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COMMON FIXED POINT THEOREMS FOR PAIRS OF HYBRID MAPS IN G-SYMMETRIC SPACES

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Abstract: The existence of the common fixed points for some pairs of hybrid maps(i.e. singled-valued and set-valued maps) satisfying some generalized contractive conditions in G-symmetric spaces are proved. The results extend and improve some results in literature.

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1. Introduction and Preliminary Definitions

Mustafa and Sims in [1] generalized the notion of a metric space to the notion of a G-metric space in the sense that each triplet of an arbitrary set is assigned a real number. Many authors also generalized the notion of metric space to different abstract spaces. Eke and Olaleru (see [2]) generalized the notion of a G-metric space to a G- partial metric space. In the same reference, they

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studied the existence of fixed points for contraction mappings in ordered G-partial metric spaces. Also, Olaleru et al in [3] recently proved the existence of fixed points for generalized Ciric-type contractive mappings in ordered G-partial metric spaces. Cartan [4] generalized the notion of metric space by omitting the triangle inequality axiom of the metric spaces to obtain what he termed symmetric spaces. Several fixed point theorems have been proved in this space (see [5], [6], [7], [8]). Inspired by this, Eke and Olaleru (see [9]) recently introduced the notion of G-symmetric spaces by omitting the rectangle inequality axiom of G-metric spaces. In this work, we prove some fixed point theorems for some multivalued maps in G-symmetric spaces.

The study of the fixed points for multivalued contraction mapping in metric space was introduced by Nadler in [16]. Thereafter many authors studied the fixed point for multivalued contractive mappings in different abstract spaces (see [1], [11], [12], [13]). The purpose of this work is to obtain some fixed point theorems involving hybrid pairs of single-valued and multi-valued mappings satisfying certain contractive conditions in the setting of G-symmetric spaces.

The following definitions and motivations will be needed in the sequel.

Definition 1.1. (see [1]) Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x)$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a generalised metric, or more specifically a G-metric on X, and the pair (X,G) is a G-metric space.

Definition 1.2. (see [4]) A symmetric on a set X is a real valued function d on $X \times X$ such that:

(i) $d(x, y) \geq 0$ and $d(x, x) = 0$ if and only if $x = y$; and

(ii) $d(x, y) = d(y, x)$.

Wilson in [14] also gave two more axioms of a symmetric d on X as:

(W_1) Given $\{x_n\}$, x and y in X, $d(x_n, x) \to 0$ and $d(x_n, y) \to 0$ imply that $x = y$;

(W_2) Given $\{x_n\}$, $\{y_n\}$ and $x \in X$, $d(x_n, x) \to 0$ and $d(x_n, y_n) \to 0$ imply that $d(y_n, x) \to 0$.

Definition 1.3. (see [15]) A mapping $T : X \to 2^X$ is called a multivalued mapping. A point $x \in X$ is called a fixed point of T if $x \in Tx$.

Definition 1.4. (see [15]) Let X be a given nonempty set. Assume that $g : X \to X$ and $T : X \to 2^X$. If $w = gx \in Tx$ for some $x \in X$, then x is called a coincidence point of g and T and w is a point of coincidence of g and T.

Definition 1.5. (see [15]) Maps $g : X \to X$ and $T : X \to 2^X$ are said to be weakly compatible if $gx \in Tx$ for each $x \in X$ implies $gTx \subseteq Tgx$.

Definition 1.6. (see [16]) Maps $g : X \to X$ and $T : X \to 2^X$ are said to be occasionally weakly compatible mappings if and only if there exists some point x in X such that $gx \in Tx$ and $gTx \subseteq Tgx$.

An occasionally weakly compatible map is weakly compatible but not vice-versa.

Proposition 1.7. (see [15]) Let X be a given nonempty set. Assume that $g : X \to X$ and $T : X \to 2^X$ are weakly compatible mappings. If g and T have a unique point of coincidence $w = gx \in Tx$, then w is the unique common fixed point of g and T.

Definition 1.8. (see [15]) Let $g : X \to X$ and $T : X \to 2^X$. The pair (g, T) satisfies property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} gx_n = t \in A = \lim_{n\to\infty} Tx_n$ for some $t \in A$ and $A \in 2^X$.

We give the following definitions and results.

Definition 1.9. A G-symmetric on a set X is a function $G_d : X \times X \times X \to R^+$ such that for all $x, y, z \in X$ the following conditions are satisfied:

$G_d(1)$ $G_d(x, y, z) \geq 0$ and $G_d(x, y, z) = 0$, if $x = y = z$;

$G_d(2)$ $0 < G_d(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

$G_d(3)$ $G_d(x, x, y) \leq G_d(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

$G_d(4)$ $G_d(x, y, z) = G_d(y, z, x) = G_d(z, x, y) =$,..., (symmetry in all three variables).

Example 1.10. Let $X = [0, 1]$ equipped with a G-symmetric defined by

$$G_d(x, y, z) = (x - y)^2 + (y - z)^2 + (z - x)^2,$$

for all $x, y, z \in X$. Then (X, G_d) is a G- symmetric space. This does not satisfied the rectangle inequality axiom of G-metric space hence it is not a G-metric space.

The analogue of axioms of Wilson [14] in G-symmetric space is as follows:

(W_3) Given $\{x_n\}$, x and y in X, $G_d(x_n, x, x) \to 0$ and $G_d(x_n, y, y) \to 0$ imply that $x = y$.

(W_4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $G_d(x_n, x, x) \to 0$ and $G_d(x_n, y_n, y_n) \to 0$ imply that $G_d(y_n, x, x) \to 0$.

Definition 1.11. Let (X, G_d) be a G-symmetric space. A G-symmetric space satisfies property (H.E) if given $\{x_n\}$, $\{y_n\}$ and $x \in X$, $G_d(x_n, x, x) \to 0$ and $G_d(y_n, x, x) \to 0$ imply that $G_d(x_n, y_n, y_n) \to 0$.

Definition 1.12. Let (X, G_d) be a G-symmetric space.

(i) (X, G_d) is G_d- complete if for every G_d- Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n\to\infty} G_d(x_n, x, x) = 0$.

(ii)$f : X \to X$ is G_d-continuous if

$$\lim_{n\to\infty} G_d(x_n, x, x) = 0$$

implies

$$\lim_{n\to\infty} G_d(fx_n, fx, fx) = 0.$$

Let (X, G_d) be a G-symmetric space, $x, y \in X$ and $A \subseteq X$, $G_d(x, y, A) = inf\{G_d(x, y, a) : a \in X\}$. CB(X) is the class of all nonempty closed and bounded subsets of X and B(X) is defined as the class of all nonempty bounded subsets of X. The diameter of $A, B, C \in CB(X)$ is denoted and defined by

$$\delta(A, B, C) = \sup\{G_d(a, b, c) : a \in A, b \in B, c \in C\}.$$

Two distinct points in a set are said to be Hausdorff if there exist two disjoint open sets such that each element is in each open set. Hausdorff assures the uniqueness of points in a multivalued set. We denote the Hausdorff G_d - distance on CB(X) by H(.,.,.), where

$$H_{G_d}(A,B,C) = \max\{\sup_{x\in A} G_d(x,C,B), \sup_{x\in B} G_d(x,A,C), \sup_{x\in C} G_d(x,A,B)\}.$$

For singlevalued sets we denote and define

$$G_d(A,B,C) = inf\{G_d(a,b,c) : a \in A, b \in B, c \in C\}.$$

Here we state the following definitions given by Zhang [17].

Assume that $F : [0,\infty) \to R$ satisfies the following:

(i) $F(0) = 0$ and $F(t) > 0$ for each $t \in (0,\infty)$ and

(ii) F is nondecreasing on $[0,\infty)$.

Define $F[0,\infty) = \{F : F$ satisfies $(i) - (ii)\}$. Let $\psi : [0,\infty) \to R$ satisfy the following

(iii) $\psi(t) < t$ for each $t \in (0,\infty)$ and

(iv) ψ is nondecreasing $[0,\infty)$.

Define $\Psi[0,\infty) = \{\psi : \psi$ satisfies $(iii) - (iv)$ above$\}$.

Tahat et al [15] proved some common fixed point theorems for pairs of hybrid mappings using weakly compatible mappings satisfying a generalized contractive conditon defined on G-metric spaces. Aliouche [18] proved the common fixed point for two pairs of hybrid mappings using the concept of T-weakly and S-weakly commuting mappings satisfying generalized contractive conditions in symmetric spaces. Abbas and Rhoades [16] proved some common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings satisfying a generalized contractive conditions of integral type in symmetric spaces. Abbas and Khan [19] further proved several common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings having a function coefficient in symmetric spaces. In this work, we prove some common fixed point theorems for pairs of hybrid mappings in G-symmetric spaces. Our results are analogue of the result of Abbas and Khan [19] for G-symmetric spaces and generalizations of other similar results in literature.

2. Main Results

Theorem 2.1. *Let (X, G_d) be a G-symmetric space satisfying (H.E). Let $g : X \to X$ and $T : X \to CB(X)$. Assume that there exists a function*

$\alpha : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r \to t^+} \alpha(r) < 1$ for every $t \geq 0$ and $\alpha(t) < t$ for each $t > 0$ such that

$$H_{G_d}(Tx, Ty, Ty) \leq \alpha(G_d(gx, gy, gy))G_d(gx, gy, gy), \tag{1}$$

for all $x, y \in X$. If $T(X) \subseteq g(X)$ and $g(X)$ is a G_d - complete subspace of X and $\{g, T\}$ satisfies the property (E.A), then g and T have a point of coincidence in X. Furthermore, if we assume that $gu \in Tu$ and $gv \in Tv$ implies that

$$G_d(gu, gv, gv) \leq H_{G_d}(Tu, Tv, Tv).$$

Then:

(i) g and T have a unique point of coincidence.

(ii) If in addition g and T are weakly compatible then g and T have a unique common fixed point.

Proof. Assume that the pair $\{g, T\}$ satisfies property (E.A). Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} G_d(gx_n, z, z) = 0$ and

$$\lim_{n\to\infty} H_{G_d}(Tx_n, A, A) = 0$$

for some $z \in A \subseteq X \in CB(X)$. By (H.E) we have that

$$\lim_{n\to\infty} G_d(gx_n, Tx_n, Tx_n) = 0.$$

Suppose that $g(X)$ is a complete subspace of X, then there exists $z = gu$ for some $u \in X$. We claim that $gu \in Tu$. If not, using (1), we have

$$H_{G_d}(Tx_n, Tu, Tu) \leq \alpha(G_d(gx_n, gu, gu))G_d(gx_n, gu, gu).$$

Letting $n \to \infty$ in the above inequality yields $H_{G_d}(gu, Tu, Tu) \leq 0$ but $H_{G_d}(gu, Tu, Tu) \geq 0$, hence $gu \in Tu$. This implies that T and g have a point of coincidence which is z and u is the coincidence point.

Next we prove the uniqueness of the coincidence point of g and T. Using (1) and this assumption which says that if $gu \in Tu$ and $gv \in Tv$, then

$$G_d(gu, gv, gv) \leq H_{G_d}(Tu, Tv, Tv),$$

we have

$$G_d(gu, gv, gv) \leq H_{G_d}(Tu, Tv, Tv) \leq \alpha(G_d(gu, gv, gv))G_d(gu, gv, gv). \tag{2}$$

Since $\alpha(G_d(gu, gv, gv)) < G_d(gu, gv, gv)$, then (2) becomes $G_d(gu, gv, gv) = 0$. i.e $gu = gv$.

Thus by (1):

$$H_{G_d}(Tu, Tv, Tv) \leq \alpha(G_d(gu, gv, gv))G_d(gu, gv, gv) = 0,$$

and $Tu = Tv$. Thus g and T have a unique point of coincidence. Since g and T are weakly compatible then by proposition 1.7, g and T have a unique common fixed point.

Corollary 2.2. *Let (X, G_d) be a G-symmetric space satisfying (H.E). Let $g : X \to X$ and $T : X \to X$. Asssume that there exists a function $\alpha : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r \to t^+} \alpha(r) < 1$ for every $t \geq 0$ such that:*

$$G_d(Tx, Ty, Ty) \leq \alpha(G_d(gx, gy, gy))G_d(gx, gy, gy), \tag{3}$$

for all $x, y \in X$. If $T(X) \subseteq g(X)$ and $g(X)$ is a G_d-complete subspace of X, then g and T have a point of coincidence. Furthermore, if g and T are weakly compatible then g and T have a unique common fixed point.

If T is changed from a set-valued map to a single-valued map, then Theorem 2.1 gives the following.

Corollary 2.3. *Let (X, G_d) be a complete G-symmetric space. Assume $T : X \to CB(X)$ satisfies the following condition*

$$H_{G_d}(Tx, Ty, Ty) \leq \alpha(G_d(x, y, y))G_d(x, y, y), \tag{4}$$

for all $x, y \in X$, where $\alpha : [0, \infty) \to [0, 1)$ satisfies $\limsup_{r \to t^+} \alpha(r) < 1$ for every $t \geq 0$. Then T has a fixed point in X.

Further, if we assume that $p \in Tp$ and $q \in Tq$ implies

$$G_d(q, p, p) \leq H_{G_d}(Tq, Tp, Tp),$$

then T has a unique fixed point.

Proof. It follows by taking g as the identity on X in Theorem 2.1.

Remark 2.4. Corollary 2.3 is an analogue of ([14], Theorem 2.1) in the context of a G - symmetric space.

Theorem 2.5. *Let f, g be self maps of a G-symmetric space X, and let T, S be maps from X into $B(X)$ such that the pairs $\{f,T\}$ and $\{g,S\}$ are occasionally weakly compatible. If*

$$F(\delta(Tx,Sy,Sy)) \leq \psi(F(M(x,y,y))) \tag{5}$$

and

$$F(\delta(Tx,Tx,Sy)) \leq \psi(F(M(x,x,y))) \tag{6}$$

for each $x,y \in X$ for which $fx \neq gy$, where

$$M(x,y,y) = \max\{G_d(fx,gy,gy), G_d(fx,Tx,Tx), G_d(gy,Sy,Sy), \delta(fx,Sy,Sy), \delta(gy,Tx,Tx)\} \tag{7}$$

and

$$M(x,x,y) = \max\{G_d(fx,fx,gy), G_d(fx,fx,Tx), G_d(gy,gy,Sy), \delta(fx,fx,Sy), \delta(gy,gy,Tx)\} \tag{8}$$

then f, g, T and S have a unique common fixed point.

Proof. Since $\{f,T\}$ and $\{g,S\}$ are occasionally weakly compatible, then there exist points $x,y \in X$ such that $fx \in Tx$, $gy \in Sy$ implies $fTx \subseteq Tfx$ and $gSy \subseteq Sgy$.

Clearly $G_d(f^2x,g^2y,g^2y) \leq \delta(Tfx,Sgy,Sgy)$, inview of the fact that $f^2x = Tfx$ and $g^2y = Sgy$, using (7) we obtain

$$\begin{aligned} M(fx,gy,gy) =& \max\{G_d(f^2x,g^2y,g^2y), G_d(f^2x,Tfx,Tfx), \\ & G_d(g^2y,Sgy,Sgy), \delta(f^2x,Sgy,Sgy), \delta(g^2y,Tfx,Tfx)\} \\ =& \max\{G_d(f^2x,g^2y,g^2y), \delta(f^2x,Sgy,Sgy), \delta(g^2y,f^2x,f^2x)\} \\ =& \max\{G_d(f^2x,g^2y,g^2y), \delta(Tfx,Sgy,Sgy), \delta(Sgy,Tfx,Tfx)\} \\ \leq& \max\{\delta(Tfx,Sgy,Sgy), \delta(Sgy,Tfx,Tfx)\}. \end{aligned}$$

Case (i). Suppose

$$\max\{\delta(Tfx,Sgy,Sgy), \delta(Sgy,Tfx,Tfx)\} = \delta(Tfx,Sgy,Sgy),$$

then we

$$F(\delta(Tfx,Sgy,Sgy)) \leq \psi(F(M(fx,gy,gy))) \leq \psi(F(\delta(Tfx,Sgy,Sgy)))$$

$$< F(\delta(Tfx, Sgy, Sgy)).$$

Case (ii). Suppose

$$\max\{\delta(Tfx, Sgy, Sgy), \delta(Sgy, Tfx, Tfx)\} = \delta(Sgy, Tfx, Tfx),$$

then substituting this inequality into (5) yields

$$F(\delta(Tfx, Sgy, Sgy)) \leq \psi(F(M(x, y, y))) \leq \psi(F(\delta(Sgy, Tfx, Tfx))) < F(\delta(Sgy, Tfx, Tfx)). \quad (9)$$

Similarly, using (8), we obtain

$$\begin{aligned} M(fx, fx, gy) &= \max\{G_d(f^2x, f^2x, g^2y), G_d(f^2x, f^2x, Tfx), \\ &\quad G_d(g^2y, g^2y, Sgy), \delta(f^2x, f^2x, Sgy), \delta(g^2y, g^2y, Tfx)\} \\ &= \max\{G_d(f^2x, f^2x, g^2y), \delta(f^2x, f^2x, Sgy), \delta(g^2y, g^2y, Tfx)\} \\ &= \max\{G_d(f^2x, f^2x, g^2y), \delta(Tfx, Tfx, Sgy), \delta(Sgy, Sgy, Tfx)\} \\ &\leq \max\{\delta(Tfx, Tfx, Sgy), \delta(Sgy, Sgy, Tfx)\}. \end{aligned}$$

Case (i). Suppose

$$\max\{\delta(Tfx, Tfx, Sgy), \delta(Sgy, Sgy, Tfx)\} = \delta(Tfx, Tfx, Sgy),$$

then we have

$$F(\delta(Tfx, Tfx, Sy)) \leq \psi(F(M(fx, fx, gy))) \leq \psi(F(\delta(Tfx, Tfx, Sgy))) < F(\delta(Tfx, Tfx, Sgy)).$$

Case (ii). If

$$\max\{G_d(f^2x, f^2x, g^2y), \delta(Tfx, Tfx, Sgy), \delta(Sgy, Sgy, Tfx)\} = \delta(Sgy, Sgy, Tfx),$$

then putting this inequality into (6) yields

$$F(\delta(Tfx, Tfx, Sgy)) \leq \psi(F(M(fx, fx, gy))) \leq \psi(F(\delta(Sgy, Sgy, Tfx))) < F(\delta(Sgy, Sgy, Tfx)).$$

By $G_d(4)$:

$$F(\delta(Tfx, Tfx, Sgy)) \leq F(\delta(Tfx, Sgy, Sgy)). \quad (10)$$

Combining (9) and (10) we obtain

$$F(\delta(Tfx, Sgy, Sgy)) < F(\delta(Tfx, Sgy, Sgy)).$$

This is a contradiction, hence $fx = gy$.

Next we show that $x = fx$. Since $fx \in Tx$ and $gy \in Sy$ then $G_d(fx, g^2y, g^2y) \leq \delta(Tx, Sfx, Sfx)$.

Using (7) and putting $x = x$ and $y = fx$, we obtain

$$\begin{aligned} M(x, fx, fx) = &\max\{G_d(fx, g^2y, g^2y), G_d(fx, Tx, Tx), \\ &G_d(g^2y, Sgy, Sgy), \delta(gy, Sgy, Sgy), \delta(g^2y, Tx, Tx)\} \\ \leq &\max\{\delta(Tx, Sfx, Sfx), \delta(Sfx, Tx, Tx)\}. \end{aligned}$$

Case (i). Suppose

$$\max\{\delta(Tx, Sfx, Sfx), \delta(Sfx, Tx, Tx)\} = \delta(Tx, Sfx, Sfx),$$

then we have

$$\begin{aligned} F(\delta(Tx, Sfx, Sfx)) \leq \psi(F(M(x, fx, fx))) &\leq \psi(F(\delta(Tx, Sfx, Sfx))) \\ &< F(\delta(Tx, Sfx, Sfx)). \end{aligned}$$

Case (ii). If

$$\max\{\delta(Tx, Sfx, Sfx), \delta(Sfx, Tx, Tx)\} = \delta(Sfx, Tx, Tx),$$

then using (5) and this inequality, we get

$$\begin{aligned} F(\delta(Tx, Sfx, Sfx)) \leq \psi(F(M(x, fx, fx))) &\leq \psi(F(\delta(Sfx, Tx, Tx))) \\ &< F(\delta(Sfx, Tx, Tx)). \quad (11) \end{aligned}$$

Similarly, using (8) we obtain

$$\begin{aligned} M(x, x, fx) = &\max\{G_d(fx, fx, g^2y), G_d(fx, fx, Tx), G_d(g^2y, g^2y, Sgy), \\ &\delta(gy, gy, Sgy), \delta(g^2y, g^2y, Tx)\} \\ \leq &\max\{\delta(Tx, Tx, Sfx), \delta(Sfx, Sfx, Tx)\}. \end{aligned}$$

Case (i). If

$$\max\{\delta(Tx, Tx, Sfx), \delta(Sfx, Sfx, Tx)\} = \delta(Tx, Tx, Sfx),$$

then we obtain

$$F(\delta(Tx,Tx,Sfx)) \leq \psi(F(M(x,x,fx))) \leq \psi(F(\delta(Tx,Tx,Sfx))) < F(\delta(Tx,Tx,Sfx)).$$

Case (ii). If

$$\max\{\delta(Tx,Tx,Sfx),\delta(Sfx,Sfx,Tx)\} = \delta(Sfx,Sfx,Tx),$$

then putting this inequality in (6) yields

$$F(\delta(Tx,Tx,Sfx)) \leq \psi(F(M(x,x,fx))) \leq \psi(F(\delta(Sfx,Sfx,Tx))) < F(\delta(Sfx,Sfx,Tx)). \quad (12)$$

Combining (11), (12) and by $G_d(4)$ gives

$$F(\delta(Tx,Sfx,Sfx)) < F(\delta(Tx,Sfx,Sfx)).$$

This is a contradiction hence $x = fx$.
Next we show that $y = gy$. Obviously $G_d(fgy,gy,gy) \leq \delta(Tgy,Sy,Sy)$.
Using (7) and letting $x = gy$ and $y = y$ in (7), we obtain

$$\begin{aligned} M(gy,y,y) =& \max\{G_d(fgy,gy,gy),G_d(fgy,Tgy,Tgy),\\ & G_d(gy,Sy,Sy),\delta(fgy,Sy,Sy),\delta(gy,Tgy,Tgy)\}\\ =& \max\{G_d(fgy,gy,gy),\delta(fgy,Sy,Sy),\delta(gy,Tgy,Tgy)\}\\ \leq& \max\{\delta(Tgy,Sy,Sy),\delta(Sy,Tgy,Tgy)\}. \end{aligned}$$

Case (i). If

$$\max\{\delta(Tgy,Sy,Sy),\delta(Sy,Tgy,Tgy)\} = \delta(Tgy,Sy,Sy),$$

then using (5) we obtain

$$F(\delta(Tgy,Sy,Sy)) \leq \psi(F(M(gy,y,y))) \leq \psi(F(\delta(Tgy,Sy,Sy))) < F(\delta(Tgy,Sy,Sy)).$$

Case (ii). If

$$\max\{\delta(Tgy,Sy,Sy),\delta(Sy,Tgy,Tgy)\} = \delta(Sy,Tgy,Tgy),$$

then using (5) with this inequality yields

$$F(\delta(Tgy, Sy, Sy)) \leq \psi(F(M(gy, y, y))) \leq \psi(F(\delta(Sy, Tgy, Tgy))) < F(\delta(Sy, Tgy, Tgy)). \quad (13)$$

Also using (8) we obtain

$$\begin{aligned} M(gy, gy, y) = &\max\{G_d(fgy, fgy, gy), G_d(fgy, fgy, Tgy), \\ &G_d(gy, gy, Sy), \delta(fgy, fgy, Sy), \delta(gy, gy, Tgy)\} \\ \leq &\max\{\delta(Tgy, Tgy, Sy), \delta(Sy, Sy, Tgy)\}. \end{aligned}$$

Case (i). Let

$$\max\{\delta(Tgy, Tgy, Sy), \delta(Sy, Sy, Tgy)\} = \delta(Tgy, Tgy, Sy),$$

then using (6) we get

$$F(\delta(Tgy, Tgy, Sy)) \leq \psi(F(M(gy, gy, y))) \leq \psi(F(\delta(Tgy, Tgy, Sy))) < F(\delta(Tgy, Tgy, Sy)).$$

Case (ii). Let

$$\max\{\delta(Tgy, Tgy, Sy), \delta(Sy, Sy, Tgy)\} = \delta(Sy, Sy, Tgy),$$

then using (6) with this inequality we have

$$F(\delta(Tgy, Tgy, Sy)) \leq \psi(F(M(gy, gy, y))) \leq \psi(F(\delta(Sy, Sy, Tgy))) < F(\delta(Sy, Sy, Tgy)). \quad (14)$$

Combining (13), (14) and by $G_d(4)$ yields

$$F(\delta(Tgy, Sy, Sy)) < F(\delta(Tgy, Sy, Sy)).$$

This is a contradiction hence $y = gy$. Therefore $x = fx = gy = y$ implies that f, g, T and S have a common fixed point.

For uniqueness, let u and v be the different common fixed point of f, g, T, S. Suppose $G_d(fu, gv, gv) \leq \delta(Tu, Sv, Sv)$ Using (7) with $x = u$ and $y = v$, we have

$$\begin{aligned} M(u, v, v) = &\max\{G_d(fu, gv, gv), G_d(fu, Tu, Tu), \\ &G_d(gv, Sv, Sv), \delta(fu, Sv, Sv), \delta(gv, Tu, Tu)\} \\ = &\max\{G_d(fu, gv, gv), \delta(fu, Sv, Sv), \delta(gv, Tu, Tu)\} \\ \leq &\max\{\delta(Tu, Sv, Sv), \delta(Sv, Tu, Tu)\}. \end{aligned}$$

Case (i). If

$$\max\{\delta(Tu,Sv,Sv),\delta(Sv,Tu,Tu)\}=\delta(Tu,Sv,Sv),$$

then using (5) we obtain

$$F(\delta(Tu,Sv,Sv))\le\psi(F(M(u,v,v)))\le\psi(F(\delta(Tu,Sv,Sv)))<F(\delta(Tu,Sv,Sv)).$$

Case (ii). If

$$\max\{\delta(Tu,Sv,Sv),\delta(Sv,Tu,Tu)\}=\delta(Sv,Tu,Tu),$$

then using (5) with this inequality yields

$$F(\delta(Tu,Sv,Sv))\le\psi(F(M(u,v,v)))\le\psi(F(\delta(Sv,Tu,Tu)))<F(\delta(Sv,Tu,Tu)).\quad(15)$$

We also make use of (8) to have

$$\begin{aligned}M(u,u,v)=&\max\{G_d(fu,fu,gv),G_d(fu,fu,Tu),\\&G_d(gv,gv,Sv),\delta(fu,fu,Sv),\delta(gv,gv,Tu)\}\\\le&\max\{\delta(Tu,Tu,Sv),\delta(Sv,Sv,Tu)\}.\end{aligned}$$

Case (i). If

$$\max\{\delta(Tu,Tu,Sv),\delta(Sv,Sv,Tu)\}=\delta(Tu,Tu,Sv),$$

then using (6) yields

$$F(\delta(Tu,Tu,Sv))\le\psi(F(M(u,u,v)))\le\psi(F(\delta(Tu,Tu,Sv)))<F(\delta(Tu,Tu,Sv)).$$

Case (ii). If

$$\max\{\delta(Tu,Tu,Sv),\delta(Sv,Sv,Tu)\}=\delta(Sv,Sv,Tu),$$

then using (6) with this inequality yields

$$F(\delta(Tu,Tu,Sv))\le\psi(F(M(u,u,v)))\le\psi(F(\delta(Sv,Sv,Tu)))<F(\delta(Sv,Sv,Tu)).\quad(16)$$

Combining (15), (16) and by $G_d(4)$ gives

$$F(\delta(Tu, Sv, Sv)) < (F(\delta(Tu, Sv, Sv)).$$

This is a contradiction, hence $u = v$.

Corollary 2.6. *Let f, g be self maps of G-symmetric space X, and let T, S be maps from X into $B(X)$ such that the pairs $\{f, T\}$ and $\{g, S\}$ are occasionally weakly compatible. If*

$$F(\delta(Tx, Sy, Sy)) \leq \psi(F(M(x, y, y))) \tag{17}$$

and

$$F(\delta(Tx, Tx, Sy)) \leq \psi(F(M(x, x, y))) \tag{18}$$

for each $x, y \in X$ for which $fx \neq gy$, where

$$M(x, y, y) = h \ \max\{G_d(fx, gy, gy), G_d(fx, Tx, Tx), G_d(gy, Sy, Sy), \frac{G_d(fx, Sy, Sy) + G_d(gy, Tx, Tx)}{2}\}, \tag{19}$$

and

$$M(x, x, y) = h \ \max\{G_d(fx, fx, gy), G_d(fx, fx, Tx), G_d(gy, gy, Sy), \frac{G_d(fx, fx, Sy) + G_d(gy, gy, Tx)}{2}\}, \tag{20}$$

and $0 \leq h < 1$, then f,g, T and S have a unique common fixed point.

Since Corollary 2.6 is a special case of Theorem 2.5 then the proof of corollary 2.6 follows from Theorem 2.5.

Remarks 2.7. Theorem 2.5 is an extension of [18], Theorem 2.1 and [16], Theorem 2.1 for symmetric spaces. Also it is an improvement of [19], Theorem 2.1 to G-symmetric spaces. Since the authors proved their results in the context of symmetric spaces.

Example 2.8. Let $X = [1, \infty)$ and G_d be a G-symmetric on X defined by $G_d(x, y, z) = \max\{(x - y)^2, (y - z)^2, (z - x)^2\}$ for all $x, y, z \in X$. Define $g : X \to X$ and $T : X \to CB(X)$ by $Tx = [1, x + 1]$ and $gx = x^2$. Also define $\alpha : [0, \infty) \to [0, 1)$ by $\alpha(t) = \frac{1}{4}$. Then:

(a) $T(X) \subseteq g(X)$

(b) g and T are weakly compatible

(c) $g(X)$ is a G_d - complete subspace of X

(d) $H_{G_d}(Tx, Ty, Ty) \leq \alpha(G_d(gx, gy, gy))G_d(gx, gy, gy)$

Taking $x_n = 1 + \frac{1}{n}$, the pair (g, T) satisfies the property (E.A) with $t = 1$, $A = [1, 2]$. Also (g, T) satisfies (H.E). All the hypothese of Theorem 2.1 are satisfied with 1 the common fixed point of g and T.

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