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The Kumaraswamy-Power Distribution: A Generalization of the Power Distribution

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Abstract

We introduce a generalization referred to as the Kumaraswamy Power distribution. The proposed model serves as a generalization of the two-parameter Power distribution using the Kumaraswamy Generalized family of distributions. We investigate some of its statistical properties; the Generalized Power distribution, Exponentiated Power distribution and the Power distribution are found to be sub-models of the proposed distribution. The method of maximum likelihood estimation is proposed in estimating the parameters of the model.

Keywords: Exponentiated Power distribution, Generalized Power distribution, Generalization, Kumaraswamy, Power distribution

1. Introduction

Let X denote a non-negative continuous random variable, distributed according to the power distribution with parameters a,b and α that is, $X \sim (a,b,\alpha)$, its probability density function (pdf) is defined by;

$$f(x)=\frac{\alpha(x-a)^{\alpha-1}}{(b-a)^{\alpha}} \quad ; \quad a \le x \le b \tag{1}$$

where;

α is the shape parameter

a,b are the boundary parameters $(a<b)$

The corresponding cumulative density function (cdf) is given by;

$$F(x)=\left(\frac{x-a}{b-a}\right)^{\alpha} \tag{2}$$

In this article, we consider the case of a two-parameter Power distribution where one of the boundary parameters is zero $(a=0)$.

Therefore, the pdf and the cdf of the Power distribution with parameters α and β are respectively given by;

$$f(x)=\frac{\alpha x^{\alpha-1}}{\beta^{\alpha}} \quad ; \quad 0 \le x \le \beta \tag{3}$$

$$F(x)=\left(\frac{x}{\beta}\right)^{\alpha} \quad ; \quad 0 \le x \le \beta \tag{4}$$

where α is the shape parameter

β is the scale parameter

A number of authors have generalized known theoretical distributions using a class of generalized distributions based on the logit of the beta random variable proposed by Eugene et al (2002) as;

$$f(x)=\frac{1}{B(a,b)}g(x)G(x)^{a-1}\{1-G(x)\}^{b-1} \tag{5}$$

The corresponding cdf is;

$$F(x)=\frac{1}{B(a,b)}\int_0^{G(x)} w^{a-1}(1-w)^{b-1}\,dw \tag{6}$$

where $a>0$ and $b>0$ are shape parameters

$$g(x)=\frac{dG(x)}{dx} \qquad \text{and} \qquad B(a,b)=\int_0^1 w^{a-1}(1-w)^{b-1}$$

Of interest to us in this article is to generalize the Power distribution using another family of generalized distributions based on the Kumaraswamy distribution as defined by Cordeiro and de Castro (2011). This is due to the fact that the cdf of the Kumaraswamy Generalized distributions do not involve any special function like the incomplete beta function ratio; thereby, making it to be more tractable than the Beta Generalized family of distributions. The pdf and the cdf of a Kumaraswamy-Generalized distribution are given respectively by;

$$f(x)=abg(x)G(x)^{a-1}\left\{1-G(x)^{a}\right\}^{b-1} \tag{7}$$

$$F(x)=1-\left\{1-G(x)^{a}\right\}^{b} \tag{8}$$

where; $a,b>0$ are additional shape parameters.

$g(x)=\frac{dG(x)}{dx}$ is an arbitrary parent continuous pdf

Arising from Equation (7), various probability models have been defined. The Kumaraswamy-Weibull distribution (Cordeiro et al; 2010), Kumaraswamy-Inverse Weibull distribution; Shahbaz et al (2012), Kumaraswamy GP distribution; Nadarajah and Eljabri (2013), Kumaraswamy-Inverse Exponential distribution; Oguntunde et al (2014) and many others are example of such in the literature.

The rest of this article is organized as follows; Section 2 introduces the proposed Kumaraswamy-Power distribution, Section 3 investigates some of its basic statistical properties including the estimation of parameters followed by a concluding remark.

2. The Kumaraswamy-Power Distribution (K-Pow Distribution)

We generate the K-Pow distribution by inserting Equations (3) and (4) into Equation (7). That is, we make the $g(x)$ and $G(x)$ in Equation (7) to be the pdf and cdf of the power distribution respectively. Hence, if a random variable X is such that; $X \sim K-Pow$ (a, b, α, β) , the pdf and the cdf are respectively given by;

$$f(x)=ab\left(\frac{\alpha x^{\alpha-1}}{\beta^{\alpha}}\right)\left[\left(\frac{x}{\beta}\right)^{\alpha}\right]^{a-1}\left\{1-\left[\left(\frac{x}{\beta}\right)^{\alpha}\right]^{a}\right\}^{b-1} \tag{9}$$

$$F(x) = 1 - \left\{ 1 - \left[\left(\frac{x}{\beta} \right)^{\alpha} \right]^{a} \right\}^{b} \tag{10}$$

For $x > 0, a > 0, b > 0, \alpha > 0, \beta > 0$

where a, b, α are shape parameters

β is the scale parameter

The possible plots for the pdf of the K-Pow distribution at some parameter values are given in Figure 1 and Figure 2;

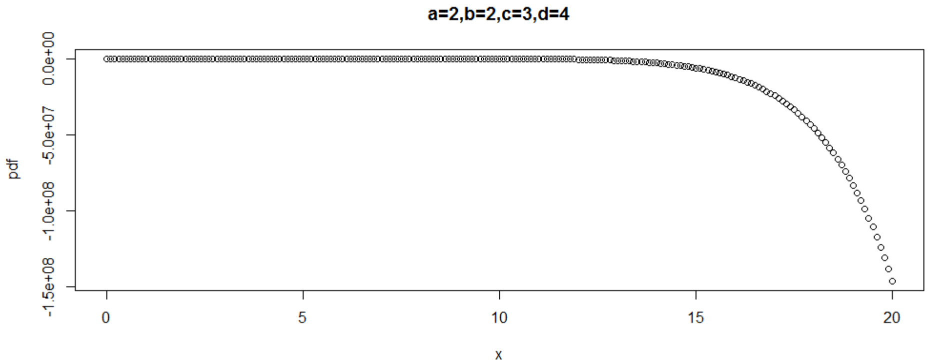

Fig. 1: The plot for the K-Pow pdf (where $c = \alpha, d = \beta$)

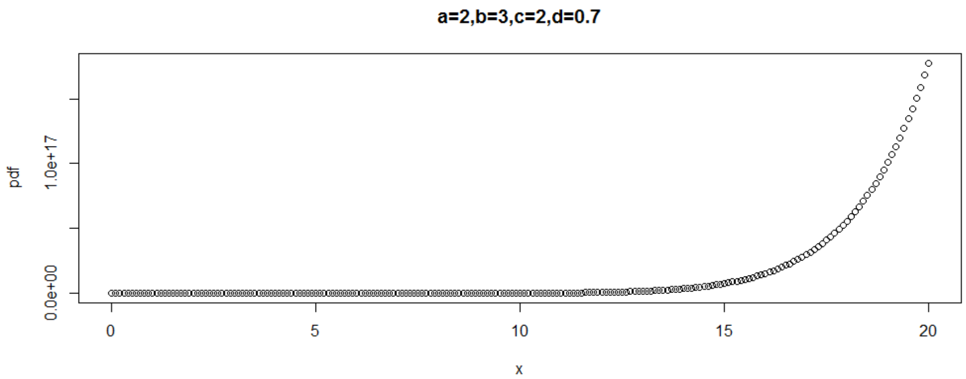

Fig. 2: The plot for the K-Pow pdf (where $c = \alpha, d = \beta$)

We observe in Figure 1 that as the value of x increases, the curve decreases. This implies that the shape of the proposed model could be “decreasing”. In Figure 2, as the value of x increases, the curve increases, we can simply say that the shape of the proposed model could be “increasing”.

Special Case:

Some distributions are found to be sub-models of the proposed K-Pow distribution. For instance;

1. For $a=1$, the K-Pow distribution reduces to give the Generalized Power distribution.
2. For $b=1$, the K-Pow distribution reduces to give the Exponentiated Power distribution.
3. For $a=b=1$, the K-Pow distribution reduces to give the baseline distribution which is the Power distribution.

2.1 General Expansion for the Density Function

Following Cordeiro and de Castro (2011), for $b>0$ real non-integer, the Kumaraswamy Generalized density can as well be written as;

$$f(x)=g(x)\sum_{i=0}^{\infty} w_i G(x)^{a(i+1)-1} \tag{11}$$

$$w_i=w_i(a,b)=(-1)^i ab\binom{b-1}{i}; \quad \text{and} \quad \sum_{i=0}^{\infty} w_i=1$$

Therefore, we can re-write the K-Pow pdf in Equation (9) as;

$$f(x)=\frac{\alpha x^{\alpha-1}}{\beta^{\alpha}}\sum_{i=0}^{\infty}(-1)^i ab\binom{b-1}{i}\left\{\left(\frac{x}{\beta}\right)^{\alpha}\right\}^{a(i+1)-1} \tag{12}$$

If $'a'$ is real non-integer, the Kumaraswamy Generalized density is given by;

$$f(x)=g(x)\sum_{i,j=0}^{\infty}\sum_{r=0}^{j} w_{i,j,r} G(x)^r \tag{13}$$

where; $w_{i,j,r}=w_{i,j,r}(a,b)=(-1)^{i+j+r} ab\binom{a(i+1)-1}{j}\binom{b-1}{i}\binom{j}{r}$; and $\sum_{i,j=0}^{\infty}\sum_{r=0}^{j} w_{i,j,r}=1$

In other words, we can re-write Equation (9) as;

$$f(x)=\frac{\alpha x^{\alpha-1}}{\beta^{\alpha}}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}(-1)^{i+j+r} ab\binom{a(i+1)-1}{j}\binom{b-1}{i}\binom{j}{r}\left\{\left(\frac{x}{\beta}\right)^{\alpha}\right\}^{r} \tag{14}$$

3. Some Properties of the K-Pow distribution

In this section, we investigate some of the basic statistical properties of the proposed model.

3.1. Reliability Analysis

In this subsection, we provide expressions for the survival function and the hazard rate of the proposed K-Pow distribution.

The survival function is mathematically given by; $S(x)=1-F(x)$

Therefore the survival function for the K-Pow distribution is given by;

$$S(x)=\left\{1-\left(\frac{x}{\beta}\right)^{a\alpha}\right\}^{b} \tag{15}$$

The plot for the survival function of the proposed K-Pow distribution at some parameter values is given in Figure 3.

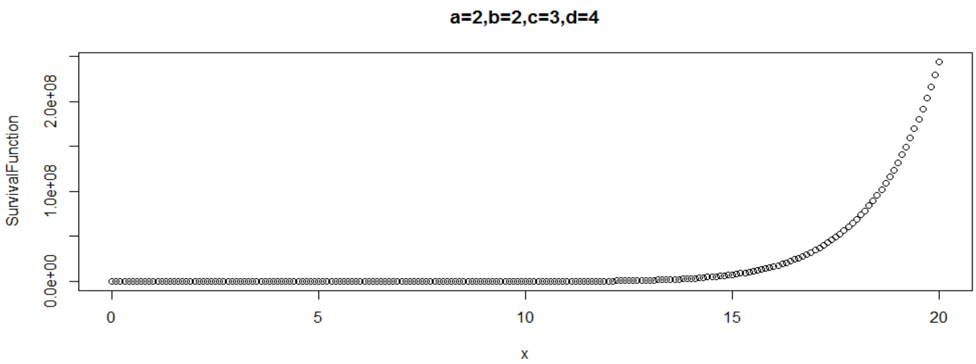

Fig. 3: The survival function for the K-Pow distribution (where; $c=\alpha, d=\beta$)

The hazard function is mathematically given by; $h(x)=\dfrac{f(x)}{1-F(x)}$.

Therefore, the hazard rate for the K-Pow distribution is given by;

$$h(x)=\frac{ab\left(\dfrac{\alpha x^{\alpha-1}}{\beta^{\alpha}}\right)\left[\left(\dfrac{x}{\beta}\right)^{\alpha}\right]^{a-1}}{\left\{1-\left(\dfrac{x}{\beta}\right)^{a\alpha}\right\}} \tag{16}$$

The possible plot for the hazard function of the proposed K-Pow distribution is given in Figure 4.

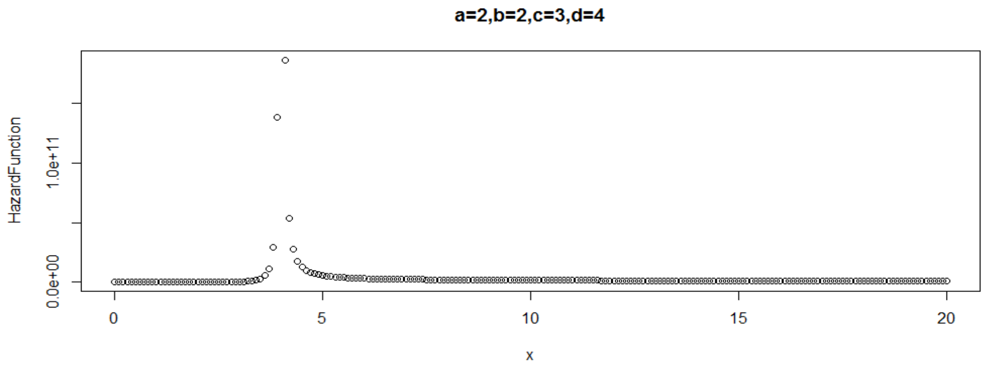

Fig. 4: The hazard function for the K-Pow distribution (where; $c = \alpha, d = \beta$)

The plot in Figure 4 shows that the shape of the hazard function for the proposed model could be unimodal (or inverted bathtub).

3.2 Moments

The k-th moment of the K-Pow distribution can be gotten directly by numerical integration of the expression in Equation (17);

$$\mu_k = ab\int x^k \left(\frac{\alpha x^{\alpha-1}}{\beta^{\alpha}}\right)\left[\left(\frac{x}{\beta}\right)^{\alpha}\right]^{a-1}\left\{1-\left[\left(\frac{x}{\beta}\right)^{\alpha}\right]^{a}\right\}^{b-1} dx \tag{17}$$

NOTE: The solution to Equation (17) can be prone to rounding off errors among other. Therefore, we can calculate the moments of the K-Power distribution in terms of the infinite weighted sums of probability weighted moments (PWMs) of the Power distribution; See Cordeiro and de Castro (2011) for details.

3.3 Parameter Estimation

In a view to estimating the parameters of the K-Pow distribution, we employ the method of the Maximum Likelihood Estimation (MLE). Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample of 'n' independently and identically distributed random variables each having a K-Pow distribution defined in Equation (9), the likelihood function L is given by;

$$L\left(\tilde{X} \mid a,b,\alpha,\beta\right) = \prod_{i=1}^{n}\left\{ab\left(\frac{\alpha x^{\alpha-1}}{\beta^{\alpha}}\right)\left[\left(\frac{x}{\beta}\right)^{\alpha}\right]^{a-1}\left\{1-\left[\left(\frac{x}{\beta}\right)^{\alpha}\right]^{a}\right\}^{b-1}\right\}$$

$$l = \log L\left(\tilde{X} \mid a, b, \alpha, \beta\right)$$

$$l = n\log a + n\log b + n\log\alpha - n\alpha\log\beta + (\alpha-1)\sum_{i=1}^{n}\log(x_i) + (a-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{\beta}\right)^{\alpha} + (b-1)\sum_{i=1}^{n}\log\left[1-\left(\frac{x_i}{\beta}\right)^{a\alpha}\right]$$

Differentiating l with respect to a, b, α and β respectively and solving the resulting non-linear simultaneous equations gives the maximum likelihood estimates of a, b, α and β respectively.

4. Conclusion

In this study, a four-parameter Kumaraswamy Power distribution has been successfully defined and some of its statistical properties have been identified. The shape of the model could be increasing or decreasing (depending on the parameter values). The hazard rate has an inverted bathtub (unimodal) shape. The method of maximum likelihood estimation is proposed for estimating the model parameters. The Generalized Power distribution, Exponentiated Power distribution and Power distribution are found to be sub-models of the proposed model. We hope the model would receive appreciable usage in reliability studies and hydrology.

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