

*Full Length Research Paper***Analysis of Hermite's equation governing the motion of damped pendulum with small displacement****Agarana M. C.\* and Iyase S. A.**

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**This paper investigates simple pendulum dynamics, putting damping into consideration. The investigation begins with Newton's second law of motion. The second order differential equation governing the motion of a damped simple pendulum is written in form of Hermite's differential equation and general solution obtained by means of power series. The results obtained are in agreement with the existing ones, and converge fast.**

**Key words:** Pendulum, Hermite's equation, dynamics, damping, angular displacement.

**INTRODUCTION**

The pendulum is a dynamical system (Broer et al., 2010). The free pendulum consists of a rod, suspended at a fixed point in a vertical plane in which the pendulum can move (Agarana and Agboola, 2015; Nelson and Olsson, 1986). When a pendulum is acted on, both by a velocity dependent damping force, and a periodic driving force, it can display both ordered and chaotic behaviors, for certain ranges of parameters (Broer et al., 2010; Randall, 2003). The free, damped pendulum has damping, dissipation of energy takes place and a possible motion is bound to converge to rest (Peters 2002, 2003; Doumashkin et al., 2004; Gray, 2011). The motion of the bob of simple pendulum is a simple harmonic motion if it is given small displacement. When the pendulum is at rest, the only force acting on its weight and tension is the string. At the position,  $\theta = 0^\circ$ , the pendulum is in a stable equilibrium. Initial conditions are the way in which a system is started. The initial conditions for the simple pendulum are the starting angles, and the initial speed. In this paper we considered small angular displacement.

The equation governing damped simple pendulum can take the form of Hermite's differential equation (Moore, 2003). Our goal in this paper is to solve the resulting Hermite's equation from the equation governing the motion of a damped simple pendulum with small displacement by means of power series. The general solution of simple harmonic motion, generally, can be determined by means of power series.

**Hermite's equation**

Hermite's differential equation is (Moore, 2003)

$$\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2py = 0 \quad (1)$$

Where p is a parameter.

The above second-order ordinary differential equation can be written as:

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$$\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + \lambda y = 0 \tag{2}$$

Where  $2p = \lambda$

This differential equation has an irregular singularity at  $\infty$ . It can be solved using the series method (Moore, 2003; Broer et al., 2010).

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} -2na_nx^n + \sum_{n=0}^{\infty} \lambda a_nx^n \tag{3}$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - 2na_n + \lambda a_n]x^n = 0 \tag{4}$$

The eventual linearly independent solutions can be written as (Moore, 2003)

$$y_1 = a_0 \left[ 1 - \frac{\lambda}{2!}x^2 - \frac{(4-\lambda)\lambda}{4!}x^4 - \frac{(8-\lambda)(4-\lambda)\lambda}{6!}x^6 - \dots \right] \tag{5}$$

$$y_2 = a_1 \left[ x + \frac{(2-\lambda)}{3!}x^3 + \frac{(6-\lambda)(2-\lambda)}{5!}x^5 + \dots \right] \tag{6}$$

**Theorem**

If the function P(x) and Q(x) can be represented by power series

$$P(x) = \sum_{n=0}^{\infty} P_n(x - x_0)^n \tag{7}$$

$$Q(x) = \sum_{n=0}^{\infty} q_n(x - x_0)^n \tag{8}$$

with positive radii of convergence  $R_1$  and  $R_2$  respectively, then any solution  $y(x)$  to the linear differential equation

$$\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0 \tag{9}$$

can be represented by a power series

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{10}$$

whose radius of convergence is less than or equal to the smaller of  $R_1$  and  $R_2$ .

**GOVERNING EQUATIONS AND SOLUTION PROCEDURES**

The equation of motion for damped, driven pendulum of mass  $m$  and length  $l$  can be written as (Agarana and Agboola, 2015):

$$ml^2 \frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + mgl\sin\theta = C\cos(\omega_D t) \tag{11}$$

Where the right hand side of Equation (1) is the driving force.

Suppose the damped pendulum is not driven, then the right hand side of Equation (1) is zero, and Equation (11) becomes:

$$ml^2 \frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + mgl\sin\theta = 0 \tag{12}$$

The three terms on the left hand side of Equation (12) are the acceleration, damping and gravitation respectively.  $\theta$  is the angular displacement,  $t$  is the time,  $l$  is the length,  $m$  is the mass,  $\gamma$  is the dissipation coefficient and  $g$  is the acceleration due to gravity. Dividing Equation (12) by  $ml^2$  we have

$$\frac{d^2\theta}{dt^2} + \frac{\gamma}{ml^2} \frac{d\theta}{dt} + \frac{g}{l} \sin\theta = 0 \tag{13}$$

For a small angular Displacement,  $\theta \approx \sin\theta$ . Therefore Equation (13) becomes:

$$\frac{d^2\theta}{dt^2} + \frac{\gamma}{ml^2} \frac{d\theta}{dt} + \frac{g}{l} \theta = 0 \tag{14}$$

At this stage, we carefully choose the values of  $t$  and  $p$  such that:

$$\frac{\gamma}{ml^2} = -2t \text{ and } \frac{g}{l} = 2p \tag{15}$$

Where  $p$  is a parameter.

Substituting Equation (15) into Equation (14) we have:

$$\frac{d^2\theta}{dt^2} - 2t \frac{d\theta}{dt} + 2p\theta = 0 \tag{16}$$

Equation (16) is Hermite's differential equation. Indeed

$$P(t) = -2t, \text{ and } Q(t) = 2p$$

Both functions being polynomials, have power series about  $t_0 = 0$  with infinite radius of convergence (Moore, 2003). The angular displacement  $\theta$  is a function of  $t$  and satisfies simple harmonic motion characteristics. Any solution  $\theta(t)$  to Equation (16) can be represented by a power series (Agarana and Agboola, 2015; Moore, 2003):

$$\theta(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n \tag{17}$$

$$\therefore \theta(t) = \sum_{n=0}^{\infty} a_n t^n, \text{ since } (t_0 = 0) \tag{18}$$

Differentiating term by term we have

$$\frac{d\theta}{dt} = \sum_{n=1}^{\infty} n a_n t^{n-1} \tag{19}$$

$$\frac{d^2\theta}{dt^2} = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} \tag{20}$$

Replacing  $n$  by  $m+2$  in Equation (10), we have

$$\frac{d^2\theta}{dt^2} = \sum_{m+2-2}^{\infty} (m+2)(m+2-1) a_{m+2} t^{m+2-2} \tag{21}$$

$$= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} t^m \tag{22}$$

And then replacing  $m$  by  $n$  once again, so that

$$\frac{d^2\theta}{dt^2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n \tag{23}$$

From Equation (19)

$$-2t \frac{d\theta}{dt} = \sum_{n=0}^{\infty} -2na_n t^n \tag{24}$$

Also from Equation (18)

$$2p\theta = \sum_{n=0}^{\infty} 2pa_n t^n \tag{25}$$

Adding together Equations (23), (24) and (25), we have

$$\frac{d^2\theta}{dt^2} - 2t \frac{d\theta}{dt} + 2p\theta = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n + \sum_{n=0}^{\infty} -2na_n t^n + \sum_{n=0}^{\infty} 2pa_n t^n \tag{26}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n + \sum_{n=0}^{\infty} (-2n+2p)a_n t^n \tag{27}$$

Since  $\theta$  satisfies Hermite's equation, we have

$$0 = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (-2n+2p)a_n] t^n \tag{28}$$

$$\Rightarrow (n+2)(n+1)a_{n+2} + (-2n+2p)a_n = 0 \tag{29}$$

$$\Rightarrow a_{n+2} = \frac{2n-2p}{(n+2)(n+1)} a_n \tag{30}$$

In order to determine the values of  $a_0, a_1, a_2, a_3, \dots$  in the above power series, the first two coefficients,  $a_0$  and  $a_1$  can be determined from the initial conditions as (Moore, 2003):

$$\theta(0) = a_0 \text{ and } \frac{d\theta}{dt}(0) = a_1 \tag{31}$$

While the other coefficients are determined by equating  $n$  to 0, 1, 2, 3, ... to obtain, from Equation (30), as (Moore, 2003):

$$a_2 = \frac{2p}{2 \cdot 1} a_0 \tag{32}$$

$$a_3 = -\frac{2(p-1)}{3!} a_1 \tag{33}$$

$$a_4 = \frac{2^2 p(p-2)}{4!} a_0 \tag{34}$$

$$a_5 = \frac{2^2 (p-1)(p-3)}{5!} a_1 \tag{35}$$

$$a_6 = \frac{-2^5 p(p-2)(p-4)}{6!} a_0 \tag{36}$$

and so forth.

Equation (28) can now be written as follows:

$$\theta = a_0 \left[ 1 - \frac{2p}{2!} t^2 + \frac{2^2 p(p-2)}{4!} t^4 - \frac{2^5 p(p-2)(p-4)}{6!} t^6 + \dots \right] + a_1 \left[ t - \frac{2(p-1)}{3!} t^3 + \frac{2^2 (p-1)(p-3)}{5!} t^5 - \frac{2^5 (p-1)(p-3)(p-5)}{7!} t^7 + \dots \right] \tag{37}$$

We now write the general solution to Equation (16) in the form

$$\theta = a_0 \theta_0(t) + a_1 \theta_1(t) \tag{38}$$

where

$$\theta_0(t) = 1 - \frac{2p}{2!} t^2 + \frac{2^2 p(p-2)}{4!} t^4 - \frac{2^5 p(p-2)(p-4)}{6!} t^6 + \dots \tag{39}$$

and

$$\theta_1(t) = t - \frac{2(p-1)}{3!} t^3 + \frac{2^2 p(p-1)(p-3)}{5!} t^5 - \frac{2^5 (p-1)(p-3)(p-5)}{7!} t^7 + \dots \tag{40}$$

$\theta_0(t)$  and  $\theta_1(t)$  form a basis for the space of solutions to Equation (16) which is hermite's equation (Moore, 2003).

For different values of the parameter  $p$ , we obtain different values of  $\theta$ . When  $p$  is a positive integer, one of the two power series will collapse, yielding a polynomial solution known as Hermite Polynomial (Moore, 2003). The initial conditions which we are choosing for the purpose of this paper are:

$$\theta(t=0) = a_0; \frac{d\theta}{dt}(t=0) = a_1 = 0$$

Where  $a_0$  is a constant whose value we will take as input.

From these initial conditions,  $a_1 = 0$  and  $a_0$  is a constant. Therefore Equation (38) becomes

$$\theta = a_0 \theta_0(t) + 0 \tag{41}$$

That is,

$$\theta = a_0 \left[ 1 - \frac{2p}{2!} t^2 + \frac{2^2 p(p-2)}{4!} t^4 - \frac{2^5 p(p-2)(p-4)}{6!} t^6 + \dots \right] \tag{42}$$

For different values of parameter  $p$  and a particularly value of constant  $a_0$ , we can see the angular displacement  $\theta$  at the different time ( $t$ ). Recall from Equation (15) that

$$p = \frac{g}{2l} \text{ and } t = \frac{-\gamma}{2mi^2} \tag{43}$$

$$\therefore \theta = a_0 \left[ 1 - \frac{g/l}{2l} \left( \frac{-\gamma}{2mi^2} \right)^2 + \frac{2^2 g/l \left( \frac{g-2}{2mi^2} \right) \left( \frac{-\gamma}{2mi^2} \right)^4}{4!} - 2^5 \left( \frac{g}{2l} \right) \left( \frac{g-2}{2l} \right) \left( \frac{g-4}{2l} \right) \left( \frac{-\gamma}{2mi^2} \right)^6 + \dots \right]$$

Also Equation (42) can be written by substituting  $\frac{g}{2l}$  for  $p$  as follows:

$$\theta = a_0 \left[ 1 - \frac{g}{2l} t^2 + \frac{1}{6} \left( \frac{g}{2l} \right) \left( \frac{g}{2l} - 2 \right) t^4 - \frac{1}{90} \left( \frac{g}{2l} \right) \left( \frac{g}{2l} - 2 \right) \left( \frac{g}{2l} - 4 \right) t^6 + \dots \right] \tag{44}$$

The general solution as given in Equation (38) implies that

$$\theta = 1 - \frac{2p}{2!} t^2 + \frac{2^2 p(p-2)}{4!} t^4 - \frac{2^5 p(p-2)(p-4)}{6!} t^6 + \dots + t - \frac{2(p-1)}{3!} t^3 + \frac{2^2 p(p-1)(p-3)}{5!} t^5 - \frac{2^5 (p-1)(p-3)(p-5)}{7!} t^7 + \dots \tag{45}$$

It is assumed that  $a_0 = 1$  and  $a_1 = 1$ .

$$\theta = 1 + t - \frac{2p}{2!} t^2 - \frac{2(p-1)}{3!} t^3 + \frac{2^2 p(p-2)}{4!} t^4 + \frac{2^2 p(p-1)(p-3)}{5!} t^5 - \frac{2^5 p(p-2)(p-4)}{6!} t^6 - \frac{2^5 (p-1)(p-3)(p-5)}{7!} t^7 + \dots \tag{46}$$

**TIME PERIOD AND DISSIPATION COEFFICIENT EFFECTS ON THE ANGULAR DISPLACEMENT**

**Time period of oscillation for small displacement**

The time period of oscillation of a simple pendulum with a small displacement is given as (Agarana and Agboola, 2015; Davidson, 1983)

$$T = 2\pi \sqrt{\frac{l}{g}} \tag{47}$$

Where  $l$  is the length of the pendulum and  $g$  is the acceleration due to gravity, respectively.

Equation (50) can be written as:

$$T = \pi \sqrt{\frac{2}{p}} \tag{48}$$

Both Equations (47) and (48) show that the period  $T$  is a function of the length of the pendulum, just as equation (48) the angular displacement is a function of the length of the pendulum (since  $p = \frac{g}{2l}$ ) and time. The time period of oscillation, for small displacement, as it affects the motion of the pendulum is determined by the length of the pendulum.

**Effect of the dissipation coefficient  $\gamma$**

From Equation (15),

$$\gamma = -2tml^2 \tag{49}$$

Also  $l = \frac{g}{2p}$  (50)

Substituting Equation (50) into Equation (40), we have

$$\gamma = -2tm \left(\frac{g}{2p}\right)^2 \tag{51}$$

$$\gamma = -\frac{ml}{2} \left(\frac{g}{2p}\right)^2, \text{ (} l \text{ taken to be } 4t) \tag{52}$$

At different values of  $m$  and particular value of  $l$ , we can see how  $\gamma$  behaves.

**NUMERICAL RESULTS AND DISCUSSION**

We illustrated the ideas presented in the previous sections by means of a more realistic numerical example. As an illustration, therefore, the following values were adopted:  $l = 10, 20, 30, 40$ ;  $\alpha_0 = 1, 2, 3, 4, 5$  and  $g = 9.81$ . We therefore determined the values of the angular displacement ( $\theta$ ) for different lengths ( $l$ ) of the pendulum and at different initial values of the angular

displacement ( $\alpha_0$ ). From Figure 1, we can see that when the length of the pendulum is 10, the value of the angular displacement ( $\theta$ ) increases as its initial value ( $\alpha_0$ ) increases. This happens at the initial time, up to time  $t=0.8$  where there is a convergence of the different values of  $\theta$  at different values of  $\alpha_0$ . However after time  $t=0.8$  the opposite became the case: as initial angular displacement increases the angular displacement decreases. The same explanation goes for the dynamic behaviour of the pendulum as shown in Figures 2, 3 and 4, but the value of the angular displacement start decreasing as the value of  $\alpha_0$  increases at different times. For instance, in Figure 2, it starts at time  $t = 0.9$ , in Figure 3, it starts at time  $t = 0.95$ , and in Figure 4, it starts at time  $t = 1$ . We therefore observe, generally, that as we increase the value of the initial angular displacement, the subsequent angular displacement increases initially and decreases after some time, depending on the length  $l$  of the pendulum. The higher the value of  $l$ , the longer it takes for the change in the motion pattern of the pendulum as regards its angular displacement.

For various values of the pendulum length ( $l$ ), the angular displacement of the pendulum for various values of the initial angular displacement  $\alpha_0$  (that is,  $\alpha_0=1, \alpha_0=2, \alpha_0=3, \alpha_0=4, \alpha_0=5$ ) considered were calculated and are plotted in Figures 5, 6, 7 and 8 as functions of time. Specifically in Figure 5, the angular displacement profile of the damped pendulum is depicted for  $\alpha_0=1$  and with the pendulum length ( $l$ ), as a parameter. The corresponding curves for  $\alpha_0=2, 3$  and 4 are shown in Figures 6, 7, and 8 respectively. Clearly, from the figures the angular displacement increases with an increase in the value of the initial angular displacement for fixed values of pendulum length. Also for specific value of the initial angular displacement, the subsequent angular displacement increases as the pendulum length increases. However, the angular displacement decreases with time irrespective of the values of the pendulum length and the initial angular displacement. In Figures 9 and 10, the angular displacement of the damped pendulum for different values of pendulum length ( $l$ ) and the period ( $T$ ) respectively, with non-zero value of the angular velocity ( $a_1$ ), is plotted as a function of time. Evidently, it can be noticed in Figure 9 that the angular displacement increases with time and as the pendulum length increases. Also in Figure 10, angular displacement increases with time and as the pendulum period increases. The similarity in Figures 9 and 10 is as a result of the fact that the period ( $T$ ) is a function of the length of the pendulum ( $l$ ). Considering the effects of damping and mass of the pendulum bob on the angular displacement of the damped pendulum; angular pendulum for various values of the damping factor and mass of the pendulum bob were calculated and plotted in Figures 11 and 12

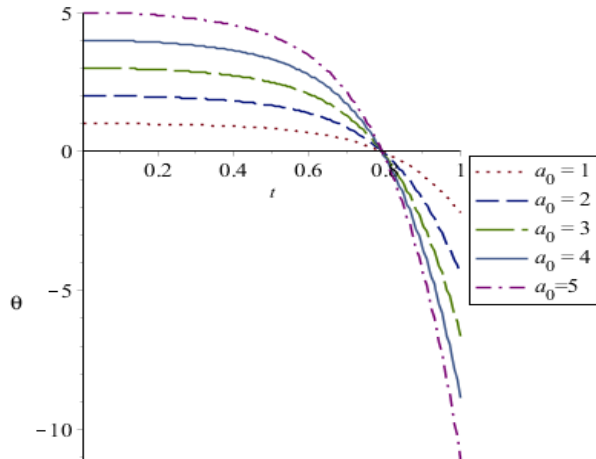


Figure 1. Angular displacement of pendulum at  $l = 10$ ,  $a_1 = 0$  and different values of  $a_0$  and time.

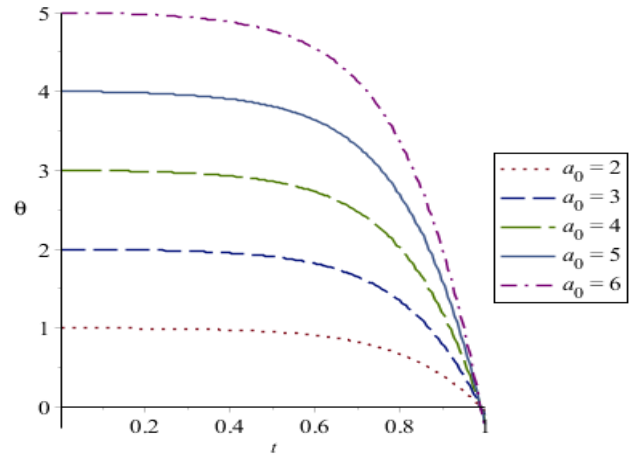


Figure 4. Angular displacement of pendulum at  $l = 40$ ,  $a_1 = 0$  and different values of  $a_0$  and time.

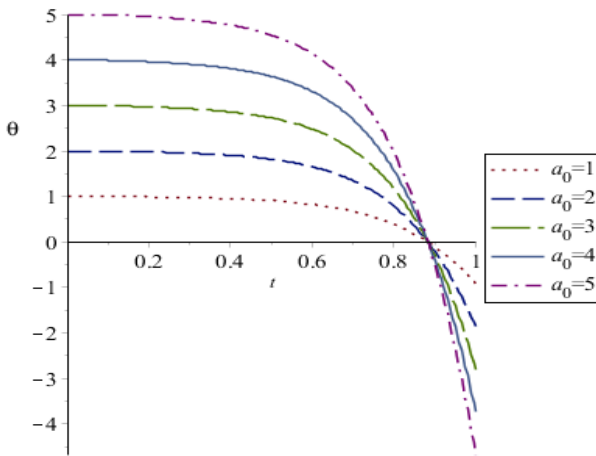


Figure 2. Angular displacement of pendulum at  $l = 20$ ,  $a_1 = 0$  and different values of  $a_0$  and time.

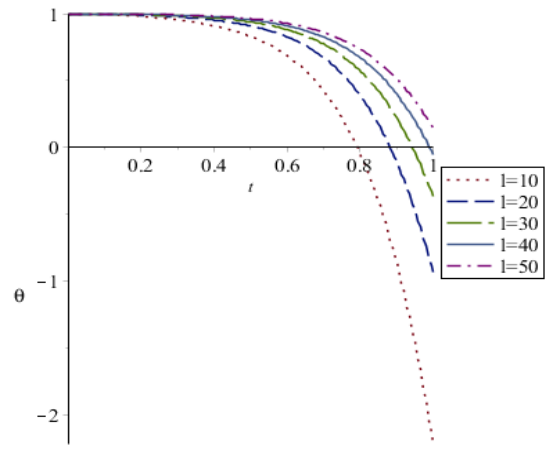


Figure 5. Angular displacement of pendulum at  $a_0 = 1$  and different lengths and time.

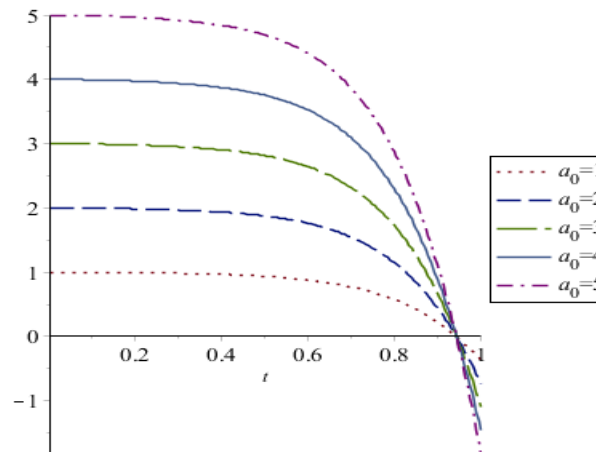


Figure 3. Angular displacement of pendulum at  $l = 30$ ,  $a_1 = 0$  and different values of  $a_0$  and time.

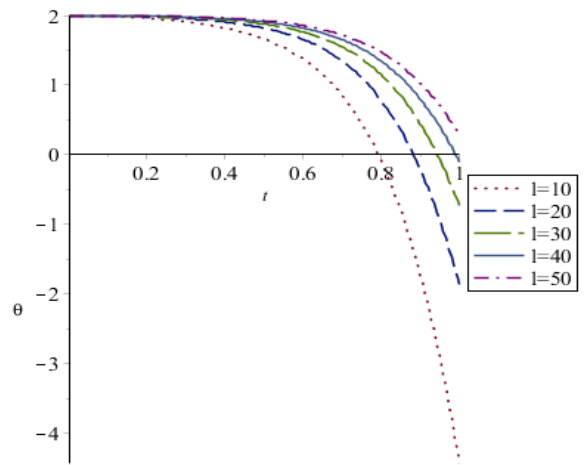
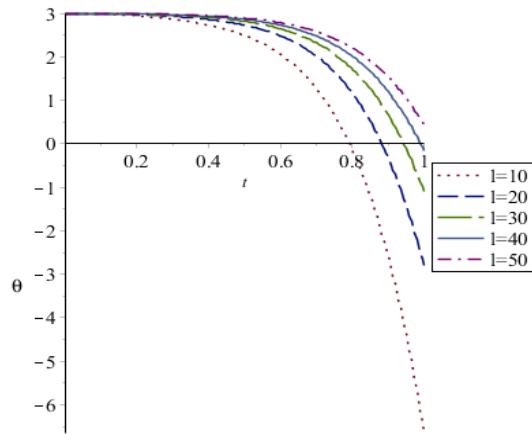
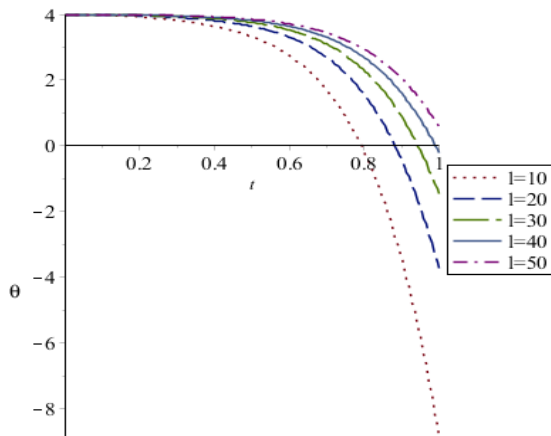


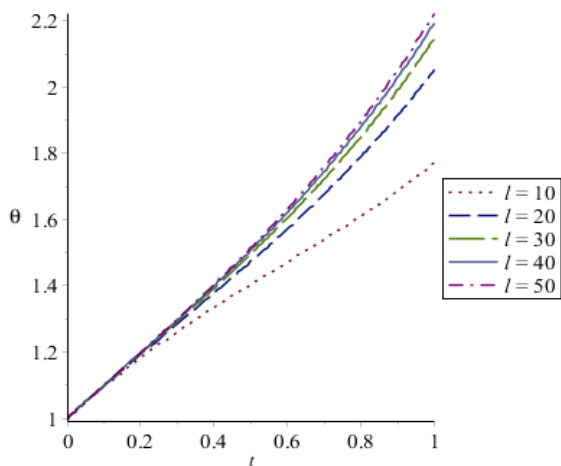
Figure 6. Angular displacement of pendulum at  $a_0 = 2$  and different lengths and time.



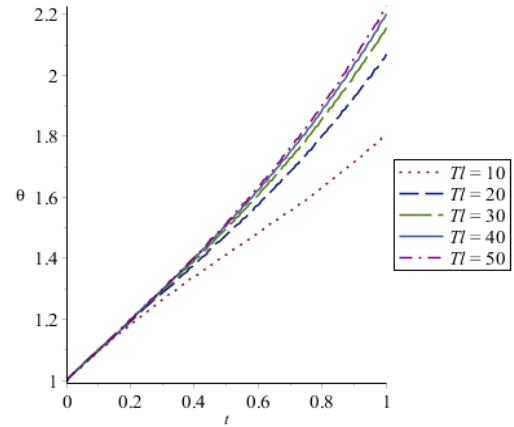
**Figure 7.** Angular displacement of pendulum at  $a_0 = 3$  and different lengths and time.



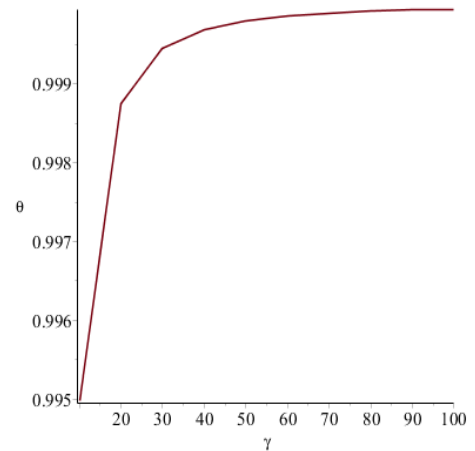
**Figure 8.** Angular displacement of pendulum at  $a_0 = 4$  and different lengths and time.



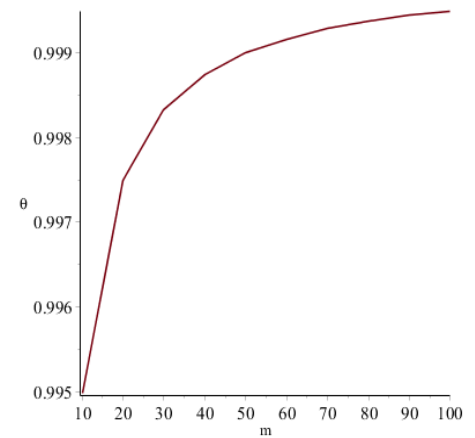
**Figure 9.** Angular displacement of pendulum with non-zero initial angular velocity and different values of initial angular displacement, time and lengths.



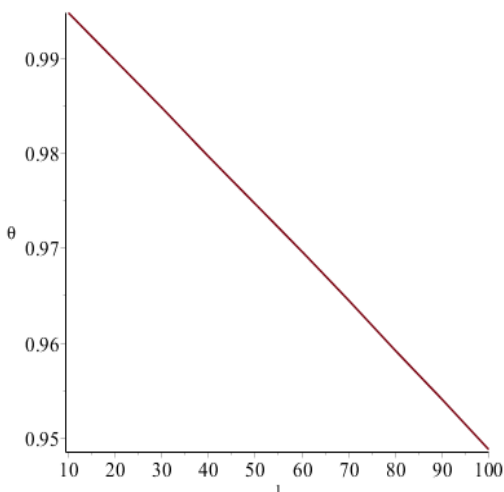
**Figure 10.** Angular displacement of pendulum with non-zero initial angular velocity and different values of initial angular displacement, time and period.



**Figure 11.** Effect of the damping factor on angular displacement of the pendulum.



**Figure 12.** Effect of the mass on angular displacement of the pendulum.



**Figure 13.** Effect of the length of pendulum on the angular displacement.

respectively. It can be seen from Figure 11 that the angular displacement increases sharply initially with an increase in the damping, then subsequently drops until becoming almost asymptotic to straight line  $\theta = 1$ , parallel to the horizontal axis. Similarly, in Figure 12 the angular displacement initially increases sharply as the mass of the pendulum bob increases, then gradually drops until becoming almost asymptotic to the straight line  $\theta = 1$ , parallel to the horizontal axis. From the two figures it implies that both the damping factor and mass of the pendulum bob have impact on the angular displacement of the pendulum. As we can see, as these parameters increase the angular displacement increases. Figure 13 shows clearly that apart from the fact that an increase in pendulum length will increase. The relationship between the angular displacement and length of the pendulum can be considered to be directly proportional and positively correlated.

## CONCLUSION

The angular displacement of a damped pendulum with small displacement is analysed on the basis of Hermite's form of governing equation of a damped simple pendulum. The general solution of the equation of motion governing damped simple pendulum, put in Hermite equation form, was obtained by means of power series. Analysis reveals that for different values of the initial angular displacement, we get different values of the subsequent angular displacement. There is a direct correlation. Also, it is revealed that the length of the pendulum affects the angular displacement; the length of

the pendulum is directly proportional to the angular displacement of the pendulum. We notice that the effect of the period of the pendulum is similar to that of the length of the pendulum. This is because the period of the pendulum is a function of the length of the pendulum. Both the damping and the mass of the bob also have effect on the angular displacement of the pendulum in almost the same manner.

## Conflict of Interest

The authors have not declared any conflict of interest.

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