, \lambda = 1.5, \theta = 1.5$

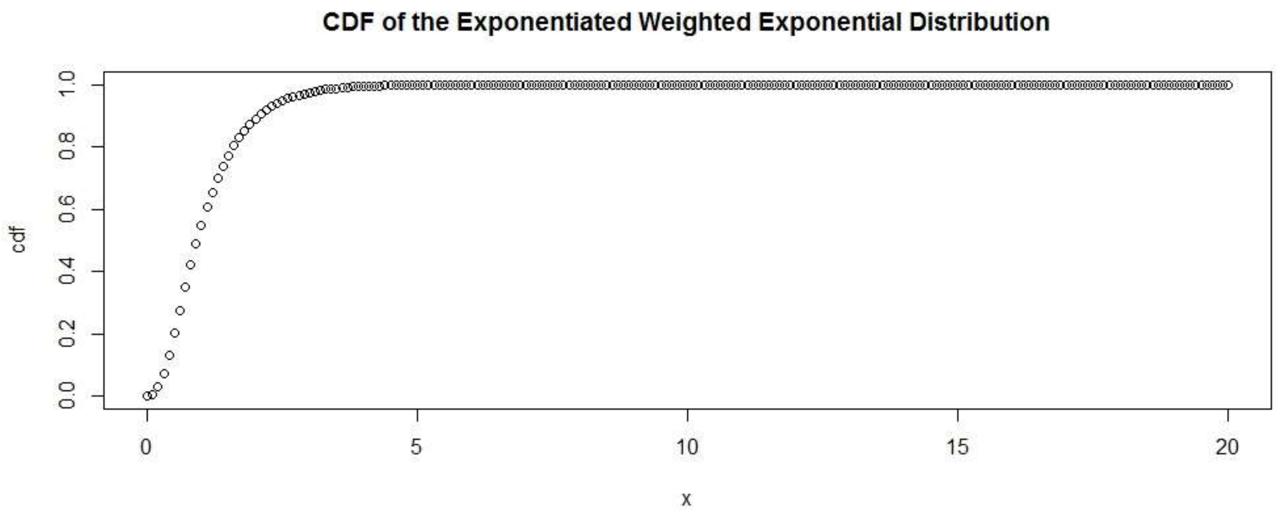

Figure 2. Plot for the cdf of the ExpWE distribution at $\alpha = 2, \lambda = 1.5, \theta = 1.5$

The plot in Figure 1 indicates that the proposed ExpWE distribution is positively skewed and the shape is unimodal (inverted bathtub). This statement is further confirmed in sub-section 3.1

Some Statistical Properties Of The Expwe Distribution

In this section, we provide expressions for some basic statistical properties of the ExpWE distribution, beginning with the asymptotic behavior.

Asymptotic Behavior

Following Oguntunde et al (2014), we seek to investigate the behavior of the ExpWE pdf as $x \to 0$ and as $x \to \infty$. To this end, we consider, $\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(x)$.

As $x \to 0$;

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left\{\theta\left(\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left[1-e^{-\lambda\alpha x}\right]\right)\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta-1}\right\}$$

$= 0$

As $x \to \infty$;

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left\{\theta\left(\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left[1-e^{-\lambda\alpha x}\right]\right)\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta-1}\right\}$$

$= 0$

Since $\lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x) = 0$, we can say that the ExpWE distribution has at least, a unique mode.

Reliability Analysis

In this subsection, we provide the expression for the reliability (or survival) function and the hazard function (or failure rate) of the proposed ExpWE distribution.

The survival function is mathematically represented by;

$$S(x) = 1 - F(x)$$

Therefore, the survival function of the ExpWE distribution is given by;

$$S(x)=1-\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta} \tag{7}$$

The plot for the survival function of the ExpWE distribution is represented in Figure 3;

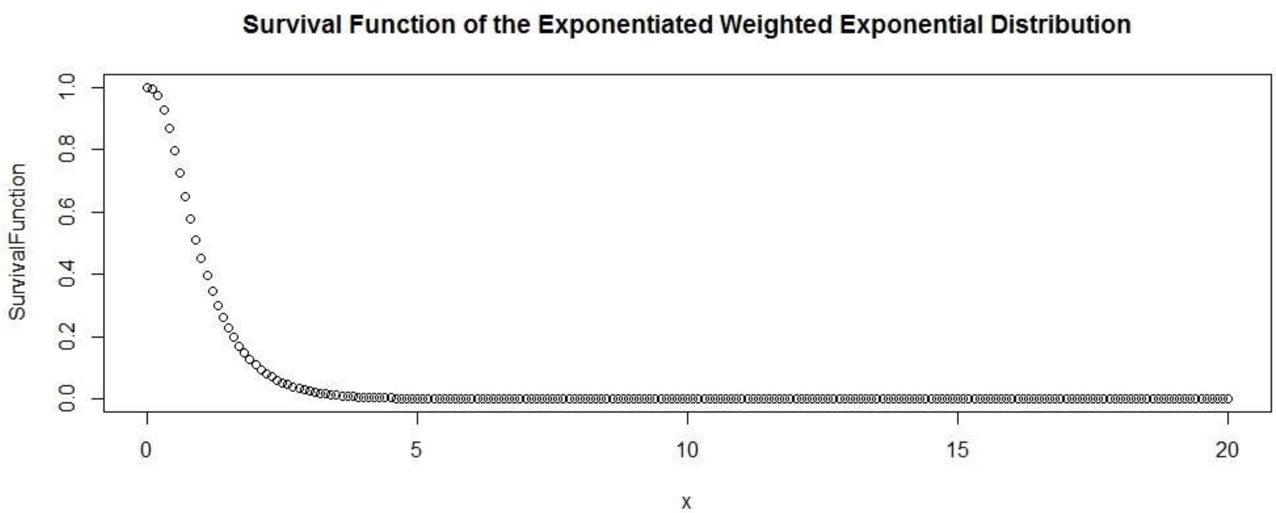

Figure 3. Plot for the survival function of the ExpWE distribution at $\alpha=2,\lambda=1.5,\theta=1.5$

The probability that a system having age $'x'$ units of time will survive up to $'x+t'$ units of time for $x>0,\alpha>0,\lambda>0,\theta>0$ and $t>0$ is given by;

$$S(t\,|\,x)=\frac{S(x+t)}{S(x)}$$

$$S(t\,|\,x)=\frac{1-\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda(x+t)}-\frac{1}{1+\alpha}\left(1-e^{-\lambda(x+t)(1+\alpha)}\right)\right]\right\}^{\theta}}{1-\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta}} \tag{8}$$

Table 1. Survival Function for Exponentiated Weighted Exponential Distribution at $\lambda=1,\theta=1$

x	$\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=4$	$\alpha=5$
0.1	0.9909441	0.9868470	0.9830099	0.9794141	0.9760426
0.2	0.9671415	0.9536903	0.9418647	0.9314436	0.9222381
0.3	0.9328248	0.9079425	0.8873596	0.8702402	0.8559221
0.4	0.8913111	0.8548830	0.8264612	0.8040662	0.7862405
0.5	0.8451819	0.7982309	0.7635958	0.7376421	0.7178794
0.6	0.7964291	0.7405680	0.7015095	0.6735678	0.6531092
0.7	0.7465736	0.6836497	0.6418437	0.6131823	0.5929032
0.8	0.6967614	0.6286345	0.5855179	0.5570823	0.5375488
0.9	0.6478404	0.5762517	0.5329850	0.5054348	0.4869803
1.0	0.6004236	0.5269256	0.4844007	0.4581648	0.4409596
1.1	0.5549390	0.4808650	0.4397357	0.4150672	0.3991732
1.2	0.5116705	0.4381295	0.3988490	0.3758731	0.3612837
1.3	0.4707900	0.3986767	0.3615369	0.3402889	0.3269562
1.4	0.4323839	0.3623977	0.3275633	0.3030182	0.2958714
1.5	0.3964733	0.3291407	0.2966806	0.2787744	0.2677315

Table 2. Survival Function for Exponentiated Weighted Exponential Distribution at $\alpha = 1, \lambda = 1$

x	$\theta=1$	$\theta=2$	$\theta=3$	$\theta=4$	$\theta=5$
0.1	0.9909441	0.9999180	0.9999993	1.0000000	1.0000000
0.2	0.9671415	0.9989203	0.9999645	0.9999988	1.0000000
0.3	0.9328248	0.9954875	0.9996969	0.9999796	0.9999986
0.4	0.8913111	0.9881867	0.9987160	0.9998604	0.9999848
0.5	0.8451819	0.9760313	0.9962892	0.9994255	0.9999111
0.6	0.7964291	0.9585589	0.9915638	0.9982826	0.9996504
0.7	0.7465736	0.9357751	0.9837237	0.9958752	0.9989547
0.8	0.6967614	0.9080464	0.9721161	0.9915445	0.9974360
0.9	0.6478404	0.8759836	0.9563265	0.9846199	0.9945838
1.0	0.6004236	0.8403387	0.9362031	0.9745083	0.9898141
1.1	0.5549390	0.8019207	0.9118426	0.9607646	0.9825379
1.2	0.5116705	0.7615343	0.8835501	0.9431341	0.9722307
1.3	0.4707900	0.7199368	0.8517877	0.9215646	0.9584912
1.4	0.4323839	0.6778119	0.8171208	0.8961948	0.9410785
1.5	0.3964733	0.6357555	0.7801687	0.8673259	0.9199276

The hazard function is mathematically given by;

$$h(x) = \frac{f(x)}{1 - F(x)}$$

Therefore, the expression for the hazard function of the ExpWE distribution is given by;

$$h(x) = \frac{\theta\left(\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left[1-e^{-\lambda\alpha x}\right]\right)\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta-1}}{1-\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta}} \quad (9)$$

We also seek to investigate the behavior of the hazard function as $x \to 0$ and as $x \to \infty$;

$$\lim_{x\to 0} h(x) = \lim_{x\to 0}\left\{\frac{\theta\left(\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left[1-e^{-\lambda\alpha x}\right]\right)\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta-1}}{1-\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta}}\right\}$$

$$= 0$$

$$\lim_{x\to \infty} h(x) = \lim_{x\to \infty}\left\{\frac{\theta\left(\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left[1-e^{-\lambda\alpha x}\right]\right)\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta-1}}{1-\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta}}\right\}$$

$$= 0$$

This implies that for ExpWE distribution, $\lim_{x\to 0} h(x) = \lim_{x\to\infty} h(x) = 0$.

The corresponding plot for the hazard function of the ExpWE distribution is represented in Figure 4;

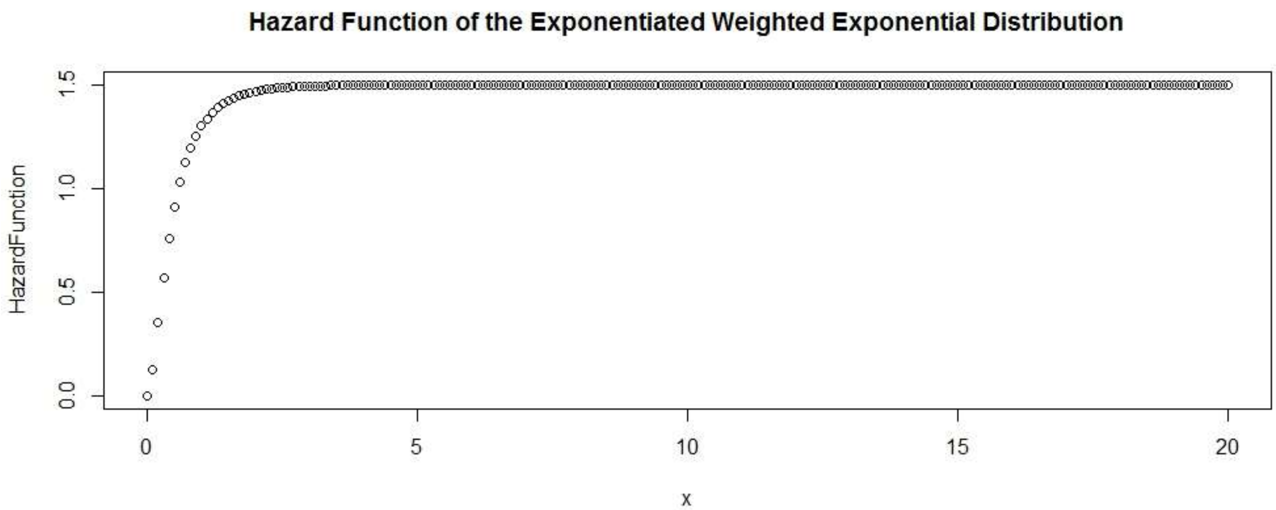

Figure 4. Plot for the hazard function of the ExpWE distribution at $\alpha = 2, \lambda = 1.5, \theta = 1.5$

Moments and Moment Generating Function

The sth moment μ_s for a continuous pdf is given by;

$$\mu_s = E\left[X^s\right] = \int_{-\infty}^{\infty} x^s f\left(x\right) dx$$

Meanwhile, in this research, we make use of a representation out of the four representations given by Alexandra et al (2012) for μ_s as follows;

For a Generalized Beta Generated distribution, the sth moment is given by;

$$E\left[X^s\right] = cB\left(a,b+1\right)^{-1} \sum_{i=0}^{\infty} \left(-1\right)^i \binom{b}{i} \tau\left(s, c\left(i+a\right)-1\right) \quad (10)$$

Setting $Q\left(u\right) = F^{-1}\left(u\right)$;

$$\tau\left(s,r\right) = \int_{-\infty}^{\infty} x^s F\left(x\right)^r f\left(x\right) dx = \int_0^1 Q\left(u\right)^s u^r du \quad (11)$$

The corresponding moment generating function (mgf) of X is given by;

$$M_X\left(t\right) = cB\left(a,b+1\right)^{-1} \sum_{i=0}^{\infty} \left(-1\right)^i \binom{b}{i} \rho\left(t, a\left(i+c\right)-1\right) \quad (12)$$

where; $$\rho\left(t,r\right) = \int_{-\infty}^{\infty} e^{(tx)} F\left(x\right)^r f\left(x\right) dx = \int_0^1 e^{\{tQ(u)\}} u^r du \quad (13)$$

If we set, $b = c = 1$ and $a = \theta$ in Equations (10) and (12), we arrive at the corresponding expressions for the Exponentiated family of distribution

With this understanding, we provide the sth moment of the ExpWE as follows;

$$E\left[X^s\right] = B\left(\theta,2\right)^{-1} \sum_{i=0}^{\infty} \left(-1\right)^i \binom{1}{i} \tau\left(s, \left(i+\theta\right)-1\right)$$

This can be simplified to give;

$$E\left[X^{s}\right]=\frac{\Gamma(\theta+2)}{\Gamma(\theta)}\sum_{i=0}^{\infty}(-1)^{i}\binom{1}{i}\tau\left(s,(i+\theta)-1\right) \tag{14}$$

where;

$$\tau(s,r)=\int_{-\infty}^{\infty}x^{s}F(x)^{r}\,f(x)\,dx=\int_{0}^{1}Q(u)^{s}\,u^{r}\,du$$

The corresponding m.g.f of the ExpWE distribution is given by;

$$M_{X}(t)=B(\theta,2)^{-1}\sum_{i=0}^{\infty}(-1)^{i}\binom{1}{i}\rho\left(t,\theta(i+1)-1\right)$$

This can be further expressed as;

$$M_{X}(t)=\frac{\Gamma(\theta+2)}{\Gamma(\theta)}\sum_{i=0}^{\infty}(-1)^{i}\binom{1}{i}\rho\left(t,\theta(i+1)-1\right) \tag{15}$$

where;

$$\rho(t,r)=\int_{-\infty}^{\infty}e^{(tx)}F(x)^{r}\,f(x)\,dx=\int_{0}^{1}e^{\{tQ(u)\}}u^{r}\,du$$

Note that;

$f(x)$ is the pdf of the ExpWE distribution as given in Equation (5)

$F(x)$ is the cdf of the ExpWE distribution as given in Equation (6)

$Q(u)$ is the quantile function (inverse cdf) of the ExpWE distribution

Parameter Estimation

In a view to estimating the parameters of the ExpWE distribution, we make use of the method of Maximum Likelihood Estimation (MLE). Let $X_1, X_2, ..., X_n$ be a random sample of n independently and identically distributed random variables each having the ExpWE distribution as defined in Equation (5), the likelihood function L is given by;

$$L\left(\underset{\sim}{X}\mid\alpha,\lambda,\theta\right)=\prod_{i=1}^{n}\left[\theta\left(\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left[1-e^{-\lambda\alpha x}\right]\right)\left\{\frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right]\right\}^{\theta-1}\right] \tag{16}$$

Let $l=\log L$;

Solving the resulting simultaneous equation of $\frac{\partial l}{\partial\alpha}=0$, $\frac{\partial l}{\partial\lambda}=0$ and $\frac{\partial l}{\partial\theta}=0$ gives the maximum likelihood estimates of α,λ and θ.

Conclusions

This article introduces a three parameter probability model called the Exponentiated Weighted Exponential distribution. The idea is to introduce an additional shape parameter to the existing Weighted Exponential distribution in other to increase its flexibility. Expressions for some basic statistical properties of the proposed model are explicitly derived. The model is positively skewed its shape is unimodal (or inverted bathtub). We hope that the model will receive appreciable usage in the fields of medicine and insurance. Further research would involve applying the proposed model to real life data sets to assess its flexibility over the Weighted Exponential distribution and to justify author’s claim.

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Conflicts of Interest

“The author declares no conflict of interest”.

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