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Existence of Fixed Points of Some Classes of Nonlinear Mappings in Spaces with Weak Uniform Normal Structure

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Abstract

In this paper, we prove some fixed point results for some classes of nonlinear mappings recently introduced by Okeke and Olaleru [5]. Our results improves several other known results in literature, including the results of Sahu *et al.* [8] and Sahu [7].

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1 Introduction and Preliminaries

Let C be a nonempty subset of a Banach space X and $S : C \to C$ a Lipschitzian mapping, we use the symbol $\sigma(S)$ to denote the exact Lipschitz constant of S, i.e.,

$$\sigma(S) = \inf\{k \in [0, \infty] : \|Sx - Sy\| \leq k\|x - y\| \text{ for all } x, y \in C\}. \tag{1.1}$$

A mapping $T : C \to C$ is said to be
(a) *nonexpansive* if $\sigma(T) = 1$,
(b) *asymptotically nonexpansive* if $\sigma(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \sigma(T^n) = 1$,
(c) *uniformly L-Lipschitzian* if $\sigma(T^n) = L$ for all $n \in \mathbb{N}$ and for some $L \in (0, \infty)$.

Sahu [7] recently introduced the following classes of nonlinear mappings as intermediate classes between the class of asymptotically nonexpansive mappings and that of mappings of asymptotically nonexpansive type (see, Goebel and Kirk [3], Kirk [4]).

Definition 1.1 [7] Let C be a nonempty subset of a Banach space E and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. A mapping $T : C \to C$ will be called *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n) \ \ \forall \ \ x, y \in C. \tag{1.2}$$

The infimum of constants k_n for which (2.18) holds will be denoted by $\eta(T^n)$ and called *nearly Lipschitz constant.* Notice that

$$\eta(T^n) = \sup\left\{\frac{\|T^n x - T^n y\|}{\|x - y\| + a_n} : x, y \in C, x \neq y\right\}. \tag{1.3}$$

A nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be
(i) *nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,
(ii) *nearly nonexpansive* if $\eta(T^n) \leq 1$ for all $n \in \mathbb{N}$,
(iii) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) \leq 1$,
(iv) *nearly uniformly k-Lipschitzian* if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,
(v) *nearly uniformly k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Inspired by the facts above, Okeke and Olaleru [5] introduced the following classes of nonlinear mappings.

Definition 1.2 Let C be a nonempty subset of a Banach space E, $\phi : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ be a continuous strictly increasing function such that $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$ and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. A mapping $T : C \to C$ will be called *ϕ-nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n.\phi(\|x - y\| + a_n) \ \ \forall \ \ x, y \in C. \tag{1.4}$$

The infimum of constants k_n for which (1.6) holds will be denoted by $\eta(T^n)$ and called *ϕ-nearly Lipschitz constant.* Notice that

$$\eta(T^n) = \sup\left\{\frac{\|T^n x - T^n y\|}{\phi(\|x-y\| + a_n)} : x, y \in C, x \neq y\right\}. \tag{1.5}$$

A ϕ-nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be

(i) *ϕ-nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,

(ii) *ϕ-nearly nonexpansive* if $\eta(T^n) \leq 1$ for all $n \in \mathbb{N}$,

(iii) *ϕ-nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) \leq 1$,

(iv) *ϕ-nearly uniformly k-Lipschitzian* if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,

(v) *ϕ-nearly uniformly k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Observe that if ϕ is identity in Definition 1.2, then we obtain the concepts introduced by Sahu [7] (see Definition 1.1 above).

Our purpose in this paper is to prove some fixed point results for the classes of nonlinear mappings defined by Okeke and Olaleru [5], as given in Definition 1.2 above.

The following definitions and lemma will be needed in this study.

Definition 1.3 [7] Let C be a nonempty subset of a Banach space E and $T : C \to C$ a mapping. T is said to be *demicontinuous* if whenever a sequence $\{x_n\}$ in C converges strongly to $x \in C$, then $\{Tx_n\}$ converges weakly to Tx.

Definition 1.4 [2] The normal structure coefficient $N(E)$ of a Banach space E is defined by

$$N(E) = \inf\left\{\tfrac{diam(C)}{r_C(C)} : C \text{ is nonempty bounded convex subset of } E \text{ with } diam\ C > 0\right\},$$

where $r_C(C) = \inf_{x\in C}\{\sup_{y\in C}\|x - y\|\}$ is the *Chebyshev radius of* C relative to itself and *diam* $(C) = \sup_{x,y\in C}\|x - y\|$ is diameter of C. The space E is said to have the *uniform normal structure* if $N(E) > 1$. A weakly convergent sequence coefficient of E is defined by

$$WCS(E) = \sup\{k : k \limsup_{n\to\infty}\|x_n\| < diam_a(\{x_n\}) \text{ for all } \{x_n\} \text{ in } E \text{ with } x_n \rightharpoonup 0\}.$$

The space E is said to have the *weak uniform normal structure* if $WCS(E) > 1$.

Definition 1.5 [1] Let C be a nonempty subset of a Banach space E. A nonempty closed convex subset D of C is said to satisfy property (ω) with respect to a mapping $T : C \to C$ if

$$\omega_T(x) \subset D \quad \text{for every} \quad x \in D, \tag{1.6}$$

where $\omega_T(x)$ denotes the set of all weak subsequential limits of $\{T^n x : n \in \mathbb{N}\}$. Moreover, T is said to satisfy the (ω)-*fixed point property* if T has a fixed point in every nonempty closed convex subset D of C which satisfies property (ω).

Lemma 1.6 [**8**] Let C be a nonempty closed convex subset of a Banach space and $T : C \to C$ a mapping such that $T^n u \to v$ as $n \to \infty$ for some $u, v \in C$. Suppose that T is demicontinuous at v. Then v is a fixed point of T in C.

2 Main Results

Theorem 2.1 Let E be a Banach space with weak uniform normal structure, C a nonempty weakly compact convex subset of E and $T : C \to C$ a ϕ-nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\limsup_{n\to\infty} \eta(T^n) < \sqrt{WCS(E)}$. Also suppose that there exists a nonempty closed convex subset M of C which satisfies property (ω) with respect to T. Then

(a) for an arbitrary $x_0 \in M$, there exists an iterative sequence $\{x_m\}$ in M defined by

$$x_m = w - \lim_{n\to\infty} T^n x_{m-1} \quad \forall m \in \mathbb{N}, \tag{2.1}$$

(b) if T is asymptotically regular on C, then there exists an element $v \in M$ such that

$\{x_m\}$ converges strongly to $v \in M$. Further, if T is demicontinuous at v, then

$v \in F(T)$.

Proof. (a) We can easily construct a nonempty closed convex separable subset C_0 of C which is invariant under each T^n (i.e. $T^n(C_0) \subset C_0$ for $n = 1, 2, \cdots$), we suppose that C itself is separable.

Due to the separability of C_0, we can select a subsequence $\{T^n x\}$ such that $\{T^n x\}$ is weakly convergent for each $x \in C$. For every $x_0 \in M \subset C$, we consider a sequence $\{T^n x_0\}$ in C. Suppose that $w - \lim_{n\to\infty} T^n x_0 = x_1 \in C$. Using property (ω) we have that $x_1 \in M$. By induction, we can construct a sequence $\{x_m\}$ in M defined by (2.1).

(b) Suppose that T is asymptotically regular on C. The weak asymptotic regularity of T ensures that $x_m = w - \lim_{n\to\infty} T^{n+r} x_{m-1}$ for each $r \in \mathbb{N}$. We are to show that $\{x_m\}$ converges strongly to a fixed point T. We set $L := \limsup_{n\to\infty} \eta(T^n)$, $D_m := \limsup_{n\to\infty} \|x_m - T^n x_m\|$ and $R_m := \limsup_{n\to\infty} \|x_{m+1} - T^n x_m\|$ for all $m = 0, 1, 2, \cdots$ Using the property of $WCS(E)$, we obtain

$$R_m = \limsup_{n\to\infty} \|x_{m+1} - T^n x_m\| \leq \frac{1}{WCS(E)} D[\{T^n x_m\}]. \tag{2.2}$$

Using the asymptotic regularity of T and the w-$l.s.c.$ of the norm $\|.\|$, we obtain

$$\begin{aligned}
D[\{T^n x_m\}] &= \limsup_{n\to\infty} (\limsup_{r\to\infty} \|T^n x_m - T^r x_m\|) \\
&\le \limsup_{n\to\infty} (\limsup_{r\to\infty} (\|T^n x_m - T^{n+r} x_m\| \\
&\quad + \|T^{n+r} x_m - T^r x_m\|)) \\
&\le \limsup_{n\to\infty} (\limsup_{r\to\infty} (\eta(T^n).\phi(\|x_m - T^r x_m\| + a_n))) \\
&= L \limsup_{r\to\infty} (\phi(\|x_m - T^r x_m\|)) \\
&\le L \limsup_{r\to\infty} (\phi(\limsup_{s\to\infty} (\|T^s x_{m-1} - T^r x_m\|))) \\
&\le L \limsup_{r\to\infty} (\phi(\limsup_{s\to\infty} (\|T^s x_{m-1} - T^{r+s} x_{m-1}\| \\
&\quad + \|T^{r+s} x_{m-1} - T^r x_m\|))) \\
&\le L \limsup_{r\to\infty} (\phi(\limsup_{s\to\infty} (\|T^s x_{m-1} - T^{r+s} x_{m-1}\| \\
&\quad + \eta(T^r)(\|T^s x_{m-1} - x_m\| + a_r)))) \\
&\le L^2 \limsup_{s\to\infty} (\phi(\|T^s x_{m-1} - x_m\|)) = L^2 \times \phi(R_{m-1}).
\end{aligned} \tag{2.3}$$

We set $\lambda := \frac{L^2}{WCS(E)} < 1$. Using (2.2), we have

$$\phi(R_m) \le \lambda \times \phi(R_{m-1}) \le \lambda^2 \times \phi(R_{m-2}) \le \cdots \le \lambda^m \times \phi(R_0) \to 0 \tag{2.4}$$

as $m \to \infty$. For each $m \in \mathbb{N}$, we obtain

$$\begin{aligned}
\|x_{m+1} - x_m\| &\le \limsup_{n\to\infty} (\|x_{m+1} - T^n x_m\| + \|T^n x_m - x_m\|) \\
&\le R_m + \limsup_{n\to\infty} (\limsup_{r\to\infty} \|T^n x_m - T^r x_{m-1}\|) \\
&\le R_m + \limsup_{n\to\infty} (\limsup_{r\to\infty} \|T^n x_m - T^{n+r} x_{m-1}\| \\
&\quad + \|T^{n+r} x_{m-1} - T^r x_{m-1}\|)) \\
&\le R_m + \limsup_{n\to\infty} (\phi(\limsup_{r\to\infty} (\eta(T^n)\times \\
&\quad (\|x_m - T^r x_{m-1}\| + a_n)))) \\
&\le (\lambda + L).\phi(R_{m-1}) \\
&\quad \cdots \\
&\le (\lambda + L)\lambda^{m-1} \times \phi(R_0).
\end{aligned} \tag{2.5}$$

We see that $\{x_m\}$ is a Cauchy sequence in M and hence there exists an element $v \in M$ such that $\lim_{m\to\infty} x_m = v$. Clearly,

$$\begin{aligned}
\|v - T^n v\| &\le \|v - x_{m+1}\| + \|x_{m+1} - T^n x_m\| + \|T^n x_m - T^n v\| \\
&\le \|v - x_{m+1}\| + \|x_{m+1} - T^n x_m\| + \eta(T^n)\times \\
&\quad \phi(\|x_m - v\| + a_n).
\end{aligned} \tag{2.6}$$

Taking limit superior as $n \to \infty$ on both sides, we obtain

$$\limsup_{n\to\infty} \|v - T^n v\| \le \|v - x_{m+1}\| + \phi(R_m) + L\|x_m - v\| \to 0,$$

as $m \to \infty$. Hence, we have that $T^n v \to v$ as $n \to \infty$. Furthermore, we assume that T is demicontinuous at v. Therefore, using Lemma 1.6, we obtain $v \in F(T)$. □

Remark 2.2 The results of Theorem 2.1 improves and generalizes several other known results in literature, including the results of Sahu *et al.* [8] and Sahu [7].

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