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# Common Fixed Point Results of Weakly Compatible Maps in G-metric Spaces

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## Abstract

We prove the existence of a unique common fixed point for two weakly compatible maps satisfying $\phi$ - conditions in G-metric spaces. Our result extends and generalizes some results in the literature.

Keywords: Common fixed point, G-metric spaces, weakly compatible maps, weak-contraction maps.

## 1 Introduction and Preliminary

Frechet [1] introduced the notion of metric spaces and are widely used in fixed point theory and applications. Different authors generalized the concept of metric spaces. Eke and Olaleru [2] introduced the concept of G-partial metric spaces which generalized the G-metric spaces in the context of partial metric spaces. Authors such as Gahler [3], Dhage [4,5,6], Matthew [7] and others in the literature also generalized the notion of metric spaces. In this work we are concerned with the generalization of the notion of the metric spaces by Mustafa and Sims [8] in which a real number is assigned to every triplets of an arbitrary set. The following definitions and motivations are found in [8]:

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**Definition 1.1:** Let $X$ be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying:

$G_1$ $G(x, y, z) = 0$ if $x = y = z$,
$G_2$ $0 < G(x, x, y)$ $\forall x, y \in X$ with $x \neq y$,
$G_3$ $G(x, x, y) \leq G(x, y, z)$ $\forall x, y, z \in X$ with $z \neq y$,
$G_4$ $G(x, y, z) = G(x, z, y) = G(y, z, x)$ (symmetry in all three variables),
$G_5$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ $\forall a, x, y, z \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or more specifically a G-metric on X, and the pair (X, G) is called G-metric spaces.

Definition 1.2: Let (X, G) be a G-metric space, and let $\{x_n\}$ a sequence of points in $X$, a point '$x$' in $X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to $x$.

**Proposition 1.3:** Let (X, G) be G-metric space, then for a sequence $\{x_n\} \subseteq X$ and point $x \in X$ the following are equivalent:

(i) $\{x_n\}$ is G-convergent to x.
(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$
(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$
(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$

**Proposition 1.4:** In a G-metric space (X, G), the following are equivalent;

(i) The sequence $\{x_n\}$ is a G-Cauchy sequence.
(ii) For every $\varepsilon > 0$, $\exists\ n \in N \ni G(x_n, x_m, x_m) < \varepsilon$, $\forall n, m \geq N$.

**Definition 1.5:** A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Banach contraction map is the basis of all other contractive maps. This map is used to establish the existence of unique fixed points for certain contraction maps defined in metric spaces and its generalizations. The existence of the fixed point for Banach contraction map was proved by Matthew [9] in partial metric spaces. Mustafa et al. [10] proved the existence of unique fixed points and common fixed points for certain contractive maps in G-metric spaces.

The concept of weak contraction was introduced by Alber and Guerre-Delabriere [11] in Hilbert space. Rhoades [12] gives a corresponding definition in metric spaces as:

A mapping $T: X \to X$, where (X, d) is a metric space is said to be weakly contractive if

$$d(Tx, Ty) \le d(x, y) - \phi(d(x, y)), \tag{1}$$

where $x, y \in X$ and $\phi:[0,\infty) \to [0,\infty)$ is continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

Aage and Salunke [13] and Shatanawed [14] proved the existence of a unique fixed point for weak contraction maps defined on G-metric spaces. The result of Aage and Salunke [13] is stated as follows:

**Theorem 1.6 [13]:** Let (X, G) be a complete G-metric space and let $T: X \to X$ be mappings satisfying:

$$G(Tx, Ty, Tz) \le G(x, y, z) - \phi(G(x, y, z)), \tag{2}$$

for all $x, y, z \in X$. If $\phi:[0,\infty) \to [0,\infty)$ is continuous and nondecreasing mapping with $\phi^{-1}(0) = 0$, $\phi(t) > 0$ for all $t \in (0,\infty)$. Then T has a unique fixed point in X.

For the fact that two maps have to commute at a point before their common fixed points can be established led to the development of some commutative maps (see. [15,16,17,18]) in which weakly compatible maps are not left out and is defined below as:

**Definition 1.7 [16]:** A point $x \in X$ is called a coincidence point of a pair of self maps S, T if there exist a point w (called a point of coincidence) in X such that $w = Sx = Tx$. Self maps S and T are said to be weakly compatible if they commute at their coincidence points, that is if $Sx = Tx$ for some $x \in X$, then $STx = TSx$.

Some authors had used the concept of weakly compatibility in proving the common fixed points of two maps in metric spaces: see ([18,19,20,21]).

In this work, we prove the common fixed point of two weakly compatible maps satisfying some weak contractive conditions in G-metric spaces. Our result generalizes the results of Aage and Salunke [13].

# 2 Results and Discussion

**Theorem 2.1:** Let (X, G) be G-metric spaces and Y a nonempty subset of X. Let $T, S: Y \to X$ be mappings satisfying:

$$G(Tx, Ty, Tz) \le G(Sx, Sy, Sz) - \phi(G(Sx, Sy, Sz)) \tag{3}$$

for all $x, y, z \in X$. If $\phi:[0,\infty) \to [0,\infty)$ is continuous and nonincreasing function with $\phi^{-1}(0)=0$, $\phi(t)>0$ for all $t \in (0,\infty)$. Suppose that T and S are weakly compatible with $T(Y) \subseteq S(Y)$. If $T(Y)$ or $S(Y)$ is a complete subspace of X, then the mappings T and S have a unique common fixed point in X.

**Proof:** Let $x_0 \in X$ be arbitrary. choose $x_1 \in X$ such that $Tx_0 = Sx_1$. Continuing this process, we can define the sequence $\{x_n\}$ by $Tx_n = Sx_{\{n+1\}}$ for some $n \in N$. Suppose $Tx_n = Tx_{\{n-1\}}$ for some $n \in N$, then we have $Tx_n = Sx_n$. Therefore $\{Tx_n\}$ is a Cauchy sequence. We assume that $Ty = Tz$ in $(3)$ and $Tx_n \neq Tx_{\{n-1\}}$ $\forall n \in N$. From (3), we have

$$G\left(Tx_{\{n-1\}}, Tx_n, Tx_n\right) \le G\left(Sx_{\{n-1\}}, Sx_n, Sx_n\right) - \phi\left(G\left(Sx_{\{n-1\}}, S_n, S_n\right)\right) \qquad (4)$$

By property of $\phi$, (4) gives

$$G\left(Tx_{\{n-1\}}, Tx_n, Tx_n\right) \le G\left(Sx_{\{n-1\}}, Sx_n, Sx_n\right) \qquad (5)$$

Similarly,

$$G\left(Sx_{\{n-1\}}, Sx_n, Sx_n\right) = G\left(Tx_{\{n-2\}}, Tx_{\{n-1\}}, Tx_{\{n-1\}}\right) \le G\left(Sx_{\{n-2\}}, Sx_{\{n-1\}}, Sx_{\{n-1\}}\right). \qquad (6)$$

From (5) and (6), this shows that $G\left(Tx_{\{n-1\}}, Tx_n, Tx_n\right)$ is monotone decreasing and consequently there exists $K \ge 0$ such that

$$G\left(Tx_{\{n-1\}}, Tx_n, Tx_n\right) \to K \text{ as } n \to \infty \qquad (7)$$

Taking $n \to \infty$ in (4), we obtain

$$K \le K - \phi(K). \qquad (8)$$

This is a contradiction, unless $K = 0$. Hence

$$G\left(Tx_{\{n-1\}}, Tx_n, Tx_n\right) \to 0 \text{ as } n \to \infty. \qquad (9)$$

Now we show that $\{Tx_n\}$ is a Cauchy sequence. Suppose $\{Tx_n\}$ is not a Cauchy sequence, then $\exists \varepsilon > 0$ for which we can find subsequence

$\{Tx_{n(k)}\}$ and $\{Tx_{m(k)}\}$ of $\{Tx_n\}$ with $\mathrm{m}(\mathrm{k}) > \mathrm{n}(\mathrm{k})$ such that

$$G\left(Tx_{n(k)},\ Tx_{m(k)}, Tx_{m(k)}\right) \geq \varepsilon. \tag{10}$$

Now,

$$G\left(Tx_{n(k)-1},\ Tx_{n(k)},\ Tx_{n(k)}\right) = G\left(Sx_{n(k)},\ Sx_{n(k)+1},\ Sx_{n(k)+1}\right).$$

This implies that,

$$G\left(Sx_{n(k)},\ Sx_{m(k)},\ Sx_{m(k)}\right) \leq G\left(Sx_{n(k)},\ Sx_{n(k)+1},\ Sx_{n(k)+1}\right) + G\left(Sx_{n(k)+1},\ Sx_{n(k)+2},\ Sx_{n(k)+2}\right)$$
$$+\ G\left(Sx_{n(k)+2}, Sx_{m(k)}, Sx_{m(k)}\right).$$

Setting $K \to \infty$ in the above inequalities and using (9) we have,

$$\lim_{K\to\infty} G\left(Sx_{n(k)},\ Sx_{m(k)},\ Sx_{m(k)}\right) = 0. \tag{11}$$

From (3) and (10) we obtain,

$$\begin{aligned}
&\varepsilon \leq G\left(Tx_{n(k)},\ Tx_{m(k)}, Tx_{m(k)}\right) \\
&\leq G\left(Sx_{n(k)},\ Sx_{m(k)},\ Sx_{m(k)}\right) - \phi(G\left(Sx_{n(k)},\ Sx_{m(k)},\ Sx_{m(k)}\right))
\end{aligned}$$

Hence,

$$\varepsilon \leq \varepsilon\text{-}\phi(\varepsilon) \text{ as } k \to \infty.$$

Clearly it is a contradiction since $\varepsilon > 0$. We must have $\varepsilon = 0$. This shows that $\{Tx_n\}$ is a Cauchy sequence in X. Since T(Y) or S(Y) is a complete subspace of X and for the fact that $T(Y) \subseteq S(Y)$, there exists a $z \in T(Y)$ such that $Sx_n \to z$ and $Tx_n \to z$ as $n \to \infty$, hence there is $x \in X$ such that $Sx = z$.

From (3) we get,

$$\begin{aligned}
&G(Tx,\ z,\ z) \leq G(Tx,\ Tx_n,\ Tx_n) + G\ (Tx_n, z, z) \\
&\leq G(Sx,\ Sx_n,\ Sx_n) - \phi(G(Sx,\ Sx_n,\ Sx_n)) + G(Tx_n,\ z,\ z) = 0.
\end{aligned}$$

But $G(Tx,\ z,\ z) \geq 0$. This implies that $Tx = z$. Thus $Sx = Tx = z$ and we have that z is a point of coincidence of S and T.

Next we show that the point of coincidence is unique. Suppose there is another point of coincidence p, and there is a coincidence point $q \in X$ such that $p = Tq = Sq$. Then by (1) we have,

$$G(z,p,p) = G(Tx,\ Tq,\ Tq) \leq G(Sx,\ Sq,\ Sq) - \phi(G(Sx,\ Sq,\ Sq)).$$

By property of $\phi$, this is a contradiction if $G(z,p,p) > 0$. Hence we have a unique point of coincidence. Since S, T are weakly compatible, then $TSx = STx$ and $Tz = Sz$. Therefore z is a coincidence point of S, T and since the point of coincidence is unique, that is $z = p$. Hence $Sz = Tz = z$, therefore z is the unique common fixed point of S, T and the proof is complete.

**Remarks 2.2:** If $S = T$ in theorem 2.1 then we have corollary 2.2. Therefore theorem 2.1 is a generalization of theorem 2.1 of Aage and Salunke [13].

**Corollary 2.3 [12]:** Let (X, G) be complete G-metric space and let $T: X \to X$ be a mapping satisfying;

$$G(Tx,\ Ty,\ Tz) \leq G(x,\ y,\ z) - \phi(G(x,\ y,\ z))$$

for all $x,y,z \in X$. If $\phi:[0,\infty) \to [0,\infty)$ is continuous and increasing function with $\phi^{-1}(0) = 0,\ \phi(t) > 0\ \forall t \in (0,\infty)$, then T has a unique fixed point in X.

**Example 2.4:** Let $X = [0,1]$ and $G(x,y,z) = |x-y| + |y-z| + |z-x|$ be a G-metric on X. Define $S, T: X \to X$ by $Tx = \frac{x}{2}$ and $Sx = \frac{5x}{3}$ with $\phi(t) = \frac{t}{2} \forall t > 0$.

Now,

$$G(Tx,Ty,Tz) = \left|\frac{x}{2} - \frac{y}{2}\right| + \left|\frac{y}{2} - \frac{z}{2}\right| + \left|\frac{z}{2} - \frac{x}{2}\right|$$

$$= \frac{1}{2}(|x-y| + |y-z| + |z-x|).$$

$$G(Sx,Sy,Sz) = \left|\frac{5x}{3} - \frac{5y}{3}\right| + \left|\frac{5y}{3} - \frac{5z}{3}\right| + \left|\frac{5z}{3} - \frac{5x}{3}\right|$$

$$= \frac{5}{3}(|x-y| + |y-z| + |z-x|).$$

Hence

$$G(Tx,\ Ty,\ Tz) \leq G(Sx,\ Sy,\ Sz) - \phi(G(Sx,\ Sy,\ Sz)).$$

The common fixed point of S and T is equal to zero and is unique.

# 3 Conclusion

The existence and uniqueness of the common fixed point for a pair of weakly compatible mappings satisfying the weak – contraction conditions in a G-metric space is proved.


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# Competing Interests

Author has declared that no competing interests exist.

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