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ON THE RESONANT OSCILLATION OF CERTAIN FOURTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract: We prove the existence of periodic solutions for the periodic boundary value problem (1.1) under some resonant conditions on the asymptotic behaviour of $\dfrac{g(t,y)}{ay}$ for $|y| \to \infty$.

The uniqueness of periodic solutions is also examined.

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1. Introduction

In this paper, we study the periodic boundary value problem

$$x^{(iv)}(t) + a \dddot{x}(t) + f(\dot{x})\ddot{x}(t) + g(t, \dot{x}(t-\tau)) + h(x) = p(t) \tag{1.1}$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3 \tag{1.2}$$

with fixed delay $\tau \in [0, 2\pi)$ where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, $p : [0, 2\pi] \longrightarrow \mathbb{R}$ and $g : [0, 2\pi] \times \mathbb{R} \longrightarrow \mathbb{R}$ are 2π-periodic in t and g satisfies caratheodory conditions. The unknown function $x : [0, 2\pi] \to \mathbb{R}$ is defined for $0 < t \leq \tau$ by $x(t-\tau) = [2\pi - (t-\tau)]$. We are specifically concerned with the existence of

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periodic solutions of (1.1) - (1.2) under some resonant conditions.

In a recent paper [4] we studied the above equation with $f(\dot{x}) = b$ and $h(x) = d$ where b and d are constants with $g(t, y)$ satisfying certain non-resonant conditions. In our present study, we will allow $g(t, y)$ to satisfy certain resonant conditions.

In what follows, we shall use the spaces $C([0, 2\pi])$, $C^k([0, 2\pi])$ and $L^k([0, 2\pi])$ of continuous, k times continuously differentiable or measurable real functions whose kth power of the absolute value is Lebesgue integrable. We shall use the Sobolev spaces $W^{4,2}_{2\pi}$ and $H^1_{2\pi}$ respectively defined by $W^{4,2}_{2\pi} = \{x : [0, 2\pi] \longrightarrow \mathbb{R} | x, \dot{x}, \ddot{x}, \dddot{x}$ are absolutely continuous on $[0, 2\pi]$, $x^{(i)}(0) = x^{(i)}(2\pi),\ i = 0, 1, 2, 3\}$ with the norm

$$|x|^2_{W^{4,2}_{2\pi}} = \sum_{i=0}^{4} \frac{1}{2\pi} \int_0^{2\pi} |x^{(i)}(t)|^2 dt$$

and

$$H^1_{2\pi} = \{x : [0, 2\pi] \longrightarrow \mathbb{R} | x$$

is absolutely continuous on $[0, 2\pi]$ and $\dot{x} \in L^2_{2\pi}\}$, with norm

$$|x|^2_{H^1_{2\pi}} = \left(\frac{1}{2\pi} \int_0^{2\pi} x(t) dt \right)^2 + \frac{1}{2\pi} \int_0^{2\pi} |\dot{x}|^2 dt$$

2. The Linear Problem

Let us consider the equation

$$x^{(iv)}(t) + a \dddot{x}(t) + b\ddot{x}(t) + c\dot{x}(t - \tau) + dx = 0 \tag{2.1}$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3 \tag{2.2}$$

where a, b, c and d are constants.

Lemma 2.1. *Let $a \neq 0$ and let*

$$(n - 1)^2 < \frac{c}{a} < n^2 \tag{2.3}$$

where $n \geq 1$ is a positive integer. Then the bvp (2.1) - (2.2) has no non trivial 2π-periodic solution.

Proof. We consider a solution of the form $x(t) = Ae^{\lambda t}$ where $\lambda = in$ with $i^2 = -1$ and $A \neq 0$ is a constant.

Then Lemma 2.1 will follow if

$$\phi(n,\tau) = -an^2 + c\cos n\tau \neq 0 \tag{2.4}$$

for all $n \geq 1$ and $\tau \in [0, 2\pi)$. By (2.3) we get

$$a^{-1}\phi(n,\tau) = -n^2 + \frac{c}{a}\cos n\tau \leq -n^2 + \frac{c}{a} < 0$$

Thus $\phi(n,\tau) \neq 0$ and the result follows.

If $x \in L^1_{2\pi}$ we shall write

$$\bar{x} = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt, \quad \tilde{x}(t) = x(t) - \bar{x} \tag{2.5}$$

so that

$$\int_0^{2\pi} \tilde{x}(t)dt = 0$$

We shall consider next the delay equation

$$x^{(iv)}(t) + a\dddot{x}(t) + b\dot{x}(t) + c(t)\dot{x}(t-\tau) + dx = 0 \tag{2.6}$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \ i = 0, 1, 2, 3 \tag{2.7}$$

where a, b, d are constants and $c \in L^2_{2\pi|}$.

Theorem 2.1. *Let $a \neq 0$ and $d \neq 0$ and set $\Gamma(t) = a^{-1}c(t)$. Suppose that*

$$0 < \Gamma(t) < 1 \tag{2.8}$$

then the bvp (2.6) - (2.7) has no nontrivial solution for every $\tau \in [0, 2\pi)$.

Proof. Let $x(t)$ be any solution of (2.6) - (2.7). Then we can verify that

$$\frac{1}{2\pi}\int_0^{2\pi} -\dot{\tilde{x}}(t)[a^{-1}(x^{(iv)} + b\ddot{x} + dx)]dt = 0$$

Thus we have

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_0^{2\pi} -\dot{\tilde{x}}(t)[a^{-1}\{x^{(iv)} + b\ddot{x}(t) + dx\} + \dddot{x} + \Gamma(t)\dot{x}(t-\tau)]dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} -\dot{\tilde{x}}(t)[\dddot{x} + \Gamma(t)\dot{x}(t-\tau)]dt \end{aligned}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\dddot{x}^2(t) - \Gamma(t)\ddot{x}(t)\ddot{x}(t-\tau)\right)dt$$

using

$$-ab = \frac{(a-b)^2}{2} - \frac{a^2}{2} - \frac{b^2}{2}$$

we have

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_0^{2\pi}(\dddot{x}^2(t) - \frac{\Gamma(t)}{2}\ddot{x}^2(t) - \frac{\Gamma(t)}{2}\ddot{x}^2(t-\tau))dt \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}\frac{\Gamma(t)}{2}(\ddot{x}(t) - \ddot{x}(t-\tau))^2 dt \end{aligned}$$

From (2.8) we have

$$\begin{aligned} 0 &\geq \frac{1}{2\pi}\int_0^{2\pi}(\dddot{x}^2(t) - \frac{\Gamma(t)}{2}(\ddot{x}^2(t) + \ddot{x}^2(t-\tau))dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\dddot{x}^2 - \Gamma(t)\ddot{x}^2(t)\right)dt \geq \delta|\dot{x}|^2_{H^1_{2\pi}} \end{aligned}$$

By Lemma 1 of [6], where we have used the fact that $\int_0^{2\pi}\ddot{x}^2(t)dt = \int_0^{2pi}\ddot{x}^2(t-\tau)dt$. Here, $\delta > 0$. Thus $\dot{x} = 0$ and since $d \neq 0$ it is clear that $x =$ constant cannot be a solution of (2.6) - (2.7). Therefore $x = 0$.

3. The Nonlinear Case

We shall now consider a preliminary Lemma which will enable us obtain a priori estimates required for our results.

Lemma 3.1. *Let all the conditions of Theorem 2.1 hold and let δ be related to $\Gamma(t)$ by Theorem 2.1. Suppose that for $V \in L^2_{2\pi}$, $0 \leq V(t) \leq \Gamma(t) + \varepsilon$ a.e. $t \in [0, 2\pi]$, $\varepsilon > 0$.*

Then

$$\frac{1}{2\pi}\int_0^{2\pi} -\ddot{x}[a^{-1}\{x^{(iv)} + b\ddot{x} + dx\} + \dddot{x} + v(t)\dot{x}(t-\tau)]dt \geq (\delta - \varepsilon)|\dddot{x}|^2_2.$$

Proof. From the proof of Lemma 2.1 we have

$$\frac{1}{2\pi}\int_0^{2\pi} -\dot{\ddot{x}}[a^{-1}\{x^{(iv)}+b\ddot{x}+dx\}+\dddot{x}+v(t)\dot{x}(t-\tau)]dt$$

$$\geq \frac{1}{2}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left[\ddot{\ddot{x}}^2-\Gamma(t)\dot{\ddot{x}}^2\right]dt\right\}$$

$$+\frac{1}{2}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left[\ddot{\ddot{x}}^2(t-\tau)-\Gamma(t)\dot{\ddot{x}}^2(t-\tau)\right]dt\right\}$$

$$-\frac{\varepsilon}{2}\cdot\frac{1}{2\pi}\int_0^{2\pi}(\dot{\ddot{x}}^2(t-\tau)+\dot{\ddot{x}}^2(t))dt$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left[\ddot{\ddot{x}}^2(t-\tau)-\Gamma(t)\dot{\ddot{x}}^2(t-\tau)\right]dt$$

$$-\frac{1}{2}\left[\frac{\varepsilon}{2\pi}\int_0^{2\pi}\dot{\ddot{x}}^2(t-\tau)dt+\frac{\varepsilon}{2\pi}\int_0^{2\pi}\dot{\ddot{x}}^2(t-\tau)\right]dt$$

$$\geq \delta|\ddot{\ddot{x}}|_2^2-\varepsilon|\ddot{\ddot{x}}|_2^2$$

$$=(\delta-\varepsilon)|\ddot{\ddot{x}}|_2^2$$

We shall next consider the nonlinear delay equation

$$x^{(iv)}(t)+a\dddot{x}(t)+f(\dot{x})\ddot{x}(t)+g(t,\dot{x}(t-\tau))+h(x)=p(t) \tag{3.1}$$

$$x^{(i)}(0)=x^{(i)}(2\pi),\quad i=0,1,2,3 \tag{3.2}$$

where $f,h:\mathbb{R}\to\mathbb{R}$ are continuous functions and $g:[0,2\pi]\times\mathbb{R}\to\mathbb{R}$ is such that $g(\cdot,x)$ is measurable on $[0,2\pi]$ for each $x\in\mathbb{R}$ and $g(t,\cdot)$ is continuous on $\mathbb{R}$ for almost each $t\in[0,2\pi]$. We assume moreover that for each $r>0$ there exists $Y_r\in L^2_{2\pi}$ such that $g(t,y)\leq Y_r(t)$ for a.e. $t\in[0,2\pi]$ and all $x\in[-r,r]$ such a g is said to satisfy Caratheodory's conditions.

Theorem 3.1. *Let $a\neq 0$. Suppose that g is a Caratheodory function with respect to the space $L^2_{2\pi}$ such that:*

(i) There exists $r>0$ such that

$$axg(t,x)\geq 0 \text{ for } |x|\geq r$$

(ii) $\limsup\limits_{|x|\to\infty} \dfrac{g(t,x)}{ax} \leq \Gamma(t)$
for $t \in [0, 2\pi]$ *with* $\Gamma(t)$ *satisfying* $0 < \Gamma(t) < 1$.

(iii) $\lim\limits_{|x|\to+\infty} sgnxh(x) = +\infty$

Suppose further that $p \in L^2_{2\pi}$, *then for arbitrary continuous function* f, *equation (3.1) has at least one* 2π*-periodic solution.*

Proof. Let $\delta > 0$ be related to Γ by Lemma 3.1. Then by (i) and (ii) there exists a constant $R > 0$ such that for a.e $t \in [0, 2\pi]$ and all y with $|y| \geq R$ we have

$$0 \leq \frac{g(t,y)}{ay} \leq \Gamma(t) + \delta/2 \tag{3.3}$$

We define

$$\tilde{Y}(t,x) = \begin{cases} (ax)^{-1}g(t,x), & |x| > R \\ (aR)^{-1}g(t,R), & 0 < x < R \\ -(aR)^{-1}g(t,-R), & -R_1 < x < 0 \\ \Gamma(t), & x = 0 \end{cases}$$

Then

$$0 \leq \tilde{Y}(t,x) \leq \Gamma(t) + \delta/2 \tag{3.5}$$

for a.e $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$.

Clearly the function

$$\tilde{g} = a\tilde{Y}(t, \dot{x}(t-\tau))\dot{x}(t-\tau)$$

is a Caratheorody function. So also is g_0 defined by

$$g_0(t, \dot{x}(t-\tau)) = g(t, \dot{x}(t-\tau)) - \tilde{g}(t, \dot{x}(t-\tau)) \tag{3.6}$$

Thus there exists $\alpha \in L^2_{2\pi}$ such that

$$|g_0(t, \dot{x}(t-\tau))| \leq \alpha(t) \tag{3.7}$$

for a.e $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$. Problem (3.1) is thus equivalent to

$$x^{(iv)}(t) + a\dddot{x}(t) + f(\dot{x})\ddot{x}(t) + \tilde{Y}(t, \dot{x}(t-\tau)\dot{x}(t-\tau) + g_0(t, \dot{x}(t-\tau)) + h(x) = p(t) \tag{3.8}$$

to which we shall apply coincidence degree theory.

Let $X = C^3[0, 2\pi]$, $Z = L^2_{2\pi}$
$\mathrm{dom}L = \{x \in X : x^{(i)}(0) = x^{(i)}(2\pi),\ \ i = 0, 1, 2, 3$ and $\dot{x}$, $\ddot{x}$, $\dddot{x}$ are absolutely continuous on $[0, 2\pi]\}$.

Define as in [6]

$$L : domL \subset X \longrightarrow Z,\ x \longrightarrow x^{(iv)} + a\ \dddot{x}$$

$$F : domL \subset X \longrightarrow Z,\ x \longrightarrow f(\dot{x})\ddot{x}$$

$$G : domL \subset X \longrightarrow Z,\ x \longrightarrow \tilde{Y}(t, \dot{x}(t-\tau))\dot{x}(t-\tau)$$

$$H : domL \subset X \longrightarrow Z,\ x \longrightarrow h(x)$$

$$A : domL \subset X \longrightarrow Z,\ x \longrightarrow \Gamma(t)\dot{x}(t-\tau)$$

$$G_0 : domL \subset X \longrightarrow Z,\ x \longrightarrow g_0(t, \dot{x}(t-\tau))$$

$$T : domL \subset X \longrightarrow Z,\ x \longrightarrow -p(t)$$

It is seen that G and G_0 are well defined and L-compact on bounded subsets of X and that L is a linear Fredholm mapping of index zero.

Therefore the proof of the theorem will follow from theorem 4.5 of [7] if we show that the possible solutions of the equation

$$Lx + \lambda Fx + (1-\lambda)Ax + \lambda Gx + \lambda G_0 x + (1-\lambda)dx + \lambda Hx + \lambda Tx = 0 \quad (3.9)$$

or equivalently the equation

$$a^{-1}[x^{(iv)} + \lambda f(\dot{x})\ddot{x}] + \dddot{x} + (1-\lambda)\Gamma(t)\dot{x}(t-\tau) + \lambda\tilde{Y}(t, \dot{x}(t-\tau))\dot{x}(t-\tau)$$

$$+(1-\lambda)a^{-1}dx + a^{-1}\lambda g_0(t, \dot{x}(t-\tau)) + a^{-1}\lambda h(x) - a^{-1}\lambda p(t) = 0 \quad (3.10)$$

where $d > 0$ are apriori bounded independently of $\lambda \in [0, 1]$.

For $\lambda = 0$ we get the equation

$$x^{(iv)} + a\ \dddot{x} + \Gamma(t)\dot{x}(t-\tau) + dx = 0$$

which by Theorem 2.1 has only the trivial solution. We observe that

$$0 < (1-\lambda)\Gamma(t) + \lambda\tilde{Y}(t, \dot{x}(t-\tau)) \le \Gamma(t) + \delta/2$$

for a.e $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$.

Hence using $V(t) = (1-\lambda)\Gamma(t) + \lambda\tilde{Y}(t, \dot{x}(t-\tau))$, Lemma 3.1 and Cauchy Schwartz inequality we get

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_0^{2\pi} -\dot{\ddot{x}}(t)\{a^{-1}[x^{(iv)} + \lambda f(\dot{x})\ddot{x}] + \dddot{x} + [(1-\lambda)\Gamma(t) \\ &\quad +\lambda\tilde{Y}(t, \dot{x}(t-\tau))]\dot{x}(t-\tau) + (1-\lambda)a^{-1}\, dx + \lambda a^{-1} g_0(t, \dot{x}(t-\tau) \\ &\quad +a^{-1}\lambda h(x) - a^{-1}\lambda p(t))\}\, dt \\ &\geq \delta/2|\ddot{x}|_2^2 - (|\alpha|_2 + |p|_2)|\dot{x}|_2 \end{aligned}$$

Using Wirtinger's inequality gives

$$0 \geq \frac{\delta}{2}|\ddot{x}|_2^2 - \beta|\ddot{x}|_2^2, \quad \text{for some } \beta > 0$$

Hence

$$|\ddot{x}|_2^2 \leq \frac{2\beta}{\delta} = \beta_1, \ \ \beta_1 > 0 \tag{3.11}$$

Thus,

$$|\dot{x}|_2 \leq \beta_2, \ \ \beta_2 > 0$$

and inequality (3.11) implies that

$$|\dot{x}|_\infty \leq \beta_3, \ \ \beta_3 > 0$$

By the continuity of f, we derive

$$|f(\dot{x})|_\infty \leq \beta_4, \ \ \beta_4 > 0.$$

Taking the average of equation (3.10) on $[0, 2\pi]$ we get by the mean value theorem

$$\begin{aligned} |(1-\lambda)dx(t^*) + \lambda h(x(t^*))| &= \left|(1-\lambda)d\frac{1}{2\pi}\int_0^{2\pi} x(t)dt + \lambda\frac{1}{2\pi}\int_0^{2\pi} h(x(t))dt\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |(1-\lambda)\Gamma(t) + \lambda\tilde{Y}(t, \dot{x}(t-\tau))||\dot{x}(t-\tau)|dt \\ &\quad +\frac{1}{2\pi}\int_0^{2\pi} \lambda|g_0(t, \dot{x}(t-\tau))|dt + \frac{1}{2\pi}\lambda\int_0^{2\pi} |p(t)|\, dt \\ &\leq \beta_3(1+\delta/2) + |\alpha|_1 + |p|_1 = \beta_5 \end{aligned}$$

for some $t^* \in [0, 2\pi]$. By (iii) we have for $k > 0$ there is $q = q_k > 0$ such that

$$|h(x)| = Sgn(x)h(x) > 0 \ \text{ for every } \ |x| > \max\{\frac{k}{d}, q\}$$

Hence for any $\lambda \in [0,1]$ we have

$$|(1-\lambda)dx + \lambda h(x)| = Sgn(x)(1-\lambda)dx + \lambda h(x) \geq (1-\lambda)k + \lambda k = k$$

for every $|x| > \max\{\frac{k}{d}, q\}$. We now choose $k > \beta_5$ and derive that

$$|x(t^*)| \leq \max\left\{\frac{k}{d}, q\right\} = \beta_6$$

Now,

$$x(t) = x(t^*) + \int_{t^*}^{2\pi} \dot{x}(t)\, dt$$

Hence

$$|x|_\infty \leq \beta_7 + 2\pi\beta_3 = \beta_7$$

multiplying (3.10) by $-\ddot{x}(t)$ and integrating over $[0, 2\pi]$ we obtain

$$| \dddot{x} |_2^2 \leq |f(\dot{x})|_\infty |\ddot{x}|_2^2 + |a^{-1}|\delta/2 + 1||\dot{x}|_2|\ddot{x}|_2 + |a|^{-1}|\alpha|_2|\ddot{x}|_2 \\ + |p|_2|\ddot{x}|_2 + |h(x)|_\infty|\ddot{x}|_2 + |d||\dot{x}|_2^2.$$

Thus

$$| \dddot{x} |_2 \leq \beta_8, \quad \beta_8 > 0$$

and

$$|\ddot{x}|_\infty \leq \beta_9, \quad \beta_9 > 0$$

Multiplying (3.10) by $x^{(iv)}(t)$ and integrating over $[0, 2\pi]$ we get

$$\begin{aligned} |x^{(iv)}|_2^2 &\leq |f(\dot{x})|_\infty |\dot{x}|_2 |x^{(iv)}|_2 + |a|^{-1}|1 + \delta/2||\dot{x}|_2|x^{(iv)}|_2 \\ &\quad + |d||x|_2|x^{(iv)}|_2 + |\alpha|_2|x^{(iv)}|_2|a|^{-1} + |h(x)|_\infty|x^{(iv)}|_2 + |p|_2|x^{(iv)}|_2 \end{aligned}$$

Therefore

$$|x^{(iv)}|_2 \leq \beta_{10}, \quad \beta_{10} > 0$$

and thus

$$| \dddot{x} |_\infty \leq \beta_{11}, \quad \beta_{11} > 0$$

Therefore

$$\begin{aligned} |x|_{C^3} &= |x|_\infty + |\dot{x}|_\infty + |\ddot{x}|_\infty + |\dddot{x}|_\infty \\ &\leq \beta_7 + \beta_3 + \beta_9 + \beta_{11} = \beta_{12} \end{aligned}$$

Choosing $\rho > \beta_{12} > 0$ we obtain the required a priori bound in $C^3[0, 2\pi]$ independently of x and λ.

4. Uniqueness Result

If in (1.1), $f(\dot{x}) = b$, $h(x) = d$ where b and d are constants, then we have the following uniqueness result.

Theorem 4.1. *Let a, b, d be constants, with $a \neq 0$, $d > 0$. Suppose g is a Caratheodory function satisfying*

$$0 < \frac{g(t, \dot{x}_2) - g(t, \dot{x}_2)}{a(\dot{x}_1 - \dot{x}_2)} \leq \Gamma(t) \tag{4.1}$$

for all $\dot{x}_1$, $\dot{x}_2 \in \mathbb{R}$, $\dot{x}_1 \neq \dot{x}_2$, where $\Gamma(t) \in L^2_{2\pi}$ is such that

$$0 < \Gamma(t) < 1$$

then for all arbitrary constant b and every $\tau \in [0, 2\pi)$ the boundary value problem

$$x^{(iv)} + a \dddot{x} + b\ddot{x} + g(t, \dot{x}(t - \tau))t \, dt = p(t) \tag{4.2}$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3 \tag{4.3}$$

has at most one solution.

Proof. Let x_1, x_2 be any two solutions of (4.2) - (4.3). Set $x = x_1 - x_2$. Then x satisfies the boundary value problem

$$a^{-1}x^{(iv)} + \dddot{x} + a^{-1}b\ddot{x} + \Gamma(t)\dot{x}(t - \tau) + a^{-1}dx = 0$$

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, 1, 2, 3$$

where the function $\Gamma(t) \in L^2_{2\pi}$ is defined by

$$\Gamma(t) = \begin{cases} \dfrac{g(t, \dot{x}_1(t - \tau)) - g(t, \dot{x}_2(t - \tau))}{\dot{x}(t)} & \text{if } \dot{x}(t) \neq 0 \\ \dfrac{1}{2} & \text{if } \dot{x}(t) = 0 \end{cases}$$

If $\dot{x}(t) = 0$ on every subset of $[0, 2\pi]$ of positive measure, then x = constant = 0 since $d \neq 0$. Hence $x_1 = x_2$. Suppose on the other hand that $\dot{x}(t) \neq 0$ on a certain subset of $[0, 2\pi]$ of positive measure. Then using the argument of theorem 2.1 we obtain that $x = 0$ and hence $x_1 = x_2$ a.e.

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