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A 5-Step Block Predictor and 4-Step Corrector Methods for Solving General Second Order Ordinary Differential Equations

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Abstract

A 5-step block predictor and 4-step corrector methods aimed at solving general second order ordinary differential equations directly will be constructed and implemented on non-stiff problems. This method, which extends the work of block predictor-corrector methods using variable step size technique possess some computational advantages of choosing a suitable step size, deciding the stopping criteria and error control. In addition, some selected theoretical properties of the method will be investigated as well as determination of the region of absolute stability. Numerical results will be given to show the efficiency of the new method.

Keywords: predictor-corrector methods, stopping criteria, region of absolute stability, variable step size technique

1.0. Introduction

Many problems of science and engineering are reduced to quantifiable form through the process of mathematical modelling. The equations arising often are expressed in terms of the unknown quantities and their derivatives. Such equations are called differential equations. Since analytical methods are not adequate for finding accurate solutions to most differential equations, numerical methods are required. The ideal objective, in employing a numerical method, is to compute a solution of specified accuracy to the differential equation. Sometimes this is achieved by computing several solutions using a method which has known error characteristics as in John

[13]. This paper considered solving directly general second order ordinary differential equations of the form Adetola and Odekunle [1]

$$y'' = f(x, y, y'), y(x_0) = y_0, y'_0(x_0) = y'_0, a \le x \le b \quad (1)$$

Numerical methods of solving (1) exists in literature. Yayaya and Badmus [24] reported that (1) can be reduced to systems of first order equations and other one-step methods for solving first order equations are used. However, Anake et al. [3] and Majid and Suleiman [18] suggested that reducing (1) to the equivalent first order system of twice the dimension equations and then solved using one-step or multistep method. This technique is very well established but it apparently will increase the dimension of the equations.

According to Jain and Iyengar [11], explicit and implicit methods combined together to obtain a new methods. Such methods is called the Predictor-Corrector Methods. Scholars such as Lambert [15] and Lambert [16] have suggested that this turns out to be an advantage in having the predictor and the corrector of the same order. Again, the predictor-corrector pair is applied in the mode of correcting to convergence which is one of the most important aspect of the predictor-corrector methods. Adetola and Odekunle [1] and Adetola et al. [2] sited the major setback of the predictor-corrector mode is the cost of developing subroutine. Furthermore, this subroutine developed are of lower order to the corrector, thus, it has great consequence on the accuracy of the corrector results.

The Block multistep methods are one of the numerical methods which have been suggested by several researchers, see Adetola and Odekunle [1], Adetola et al. [2], James et al. [12], Majid et al. [17], Majid and Suleiman [18] and Zarina et al. [25]. The commonly block methods used to evaluate (1) can be categorise as one-step block method and multistep block method. Again, block methods was proposed by scholars to cater for the shortcoming of predictor-corrector method, since block method provides solutions at each grid within the interval of integration without overlapping thereby eradicating the idea of subroutine.

Scholars such as Adetola and Odekunle [1], Ehigie et al. [5], Ismail et al. [9] and Ken et al. [14]proposed block multistep methods which were applied in predictor-corrector mode. Block multistep methods have the advantage of evaluating simultaneously at all points with the integration interval, thereby reducing the computational burden when evaluation is needed at more than one point within the grid. Again, starting values are provided by Taylor series expansion in order to compute the corrector method.

Researchers in Adetola and Odekunle [1], Ehigie et al. [5], Ismail et al. [9] and Ken et al. [14]implemented block predictor-corrector method in which at each practical application of the method, the method was only intended to predict and correct the results generated. In this paper, the motivation is stemmedfrom the fact that block predictor-corrector methods applied by different authors never surpass its advantage as suggested above, which makes the block predictor-corrector method to be under-utilized. Hence, there is a need to propose a type of block predictor-corrector method in the form of 5-step block predictor (explicit Adams-Bashforth) and 4-step corrector methods (implicit Adams-Moulton) implemented using variable step size technique.

This method possess the following advantages such as changing the stepsize and determining on a suitable step size for the block predictor-corrector method, choosing the stopping criteria and error control or minimization.

2.0. Justification

We first state the theorem that demonstrates the uniqueness of solutions of higher order ordinary differential equations.

Theorem 1 (Existence and Uniqueness)

Let $f(x,y)$ be defined and continuous for all points (x,y) in the region D defined by $a \le x \le b,\ -\infty \le x \le \infty,$ where a and b are finite, and let there exists a constant L such that for any $x \in [a,b]$ and any two numbers y and y^{*},

$$\left| f(x,y) - f(x,y^{*}) \right| \le L\left| y - y^{*} \right|.$$

This condition is known as Lipchitz condition. Then there exists exactly one function $y(x)$ with the following four properties:

(i) $y(x)$ is continuous and differentiable for $x \in [a,b]$,

(ii) $y'(x) = f(x, y(x)),\ x \in [a,b]$

(iii) $y''(x) = f(x, y(x), y'),\ x \in [a,b]$,

(iv) $y(a) = \eta$ and $y'(x) = \eta'$.

See Ken et al. [14] and Wendy [23] for details.

Theorem 2 (Weierstrass)

The Weierstrass approximation theorem states that a continuous function f(x) over a closed interval $[a,b]$ can be approximated by a polynomial $P_n(x), [a,b]$ of degree n, such that

$$\left| f(x) - P_n(x) \right| < \varepsilon,\ x \in [a,b].$$

Where $\varepsilon > 0$ is a small quantity and n is sufficiently large, see Jain et al. [10].

3.0. Theoretical Procedure

In this proposed study, we seek to examine directly the general second order ODEs of the form

$$y'' = f(x,y,y'), y(x_0) = y_0, y'(x_0) = y'_0, a \le x \le b \qquad (2)$$

The solution to (2) may written as

$$\sum_{i=0}^{j} \alpha_i y_{i+j} = h^2 \sum_{i=0}^{j} \beta_i f_{i+j} \qquad (3)$$

Where $f_{i+j} \equiv f(x_{i+j}, y(x_{i+j}))$, α_i and β_i are constant and assume that $\alpha_i \neq 0$, $|\alpha_0| + |\beta_0| > 0$. Since (3) can be multiplied by the same constant without altering the relationship α_i *and* β_i are arbitrary to the extent of a multiplication constant. The arbitrariness has been removed by assuming that $\alpha_j = 1$. Method (3) is explicit if $\beta_j = 0$ and implicit if $\beta_j \neq 0$ as introduced in Ken et al. [14], Lambert [15] and Lambert [16].

This study is focused on the use of Adam's method of variable step size technique in developing a type of 5-step block predictor and 4 step corrector methods for solving general second order ODEs forthwith. The method will be constructed based on interpolation and collocation approach using power series as the approximate solution of the problem as stated in Ehigie et al. [5], Faires and Burden [7], Lambert [15] and Lambert [16]. Thus,this power series solution can be written in the form

$$y(x) = \sum_{i=0}^{j} a_i \left(\frac{x - x_n}{h} \right)^i \tag{4}$$

3.1. Formulation of the Method

According to Ismail et al. [9] and Ken et al. [14] , in a 2-point block method, the interval $[a, b]$ is divided into subintervals of blocks with each interval containing two points, i.e. x_n and x_{n-1} in the first block while x_{n+1} and x_{n+2} in the second block where solutions to (2) are to be computed. The method will formulate two new evenly spaced solution values concurrently. Similarly, this can be extended to a 3-point one block method where the backward and forward values are the points of interpolation and collocation as well as evaluation.

3.2. Representation of r-Point Block Method

From Fatunla [6] and Ken et al. [14], the k-point block method for (3) is given by the matrix finite difference equation

$$A^{(0)} Y_m = \sum_{i=0}^{j} A^{(i)} Y_{m-i} + h^2 \sum_{i=0}^{j} B^{(i)} F_{m-i} \tag{5}$$

Where $Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ . \\ . \\ . \\ y_{n+r} \end{bmatrix}$, $F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ . \\ . \\ . \\ f_{n+r} \end{bmatrix}$, $Y_{m-1} = \begin{bmatrix} y_{n-r+1} \\ y_{n-r+2} \\ . \\ . \\ . \\ y_n \end{bmatrix}$, $F_{m-1} = \begin{bmatrix} f_{n-r+1} \\ f_{n-r+2} \\ . \\ . \\ . \\ f_n \end{bmatrix}$,

$(for\ \ n = mr, m = 0,1,\ldots 1)$,

$A^{(i)}$ and $B^{(i)}$ are $r \times r$ matrices. It is assumed that matrix finite difference equation is normalized so that $A^{(0)}$ is an identity matrix. The block scheme is explicit if the coefficient matrix $B^{(0)}$ is a null matrix.

3.3. Derivation of 5-Step Block Multistep Predictor Method

As in Adetola and Odekunle [1] and Ehigie et al. [5], interpolating (4) at $x = x_{n-i}$ for $i = 0(1)j$ and collocating (4) at $x = x_{n-i}$ for $i = 0(1)j$ gives a system of equations which can be expressed as $AX = U$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -x_{n-1} & x_{n-1}^2 & x_{n-1}^3 & x_{n-1}^4 & x_{n-1}^5 & x_{n-1}^6 \\ 0 & 0 & 2x_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 2x_{n-1}^2 & -6x_{n-1}^3 & 12x_{n-1}^4 & -20x_{n-1}^5 & 30x_{n-1}^6 \\ 0 & 0 & 2x_{n-2}^2 & -12x_{n-2}^3 & 48x_{n-2}^4 & -160x_{n-2}^5 & 480x_{n-2}^6 \\ 0 & 0 & 2x_{n-3}^2 & -18x_{n-3}^3 & 108x_{n-3}^4 & -504x_{n-3}^5 & 2430x_{n-3}^6 \\ 0 & 0 & 2x_{n-4}^2 & -24x_{n-4}^3 & 192x_{n-4}^4 & -1280x_{n-4}^5 & 7680x_{n-4}^6 \end{bmatrix}$$

$$X = \left[a_0, a_1, a_2, a_3, a_4, a_5, a_6\right]^T$$

$$U = \left[y_n, y_{n-1}, y_{n-2}, y_{n-3}, y_{n-4}, y_{n-5}, y_{n-6}\right]^T \quad (6)$$

Solving (6) and substituting the values of (6) into (4) gives a continuous linear multistep method of the form

$$y(x) = \sum_{i=0}^{j} \alpha_i y_{i+1} + h^2 \sum_{i=0}^{j} \beta_i f_{i+1} \quad (7)$$

Evaluating (7) at points $x = x_{n+i}$ for $i = 1(1)j$, we obtain the convergent 5-step block multistep predictor method as

$$y_{n+1} = 2y_n + y_{n-1} h^2\left[\frac{299}{240} f_n - \frac{11}{15} f_{n-1} + \frac{97}{120} f_{n-2} - \frac{2}{5} f_{n-3} + \frac{19}{240} f_{n-4}\right]$$

$$y_{n+2} = 3y_n + 2y_{n-1} h^2\left[\frac{639}{80} f_n - \frac{787}{60} f_{n-1} + \frac{547}{40} f_{n-2} - \frac{139}{20} f_{n-3} + \frac{337}{240} f_{n-4}\right] \quad (8)$$

$$y_{n+3} = 4y_n + 3y_{n-1} h^2\left[\frac{3667}{120} f_n - \frac{342}{5} f_{n-1} + \frac{1507}{20} f_{n-2} - \frac{596}{15} f_{n-3} + \frac{329}{40} f_{n-4}\right]$$

Adopting Fatunla [6] and Ken et al. [14], the 5-step block multistep predictor method can be written in matrix finite difference equation as

$$A^{(0)} Y_m = A^{(1)} Y_{m-1} + h^2 B^{(1)} F_{m-1} \quad (9)$$

Differentiating (7) once and evaluating at the same three discrete points of $i = 1,2,3$ for $x = x_{n+i}$, $i = 1,2,3$, for $x = x_{n+i}$, we obtain a block of first order derivative which can be used to determine the derivative term in the initial value problem (2), seen in Ehigie et al. [5].

$$y'_{n+1} = \frac{1}{h}\left[y'_n - y'_{n-1}\right] + h\left[\frac{4169}{1440} f_n - \frac{313}{90} f_{n-1} + \frac{55}{16} f_{n-2} - \frac{76}{45} f_{n-3} + \frac{481}{1440} f_{n-4}\right]$$

$$y'_{n+1} = \frac{1}{h}\left[y'_n - y'_{n-1}\right] + h\left[\frac{1959}{160} f_n - \frac{9449}{360} f_{n-1} + \frac{6737}{240} f_{n-2} - \frac{349}{24} f_{n-3} + \frac{4283}{1440} f_{n-4}\right] \quad (10)$$

$$y'_{n+1} = \frac{1}{h}\left[y'_n - y'_{n-1}\right] + h\left[\frac{52153}{1440} f_n - \frac{951}{10} f_{n-1} + \frac{26089}{240} f_{n-2} - \frac{2639}{45} f_{n-3} + \frac{1183}{96} f_{n-4}\right]$$

3.4. Derivation of 4- Step Block Multistep Corrector Method

Interpolating (4) at $x = x_{n-i}$ for $i = 0(1)j$ and collocating (4) at $x = x_{n-i}$, $x = x_{n+i}$ for $i = 0(1)j$ gives a system of equations which can be expressed as $AX = U$ as discussed above.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -x_{n-1} & x_{n-1}^2 & x_{n-1}^3 & x_{n-1}^4 & x_{n-1}^5 & x_{n-1}^6 \\ 0 & 0 & 2x_{n+1}^2 & -6x_{n-1}^3 & 12x_{n-1}^4 & -20x_{n-1}^5 & 30x_{n-1}^6 \\ 0 & 0 & 2x_{n-3}^2 & -18x_{n-3}^3 & 108x_{n-3}^4 & -504x_{n-3}^5 & 2430x_{n-3}^6 \\ 0 & 0 & 2x_{n+1}^2 & 6x_{n+1}^3 & 12x_{n+1}^4 & 20x_{n+1}^5 & 30x_{n+1}^6 \\ 0 & 0 & 2x_{n+2}^2 & 12x_{n+2}^3 & 48x_{+2}^4 & 160x_{n+2}^5 & 480x_{n+2}^6 \\ 0 & 0 & 2x_{n+3}^2 & 18x_{n+3}^3 & 108x_{n+3}^4 & 504x_{n+3}^5 & 2430x_{n+3}^6 \end{bmatrix}$$

$$X = \left[a_0, a_1, a_2, a_3, a_4, a_5, a_6\right]^T$$
$$U = \left[y_n, y_{n-1}, f_{n-1}, f_{n-3}, f_{n+1}, f_{n+2}, f_{n+3}\right]^T \quad (11)$$

Solving (11)and substituting the values of (11) into (4) yields a continuous linear multistep method of the form

$$y(x) = \sum_{i=0}^{j} \alpha_i y_{i+1} + h^2 \sum_{i=0}^{j} \beta_i f_{i+1} \quad (12)$$

Evaluating (12) at points $x = x_{n+i}$ for $i = 1(1)j$, we obtain the convergent 4-step block multistep corrector method as

$$y_{n+1} = 2y_n + y_{n-1} h^2\left[\frac{191}{480} f_{n-1} - \frac{17}{800} f_{n-3} + \frac{487}{480} f_{n+1} - \frac{37}{75} f_{n+2} + \frac{49}{480} f_{n+3}\right]$$

$$y_{n+2} = 3y_n + 2y_{n-1} h^2\left[\frac{199}{240} f_{n-1} - \frac{9}{200} f_{n-3} + \frac{59}{20} f_{n+1} - \frac{71}{75} f_{n+2} + \frac{17}{80} f_{n+3}\right] \quad (13)$$

$$y_{n+3} = 4y_n + 3y_{n-1} h^2\left[\frac{101}{80} f_{n-1} - \frac{83}{1200} f_{n-3} + \frac{397}{80} f_{n+1} - \frac{14}{25} f_{n+2} + \frac{97}{240} f_{n+3}\right]$$

As usual, Fatunla [6] and Ehigie et al. [14] stated that the 4-step block multistep corrector method can be written in matrix finite difference equation as

$$A^{(0)} Y_m = A^{(1)} Y_{m-1} + h^2 \left[B^{(1)} F_m + B^{(2)} F_{m-1}\right] \quad (14)$$

Differentiating (12) once and evaluating at the same three discrete points of $i = 1,2,3$ for $x = x_{n+i}$ we have a block of first order derivative which can be used to determine the derivative term in the initial value problem (2).

$$
\begin{aligned}
y'_{n+1} &= \frac{1}{h}\left[y'_n - y'_{n-1}\right] + h\left[\frac{1277}{2880}f_{n-1} - \frac{361}{14400}f_{n-3} + \frac{509}{320}f_{n+1} - \frac{287}{450}f_{n+2} + \frac{371}{2880}f_{n+3}\right] \\
y'_{n+1} &= \frac{1}{h}\left[y'_n - y'_{n-1}\right] + h\left[\frac{121}{288}f_{n-1} - \frac{9}{400}f_{n-3} + \frac{253}{120}f_{n+1} - \frac{43}{450}f_{n+2} + \frac{43}{480}f_{n+3}\right] \\
y'_{n+1} &= \frac{1}{h}\left[y'_n - y'_{n-1}\right] + h\left[\frac{431}{960}f_{n-1} - \frac{377}{14400}f_{n-3} + \frac{1831}{960}f_{n+1} + \frac{107}{150}f_{n+2} + \frac{263}{576}f_{n+3}\right]
\end{aligned}
\qquad (15)
$$

4.0. Investigation of the Theoretical Properties of the Methods

4.1. Order of the Method

Definition 1. EmbracingJain et al. [10], Lambert [15] and Lambert [16], the linear k-step method of (8) and (13) with associated difference operator

$$L[y(x);h] = \sum_{i=0}^{j}\left[\alpha_i y(x+ih) - h^2\beta_i y''(x+ih)\right] \qquad (16)$$

where $y(x)$ is an arbitrary function, continuously differentiable on an interval $[a,b]$. If we assume that $y(x)$ has a many higher derivatives as we require, then, expanding using Taylor series about the point x, we obtain

$$C_0 = \alpha_0 + \alpha_1 + \ldots + \alpha_j$$

$$C_1 = \alpha_1 + 2\alpha_2 + \ldots + j\alpha_j$$

$$C_2 = \frac{1}{2}\left(\alpha_1 + 2^2\alpha_2 + \ldots + j^2\alpha_j\right) - \left(\beta_0 + \beta_1 + \ldots + \beta_j\right)$$

$$C_q = \frac{1}{q!}\left(\alpha_1 + 2^q\alpha_2 + \ldots + j^q\alpha_j\right) - \frac{1}{(q-2)!}\left(\beta_1 + 2^{q-2}\beta_2 + \ldots + j^{q-2}\beta_j\right),\ q = 3,4,\ldots.$$

Following Lambert [15], we say that the method has order p if

$$C_0 = C_1 = C_2 = \ldots = C_{p+1} = 0,\ C_{p+2} \neq 0$$

C_{p+2} is then the error constant and $C_{p+2}h^{p+2}y^{(p+2)}(x_n)$ the principal local truncation error at the point x_n.

Combining Jain et al. [10], Lambert [15] and Lambert [16], we noticed that the block multistep method of (8) and (13) has order p, if $C_0 = C_1 = C_2 = \ldots = C_{p+1} = 0, C_{p+2} \neq 0$.

Therefore, we concluded that the methods (8) and (13) have order p=5 and error constants given by the vectors, $C_7 = \left[\frac{3}{40}, \frac{353}{240}, \frac{581}{60}\right]^T$ and

$$C_7 = \left[-\frac{37}{300}, -\frac{157}{600}, -\frac{157}{150}\right]^T$$

4.2. Convergence

Agreeing to Hairer et al. [8], Ken et al. [14] and Lambert [15], if the multistep method

$$\sum_{i=0}^{j} \alpha_i y_{n-i} = h^2 \sum_{i=0}^{j} \beta_i f_{n-i} \tag{17}$$

is convergent, then it is necessarily

(i) stable and

(ii) consistent (i.e. of order 1:

$$\rho(1) = 0, \rho'(1) = \sigma(1), \rho''(1) = 2\sigma(1)) \tag{18}$$

4.3. Zero Stability

Theorem 3 (First Root Condition)

From Bruce [4] and Ken et al. [14], the multistep methods (8) and (13) are stable if all the roots r_j of the characteristic polynomial $\rho(r)$ satisfy $|r_i| \le 1$ and $|r_i| = 1$ if then r_i must be a simple root.

Definition 2. As in Hairer et al. [8] and Mohammed et al. [19], the multistep method (17) is called stable, if the generating polynomial

$$\rho(r) = \det\left(A^{(0)} - E\right) \tag{19}$$

satisfies the first root condition, i.e.,

(i) The roots of $\rho(r)$ lie on or within the unit circle;

(ii) The roots on the unit circle are simple.

In order to analyze the methods for zero-stability, equation (8) and (13) are both normalize and written as a block method given by the matrix finite difference equations as discussed in Mohammed et al. [19]

$$\begin{aligned} A^{(0)} Y_m &= A^{(1)} Y_{m-1} + h^2 \left(B^{(1)} F_m + B^{(2)} F_{m-1}\right) \\ A^{(0)} Y_m &= A^{(1)} Y_{m-1} + h^2 \left(B^{(0)} F_m + B^{(1)} F_{m-1}\right) \end{aligned} \tag{20}$$

In addition, the zero stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$,

$$\rho(r) = r^{z-\sigma}\left(r-1\right)^{\sigma}, \tag{21}$$

where σ is the order of the differential equation, z is the order of the matrix $A^{(0)}$ and E, see Adetola et al. [2], Mohammed et al. [19] and Sani [20] for details.
For our method

$$\rho(r) = \left[r\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 2 \\ 0 & -2 & 3 \\ 0 & -3 & 4 \end{bmatrix} \right] = 0$$

Solving the matrix equation above, if

$$\rho(r) = \det \left[A^{(0)} r^2 - A^{(1)} \right] \tag{22}$$

gives

$$\rho(r) = r(r-1)^2. \tag{23}$$

Hence, our method is zero stable according to [8, 14].

4.4 Consistency

Theorem 4

According to Bruce [4], Ken et al. [14] and Lambert [15], a linear multistep method is consistent if it has order greater than or equal to 1. Thus

$$\begin{aligned} &\sum_{i=0}^{j} a_i = 0, \\ &\sum_{i=0}^{k} i a_i + \sum_{i=0}^{k} b_i = 0 \end{aligned} , \tag{24}$$

In terms of the characteristic polynomial, the method is consistent if and only if

$$\rho(1) = 0, \rho'(1) = \sigma(1), \rho''(1) = 2\sigma(1). \tag{25}$$

Definition 3. The linear multistep method (17) is said to be consistent provided its error order p satisfies $p \geq 1$. It can be shown that this implies that the first and second characteristics polynomial are fulfilled as seen in (24).
Since the block multistep methods (8) and (13) are consistent as it has order $p > 1$. Adopting Hairer et al. [8] and Lambert [15], we can deduce the convergence of the block multistep methods (8) and (13).

4.5. Region of Absolute Stability of the Method

Theorem 5. (Second Root Condition)

From Bruce [4], the linear multistep method (17) is absolutely stable if all the roots r_j of the characteristic polynomial

$$\phi(r) = \rho(r) - z\sigma(r) \tag{26}$$

satisfy $|r_j| \leq 1$.

Definition 4. From Adetola et al. [2], Ken et al. [2] and Lambert [15], the linear multistep method (17) is said to be absolutely stable for a given $\bar{h}$ if, for that $\bar{h}$, all the roots r_s of (26) satisfy $|r_s| \le 1,\ s = 1,2,\ldots,j$; where $\bar{h} = \lambda^2 h^2$ and $\lambda = \frac{\partial f}{\partial y}$.

However, we choose and follow the boundary locus method to determine the region of absolute stability of the block methods and to obtain the roots of absolute stability, we substitute the test equation $y'' = -\lambda^2 y$ into the block formula to obtain

$$\rho(r) = \det\left(A^{(0)}Y_m(r) - A^{(1)}Y_{m-1}(r) - \left(B^{(0)}F_m(r)h^2\lambda^2 + B^{(1)}F_{m-1}(r)h^2\lambda^2\right)\right) = 0 \qquad (27)$$

Substituting $\bar{h} = 0$ in (27), we obtain all the roots of the derived equation to be equal to 1; hence, according to [4] defined on **theorem 5**, the block method is absolutely stable.

Therefore, the boundary of the region of absolute stability is given by

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} = \frac{r^2 - 2r + 1}{\frac{7}{6}r^3 - \frac{5}{12}r^2 + \frac{1}{3}r - \frac{1}{12}} \qquad (28)$$

Let $r = e^{i\theta} = \cos\theta + i\sin\theta$, therefore (1.4.13) becomes

$$\bar{h}(\theta) = \frac{\cos 3\theta - 3\cos\theta + 2}{\frac{17}{80}\cos 6\theta - \frac{71}{75}\cos 5\theta + \frac{59}{20}\cos 4\theta - \frac{199}{240}\cos 2\theta + \frac{9}{200}} \qquad (29)$$

(4.3)

Evaluating (1.4.14) at 120^0 within $[0, 180^0]$, which gives the interval of absolute stability to be $[-3.96, 0]$ after evaluation at interval of $\bar{h}(\theta)$. The stability region is shown in **Figure 1** and the enclosed region inside the boundary in **Figure 1** demonstrate the region of absolute stability of the proposed method.

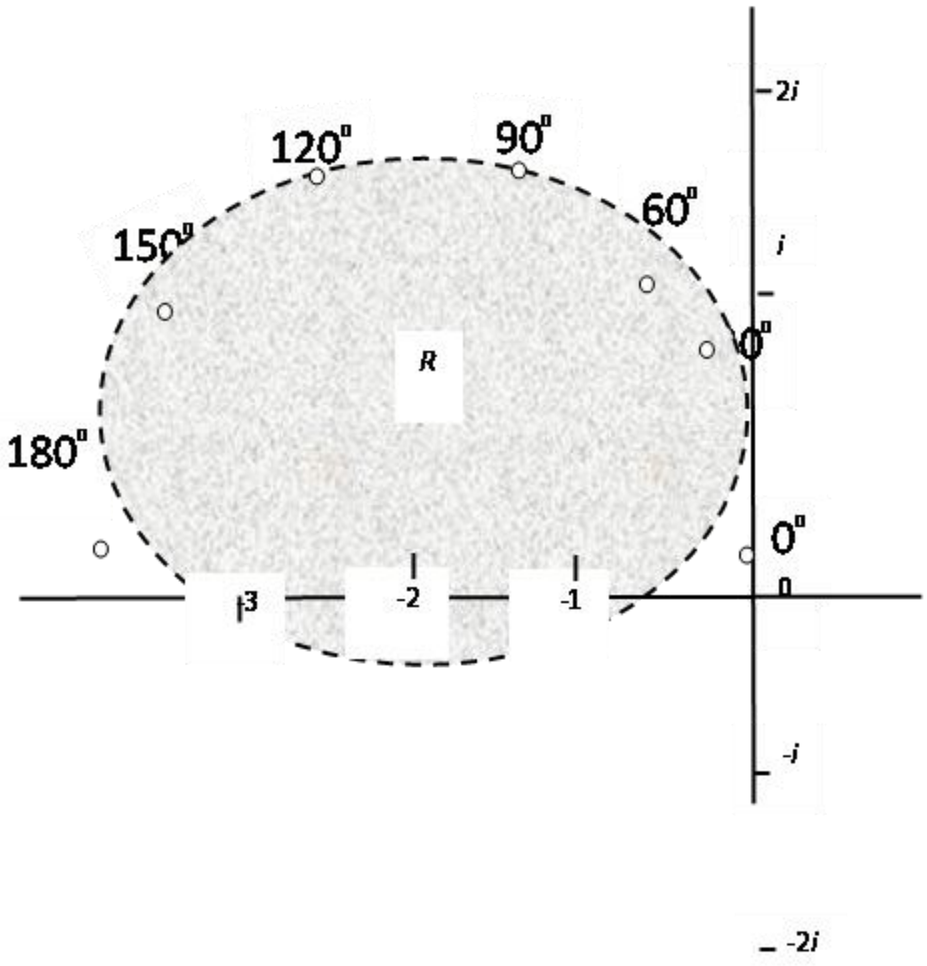

Fig. 1 showing the region of absolute stability of the block predictor-corrector mode, since the root of the stability polynomial is $r \leq 1$.

Note: Figure 1 is a freehand drawing.

4.6. Implementation of the Variable Step-Size Technique

Adopting Faires and Burden [7] and Lambert [15]:

(i) Predictor-Corrector techniques always generate two approximations at each step, so they are natural candidates for error-control adaptation.

(ii) To demonstrate the error-control procedure, a variable step-size predictor-corrector methods using 5-step explicit Adams-Bashforth method as predictor while the 4-step implicit Adams-Moulton method as corrector methods are constructed.

Firstly, the 5-step predictor Local Truncation Error (LTE) is

$$\frac{y(t_{i+1}) - w_{i+1}^{p}}{h} = C_{p+2}\, y^{(6)}(\xi_i)\, h^{6} \qquad (30)$$

Secondly, the 4-step corrector Local Truncation Error (LTE) is

$$\frac{y(t_{i+1}) - w^{c}_{i+1}}{h} = -C_{p+2}\, y^{(6)}(\xi)\, h^{6} \tag{31}$$

Where the 5-step predictor and 4-step corrector methods use this assumption such that the approximations $w_0, w_1 \cdots, w_i$ are all exact, w^{p}_{i+1} and w^{c}_{i+1} represents the predicted and corrected approximations given by the 5-step predictor and 4-step corrected methods.

To proceed further, we must make the assumption that for small valuesof h, we have

$$y^{(6)}(\xi_i) \approx y^{(6)}(\overline{u}_i) \tag{32}$$

The effectiveness of the error-control technique depends directly onthis assumption.

On subtracting (30) from (31) and combining the local truncation error estimates, we have

$$\frac{w^{c}_{i+1} - w^{p}_{i+1}}{h} \approx C_{p+2}\, y^{(6)}(\xi)\, h^{6} \tag{33}$$

Therefore, eliminating term involving $y^{(6)}(\xi)h^{6}$ in (31) yields finally the following approximation to the 4-step corrector local truncation error:

$$|\tau_{i+1}|(h) \approx C_{p+2}\frac{\left|w^{c}_{i+1} - w^{p}_{i+1}\right|}{h} < \varepsilon \tag{32}$$

Equation (32) is Adam's estimate for correcting to convergence which is bounded by a prescribed tolerance ε.

In addition, the error estimate (32) is used to decide whether to accept the results of the current step or to redo the step with a smaller step size. The step is accepted based on a test as described by (32) as seen in Uri and Linda [21].

As in Uri and Linda [21] and Zarina et al. [25], varying the step size is crucial for the effective performance of a discretization method. Step size adjustment for 5-step predictor and 4-step corrector block multistep methods using variable step has been stated earlier. On the given step, the user will provide a prescribed tolerance. In the block multistep, variable step-size strategy codes, the block solutions are accepted if the local truncation error, LTE is less than the prescribed tolerance. If the error estimate is greater than the accepted prescribed tolerance, the value of τ_{i+1} is rejected, the step is repeated with halving the current step size or otherwise, the step is multiply by 2. The error controls for the code was at the first point in the block because in general it had given us better results according to the new method.

Furthermore, equation (32) guarantees the convergence criterion of the method during the test evaluation.

Finally, a number of approximation assumptions have been made in thisdevelopment, so in practice a new step size (qh) is chosen conservatively, often as

$$qh = \left(\frac{\varepsilon}{2\left| w_{i+1}^{c} - w_{i+1}^{p} \right|} \right)^{\frac{1}{4}} \tag{33}$$

Equation (33) is used in deciding a new step size for the method.

5.0. Test Problems

The performance of the 5-step block predictor and 4-step corrector methods was carried out on non-stiff problems. For problem 1 and 2 the following tolerances 10^{-6}, 10^{-8}, 10^{-10}, 10^{-12}, and 10^{-14} was used to compare the performance of the newly proposed method with other existing methods as in [22].

Problem 1: Oscillation problem

The first problem to be considered is nonlinear and was extracted from Vigo and Ramos [22]. This was solved with idea of Adam's (block multistep, predictor-corrector) methods using variable step size technique. Falkner methods in predictor-corrector mode (PEC) using variable step size was a multistep scheme. Table 1 displays results of the comparisons for the code of variable-order, variable-step (VOVS), the variable step size Stormer method with orders n=6, 8, Falkner's method and Adam's method in predictor-corrector mode with the same order, variable step size. This problem is represented in the form

$$y'' + \sinh y = 0, \; y(0) = 0, \; y'(0) = 1$$

Problem 2: Van der Pol oscillator

Problem 2 was extracted from Vigo and Ramos [22]. However, Falkner method of order eight was designed and executed on k-step predictor-corrector methods using variable step size represented in multistep form. The newly proposed block multistep, predictor-corrector methods belongs to the family of Adams and was created to solve general second order ODEs using variable step size technique. Nevertheless, the well-known Vander Pol oscillator is given by

$$y'' - 2\xi(1 - y^2)y' + y = 0, \; y(0) = 0, \; y' = 0.5, \; x \in [0,400],$$

where $\xi = 0.025$ was solved without the damping term.

The computer codes are written in Mathematica and implemented on windows operating system using Mathematica 9 kernel. The computational results for problem 1-2 in Tables 1-2 are computed using the proposed method as well as the method in Vigo and Ramos [22].

5.1. Numerical Results

Notations

TOL: Tolerance Level
MTD: Method Employed
MAXE: Magnitude of the Maximum Error of the Computed Solution
ABMPC: Adam's Block Multistep Predictor-Corrector Methods

Table 1. Numerical results of Vigo and Ramos [22] and ABMPC for solving problem 1.

MTH	TOL	Maximum Errors
VOVS	10^{-6}	$15.23.10^{-3}$
STOR(6)	10^{-6}	$2.91.10^{-5}$
FALK(6)	10^{-6}	$2.16.10^{-5}$
ABMPC(5)	10^{-6}	$2.95847.10^{-6}$
VOVS	10^{-8}	$5.25.10^{-4}$
STOR(8)	10^{-8}	$6.92.10^{-6}$
FALK(8)	10^{-8}	$1.48.10^{-7}$
ABMPC(5)	10^{-8}	$4.83832.10^{-8}$

Table 2. Numerical results of Vigo and Ramos [22] and ABMPC for solving problem 2

MAXE	TOL	MAXE
$1.0685.10^{-4}$	10^{-6}	$4.16446.10^{-6}$
$1..7739.10^{-6}$	10^{-8}	$8.67599.10^{-8}$
$2.6132.10^{-8}$	10^{-10}	$6.41385.10^{-10}$
$1.1526.10^{-9}$	10^{-12}	$6.27726.10^{-12}$
$2.0986.10^{-12}$	10^{-14}	$1.30875.10^{-13}$

6.0. Discussions and Conclusion

From Table 1, Vigo and Ramos [22] was implemented using Falkner's method of variable step size technique which is a multistep scheme. The implementation of an explicit and implicit multistep (predictor-corrector) method as a single numerical solution cannot be compared to a block predictor-corrector methods whose solution points runs simultaneously in generating the required results. Hence, the newly

proposed 5-step predictor and 4-step corrector methods are preferable applying variable step size technique introduced by Adam's. Again, from Table 2, Vigo and Ramos [22] executed Falkner's method of explicit and implicit methods employing variable step size technique. Moreover, this cannot be compared with the result of the newly proposed block multistep, 5-step predictor and 4-step corrector methods which yields better accuracy in terms of the maximum error at all tested tolerance levels, since it was implemented using variable step size technique. In addition, this gives a better result at all tested tolerance levels.
Finally, the implementation of the Adam's method of variable step size technique comes with a huge task especially deciding on a suitable new step size for executing the problem. Although, this is the price of variable step size technique, but nevertheless, this yields the desired result with better accuracy.

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