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Approximate Solution of Multipoint Boundary Value Problems

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Abstract: This study applies the Differential Transform Method (DTM) to obtain the approximate solution of multipoint boundary value problems. Two examples are solved to illustrate the efficiency of the method. Comparison with the solution obtained by Adomian Decomposition Method revealed that the DTM is an excellent method for this type of problems.

Key words: Multipoint boundary value problems, Differential Transform Method, Adomian Decomposition Method, series solution, efficiency

INTRODUCTION

Boundary value problems occur in many areas such as engineering, technology control, optimization, etc. Specifically, in engineering it occurs in optimal control, beam deflections, fluid dynamics, hydrodynamics and hydro magnetics stability. Numerical analysis of boundary value problems has been studied by researchers like Wazwaz (2001), He (2003) and Adesanya *et al*. (2013).

Multipoint boundary value problems are boundary value problems with greater boundary conditions than the order of the differential equation. This type of differential equation has received greater attention in the recent years owing to its wide applications. For example, the vibrations in a wire of uniform cross sections and composed of materials with different densities, theory of elasticity and fluid flow through porous plate. Much research has been carried out in the area of the theoretical analysis of second order (Liang *et al*., 2008; Li and Shen, 2006) and third order multipoint boundary value problems (Sun, 2005; Shi *et al*., 2014; Yang, 2008). Few numerical analysis of multipoint boundary value problems are available in literature these include: Tatari and Dehghan (2006), Das *et al*. (2010), Liu and Wv (2002) and Akram *et al*. (2013). Differential Transform Method (DTM) is applied in this work and the results are compared with numerical solution by Adomian Decomposition Method (ADM). The DTM was first introduced by Zhou. This method has been adopted by numerous researchers like Edeki *et al*. (2014, 2015) to various physical situations because of its accuracy, simplicity and rapid convergence to the exact solution.

MATERIALS AND METHODS

Analysis of Differential Transform and Adomian Decomposition Methods

Fundamentals of Differential Transform Method: Let the arbitrary function y = u(t) be expressed in Taylor series about a point $t = t_0$ as:

$$u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[\frac{d^k u}{dt^k} \right]_{t=t_0} \tag{1}$$

with the differential transformation of U(k) given as:

$$U(k) = \frac{1}{k!} \left[\frac{d^k u}{dt^k} \right]_{t=t_0} \tag{2}$$

We then obtain the inverse differential transform of U(k) as:

$$u(t) = \sum_{k=0}^{\infty} t^k Y(k) \tag{3}$$

The following theorems can be obtained from Eq. 1-3 (Table 1):

(i) If $u(t) = v_1(t) \pm v_2(t)$ then, $U(k) = V_1(k) \pm V_2(k)$
(ii) If $u(t) = \alpha v_1(t)$ then, $U(k) = \alpha V_1(k)$
(iii) If $u(t) = d^n v_1(t)/dx^n$ then, $U(k) = (k+n)!/k!\ V_1(k+n)$
(vi) If $u(t) = v_1(t) v_2(t)$ then, $U(k) = \sum_{n=0}^{k} V_2(n) V_1(k-n)$
(v) If $u(t) = t^b$ then, $U(k) = \delta(k-b)$ where

$$\delta(k-b) = \begin{cases} 1, & \text{if } k = b \\ 0, & \text{if } k \neq b \end{cases}$$

(vi) If $u(t) = v_1\, dv_2(t)/dt$ then, $U(k) = \sum_{n=0}^{k} (k-n+1) V_1(n) V_2(k-n+1)$

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Table 1: The basic rules for transforming the boundary conditions

Boundary condition	Transformed boundary condition	Boundary condition	Transformed boundary condition
$y(0) = 0$	$Y(0) = 0$	$y(t) = 0$	$\sum_{n=0}^{k} T^k Y(k) = 0$
$y'(0) = 0$	$Y(1) = 0$	$y'(t) = 0$	$\sum_{n=0}^{k} kT^{k-1} Y(k) = 0$
$y''(0) = 0$	$y(2) = 0$	$y''(t) = 0$	$\sum_{n=0}^{k} k(k-1)T^{k-2} Y(k) = 0$

Fundamentals of Adomian Decomposition Method: Consider the differential equation:

$$Ly + Ry + Ny = g \tag{4}$$

Where:

L = Third order operator
R = Linear differential operator
Ny = The non-linear term
g = The source term

L is a fourth order operator and it is assumed to be invertible. Applying its inverse operator L^{-1} to both sides of Eq. 4 and using the boundary conditions, we obtain:

$$y(t) = y(0) + ty'(0) + \frac{t^2}{2!}y''(0) + \frac{t^3}{3!}y'''(0) + L^{-1}(g) - L^{-1}(Ry) - L^{-1}(Ny) \tag{5}$$

Then, y(t) can be written as:

$$y(t) = \sum_{n=0}^{\infty} y_n \tag{6}$$

The 0th component $y_0(t)$ include all the terms from the boundary conditions and the source term g, i.e.:

$$y_0(t) = y(0) + ty'(0) + \frac{At^2}{2!} + \frac{Bt^3}{3!} + L^{-1}(g) \tag{7}$$

with constants:

$$A = y''(1),\ \ B = y'''(1) \tag{8}$$

which will be determined at t = 1. The remaining terms are embedded in the recursive relation:

$$y_{n+1}(t) = L^{-1}(Ry_n) - L^{-1}(Ny_n),\ n \geq 0 \tag{9}$$

The non-linear term is determined by an infinite series of Adomian polynomials which can be calculated using:

$$A_n = \frac{1}{n!}\frac{d^n}{dt^n}\left[N\left(\sum_{n=0}^{\infty} ty_n\right)\right]_{t=0} \tag{10}$$

RESULTS AND DISCUSSION

Examples: The methods are applied to the two examples with multipoint boundary conditions.

Example 1: Consider the third order differential equation that shows the shear deformation of sandwich beams:

$$y'''(t) - k^2 y'(t) - a = 0 \tag{11}$$

with the boundary conditions:

$$y'(0) = y'(1) = y(0.5) = 0 \tag{12}$$

The constants k = 5 and a = 1. The theoretical solution of the equation is given as:

$$y(t) = \frac{a}{k^3}\left(\sinh\frac{k}{2} - \sinh kt\right) + \frac{a}{k^2}\left(t - \frac{1}{2}\right) + \frac{a}{k^3}\tanh\frac{k}{2}\left(\cosh kt - \cosh\frac{k}{2}\right) \tag{13}$$

Solution by Differential Transform Method: Transforming Eq. 11 with the boundary conditions Eq. 12, we obtain:

$$Y(k+3) = \frac{1}{(k+3)!}\left(25(k+1) - \delta(k)\right) \tag{14}$$

$$Y(0) = A,\ Y(1) = 0,\ Y(2) = \frac{B}{2!} \tag{15}$$

Substituting Eq. 15 in Eq. 14, yields the following:

$$Y(3) = -\frac{1}{6},\ Y(4) = \frac{25B}{24},\ Y(5) = -\frac{5}{24},\ Y(6) = \frac{125B}{144}, Y(7) = -\frac{125}{1008},$$
$$Y(8) = \frac{3125B}{8064},\ Y(9) = -\frac{3125}{72576},\ Y(10) = \frac{15625B}{145152},...$$

Collecting the terms together, we have:

$$y(t) = A+\frac{B}{2}t^2-\frac{1}{6}t^3+\frac{25B}{24}t^4-\frac{5}{24}t^5+\frac{125B}{144}t^6+... \quad (16)$$

We now impose the boundary conditions at t = 1 and t = 0.5 using Table 1 to obtain the following system of equations:

$$\begin{aligned}&-0.02840163002+A+0.2052914037B=0\\&-\frac{56048557}{19160064}+\frac{2624983B}{177408}=0\end{aligned} \quad (17)$$

Solving the system of equations above yields the constants A and B:

$$A = -0.01210708562,\ B = 0.1973228596 \quad (18)$$

Using Eq. 17 in Eq. 16 gives the series solution as:

$$\begin{aligned}y(t) = &-0.012107085625+0.09866142980t^2-\frac{1}{6}t^3+\\&0.2055446454t^4-\frac{5}{24}t^5+0.1712872045t^6-...\end{aligned} \quad (19)$$

Solution by Adomian Decomposition Method: Applying L^{-1} on Eq. 11 and imposing the boundary conditions Eq. 12 at t = 0, we obtain:

$$y(t) = y(0)+ty'(0)+\frac{t^2}{2}y''(0)-L^{-1}(1)+25L^{-1}(y') \quad (20)$$

the constants A = y(0) and B = y''(0) are to be determined later. The 0th component is identified as:

$$y_0(t) = A+\frac{Bt^2}{2}-L^{-1}(1) = A+\frac{Bt^2}{2}-\frac{t^3}{6} \quad (21)$$

and recursive relation is given as:

$$y_{n+1}(t) = 25L^{-1}(y') = 25L^{-1}(A_n) \quad (22)$$

at n = 0, 1, 2, ..., we obtain:

$$\begin{aligned}y_1 &= 25L^{-1}(y'_0) = \frac{25B}{24}t^4-\frac{5}{24}t^5\\y_2 &= 25L^{-1}(y'_1) = \frac{125B}{144}t^6-\frac{125}{1008}t^7...\end{aligned} \quad (23)$$

Collecting all the terms together, we have:

$$\begin{aligned}y(t) = &A+\frac{B}{2}t^2-\frac{t^3}{6}+\frac{25B}{24}t^4-\frac{5}{24}t^5+\\&\frac{125B}{144}t^6-\frac{125}{1008}t^7+\frac{3125B}{8064}t^8+...\end{aligned} \quad (24)$$

To evaluate the constants A and B, we apply the boundary conditions at t = 1 for y(t) and t = 0.5 for y''(t) which yield the system of Eq. 25:

$$\begin{aligned}&14.84064211556B+A = 2.92839794099\\&0.2052915793+A = 0.02840163584\end{aligned} \quad (25)$$

Solving the system of equations gives (Table 2 and Fig. 1):

$$A = -0.01210708565,\ B = 0.1973228596 \quad (26)$$

We obtain the series solution by substituting Eq. 26 in Eq. 24, giving:

$$\begin{aligned}y(t) = &-0.012107085655+0.09866142980t^2-\frac{1}{6}t^3+\\&0.2055446454t^4-\frac{5}{24}t^5+0.1712872045t^6-\frac{125}{1008}t^7+...\end{aligned} \quad (27)$$

Table 2: Exact and approximate values for example 1

t	Exact solution	DTM solution	ADM solution	DTM error	ADM error
0	-0.012107086	-0.012107086	-0.012107086	5.211E-12	3.52E-11
0.1	-0.011268507	-0.011268507	-0.011268507	5.368E-12	3.54E-11
0.2	-0.009222206	-0.009222206	-0.009222206	5.897E-12	3.59E-11
0.3	-0.006466868	-0.006466868	-0.006466868	6.995E-12	3.7E-11
0.4	-0.003320195	-0.003320195	-0.003320195	9.038E-12	3.9E-11
0.5	0	-1.26704E-11	-4.26704E-11	1.267E-11	4.27E-11
0.6	0.003320195	0.003320195	0.003320195	1.893E-11	4.89E-11
0.7	0.006466868	0.006466868	0.006466868	2.948E-11	5.95E-11
0.8	0.009222206	0.009222206	0.009222206	4.693E-11	7.69E-11
0.9	0.011268507	0.011268507	0.011268507	7.537E-11	1.05E-10
1	0.012107086	0.012107085	0.012107085	1.213E-10	1.51E-10

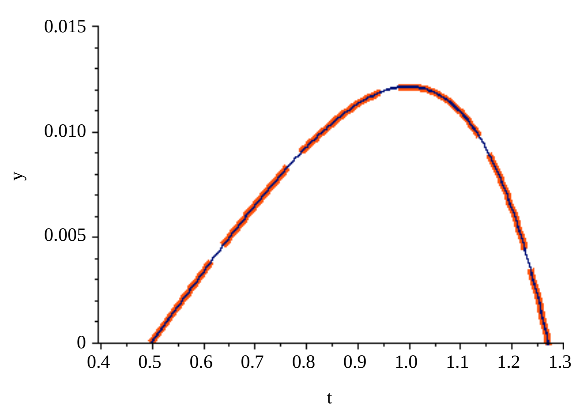

Fig. 1: Plot of DTM and ADM solution of example 1

Example 2: We now consider a four point non-linear differential equation:

$$y''''(t) = 4t^7+24-yy' \quad (28)$$

The boundary conditions are:

$$y(0) = 0,\ y(1) = 1,\ y''(0.5) = 3,\ y'''(0.25) = 6 \quad (29)$$

and the theoretical solution of Eq. 28 is given as:

$$y(t) = t^4 \quad (30)$$

Solution by Differential Transforms Method: We begin by transforming Eq. 28 and 29:

$$Y(k+4) = \frac{1}{(k+4)!}\left(-\sum_{r=0}^{k}(k-r+1)Y(r)Y(k-r+1) + 4\delta(k-7)+24\delta(k)\right) \quad (31)$$

$$Y(0) = 0,\ Y(1) = A,\ Y(2) = \frac{B}{2!},\ Y(3) = \frac{D}{3!} \quad (32)$$

Using the transformed boundary conditions Eq. 32 in the recurrent relation Eq. 31, we obtain:

$$Y(4) = 1,\ Y(5) = -\frac{A^2}{120},\ Y(6) = -\frac{AB}{240}$$
$$Y(7) = -\frac{AD}{1260}-\frac{B^2}{1680},\ Y(8) = -\frac{A}{336}-\frac{BD}{4032}$$
$$Y(9) = \frac{A^3}{60480}-\frac{B}{1008}-\frac{D^2}{36288},\ Y(10) = \frac{A^2B}{864400}-\frac{D}{4320}$$

Collecting the terms together, leads to:

$$y(t) = At + \frac{B}{2}t^2 + \frac{D}{6}t^3 + t^4 - \frac{A^2}{120}t^5 - \frac{AB}{240}t^6 + \left(-\frac{AD}{1260}-\frac{B^2}{1680}\right)t^7 + ... \quad (33)$$

To evaluate constants A, B and D, we refer to Table 1 by imposing the boundary conditions at t = 1, t = 0.5 and t = 0.25. This yield the following:

$$A = 0,\ B = 0,\ D = 0 \quad (34)$$

Substituting Eq. 34 in Eq. 33 yields the exact solution:

$$y(t) = t^4 \quad (35)$$

Solution by Adomian Decomposition Method: Applying L^{-1} on Eq. 28 and imposing the boundary conditions Eq. 29 at x = 0, we obtain:

$$y(t) = y(0) + ty'(0) + \frac{t^2}{2!}y''(0) + \frac{t^6}{3!}y'''(0) - L^{-1}(4t^7) + L^{-1}(24) - L^{-1}(yy') \quad (36)$$

The 0th component is written as:

$$y_0(t) = Bt + \frac{C}{2!}t^2 + D\frac{t^3}{3!} + \frac{t^{11}}{1980} + t^4 \quad (37)$$

And the recursive relation is of the form:

$$y_{n+1}(t) = 25L^{-1}(yy') = 25L^{-1}(A_n) \quad (38)$$

the Adomian polynomials for the non-linear part of the form yy' is written as follows:

$$A_0 = y_0y'_0$$
$$A_1 = y_0y'_1 + y_1y'_0 \quad (39)$$
$$A_2 = y_0y'_2 + y_1y'_1 + y_2y'_0$$
$$\ldots\ldots\ldots$$

Using Eq. 39 in Eq. 38, we obtain the following iterates:

$$y_1 = 25L^{-1}(y_0y'_0)$$
$$y_2 = 25L^{-1}(y_0y'_1 + y_1y'_0) \quad (40)$$
$$\ldots\ldots\ldots$$

In this example, only three terms are used because of the large computations involved. More iterates will enhance the convergence. To evaluate constants B, C and D, we apply the boundary conditions Eq. 29 at t = 1, t = 0.5 for y"(t) and t = 0.5 for y"(t) and solving the resulting system of equations, we obtain (Table 3 and Fig. 2):

Table 3: The exact and approximate values for example 2

t	Exact solution	ADM solution	DTM solution	ADM error	DTM error
0	0	0	0	0	0
0.1	0.0001	1E-04	0.0001	1.675E-12	0
0.2	0.0016	0.0016	0.0016	3.35E-12	0
0.3	0.0081	0.0081	0.0081	5.026E-12	0
0.4	0.0256	0.0256	0.0256	6.702E-12	0
0.5	0.0625	0.0625	0.0625	8.379E-12	0
0.6	0.1296	0.1296	0.1296	1.006E-11	0
0.7	0.2401	0.2401	0.2401	1.173E-11	0
0.8	0.4096	0.4096	0.4096	1.334E-11	0
0.9	0.6561	0.6561	0.6561	1.387E-11	0
1	1	1	1	0	0

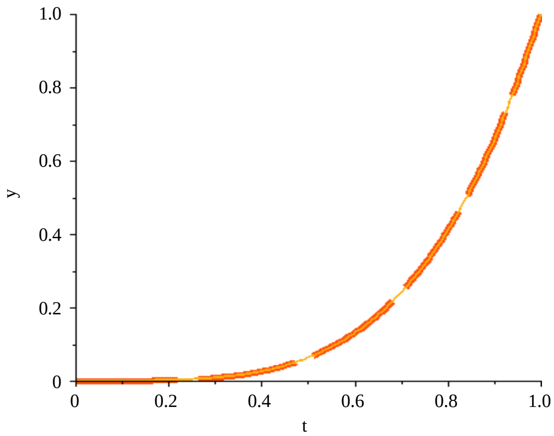

Fig. 2: Plot of DTM and ADM solution of example 2

$$A = -1.674841180\times10^{-11},\ B = -3.545612342\times10^{-14}\ \text{and}\ D = -1.636036248\times10^{-14} \quad (41)$$

The series solution is written:

$$y(t) = -1.674841180\times10^{-11}t-1.772806171\times10^{-14}t^2-2.726727080\times10^{-15}t^3+t^4-2.337577482\times10^{-24}t^5+... \quad (42)$$

CONCLUSION

In this research, Differential Transform Method has been applied to obtain the numerical solution of multipoint boundary value problems. Comparison was made with the results obtained by Adomian Decomposition Method. The two examples discussed the revealed the simplicity and effectiveness of DTM to handle various problems after the transformation of the differential equation, unlike ADM that requires the calculation of Adomian polynomials.

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