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Adopting a Variable Step Size Approach in Implementing Implicit Block Multi-Step Method for Non-Stiff Ordinary Differential Equations

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Abstract: In this study, a variable step size approach is adopted in implementing Implicit Block Multi-step Method for solving non-stiff ordinary differential equations. This idea has many computational advantages when compared with other methods. They include designing a suitable step size/changing the step size, stopping criteria (prescribed tolerance level) and error control/minimization. This approach utilizes the estimates of the principal local truncation error on a pair of explicit and implicit of Adams type formulas which are implemented in $P(CE)^m$ mode. Gauss Seidel approach is employed for the implementation of the proposed method. Numerical experiments are given to show the efficiency of the method.

Key words: Variable step size technique, Implicit Block Multi-step Method, non-stiff ODEs, prescribed tolerance level, Gauss Seidel approach

INTRODUCTION

Linear multi-step methods generally come in families. The most popular for non-stiff problems is the Adams family while stiff problems belong to Backward Differentiation Formula (BDF) family as reported by Uri and Linda. According to Cash and Semnani (1993), the most two commonly used classes of formulae for the numerical solution of non-stiff initial-value problem are Adams and Runge-Kutta Methods. Nevertheless, Runge-Kutta and Adams formulae are frequently quite effective for solving non-stiff ODEs both types of methods possesses certain well-known computational disadvantages.

In this study, the intent has to do with Implicit Block Multi-step Method for the numerical integration of non-stiff ODEs of the form:

$$y'''(x) = f(x, y, y', y''),\ y(a) = \alpha,\ y'(a) = \beta,\quad y'''(a) = \Psi,\ x\in[a, b] \text{ and } f: R\times R^m \to R^m \tag{1}$$

The solution to Eq. 1 is in general, written as:

$$\sum_{i=1}^{j} \alpha_i y_{n+i} = h^3 \sum_{i=1}^{j} \beta_i f_{n+i} \tag{2}$$

where, the step size is h, $\alpha_j = 1$, α_i, $i = 1, ..., j$, β_j are unknown constants which are uniquely specified such that the formula is of order j as discussed by Akinfenwa *et al*. (2013).

We assume that $f\in R$ is sufficiently differentiable on $x\in[a, b]$ and satisfies a global Lipchitz condition, i.e., there is a constant $L\geq 0$ such that:

$$|f(x, y)-f(x, \bar{y})| \leq L|y-\bar{y}|,\ \forall y, \bar{y}\in R$$

Under this presumption (Eq. 1) assured the existence and uniqueness defined on $x\in[a, b]$ as discussed by Xie and Tian (2014).

Equation 1 has been virtually used in a broad variety of real life applications mostly in science and engineering field and other areas of applications. Researchers have suggested that the reduction of Eq. 1 to the system of first-order equations will be computationally expensive and wastage of human effort (Awoyemi, 2003; Mehrkanoon, 2011; Adesanya *et al*., 2013; Anake *et al*., 2013; Awoyemi *et al*., 2014). While others prefer to solve special third-order ODEs. This approach is well established in literature as discussed by Olabode and Yusuph (2009) and Mechee *et al*. (2013). Thus, there

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is the need to adopt a variable step size approach for solving directly (Eq. 1) based on Implicit Block Multi-step Method.

In the recent past, some researchers such as Awoyemi (2003), Olabode and Yusuph (2009), Adesanya *et al.* (2013), Anake *et al.* (2013) and Awoyemi *et al.* (2014), just to mention a few have examined and proposed a better method for solving general and special third order ODEs directly. Mehrkanoon implemented directly variable step size block multistep method for solving general third order ODEs. The method which combined a pair of explicit and implicit of Adams type formula implemented in $P(CE)^m$ mode. Moreover, this is directed towards backward differentiation formula. Majid developed two-point four step direct implicit block method in simple form of Adams-Moulton Method for solving directly the general third order (ODEs) applying variable step size. Again, this is addressed in the direction of stiff ODEs. Adesanya *et al.* (2013) implemented a new hybrid block method for the solution of general third order initial value problems of ODEs with fixed step size method. You and Chen (2013) compare a two-stage explicit RKT Method of fourth order and a three-stage explicit method of fifth order together with an implicit RKT Methods are considered as well. Awoyemi *et al.* (2014) constructed a five step P-Stable Method for the numerical integration of third order ODEs using fixed step size approach. Zunir and Kuboye developed an accurate implicit block method for integrating third order ODEs via interpolation and collocation of power series approximate solution.

Definition 1: According to Akinfenwa *et al.* (2013), a Block by Block Method is a method for computing vectors Y_0, Y_1, ... in sequence. Let the r-vector (r is the number of points within the block) Y_μ, F_μ and G_μ for n = mr, m = 0, 1, ... be given as $Y_w = (y_{n+1}, ..., y_{n+r})^T$, then the l-block r-point methods for (Eq. 1) are given by:

$$Y_w = \sum\nolimits_{i=1}^{j} A^{(i)} Y_{w-i} + h\sum\nolimits_{i=1}^{j} B^{(i)} F_{w-i}$$

where, $A^{(i)}$, $B^{(i)}$, i = 0, ..., j are r by r matrices as introduced by Fatunla (1990). Thus, from the above definition, a block method has the advantage that in each application, the solution is approximated at more than one point simultaneously. The number of points depends on the structure of the block method. Therefore, applying these methods can give quicker and faster solutions to the problem which can be managed to produce a desired accuracy (Majid and Suleiman, 2007; Mehrkanoon *et al.*, 2010). The main aim of this study is to propose an Implicit Block Multi-step Method for solving directly (Eq. 1) adopting variable step size approach on non-stiff ODEs. This approach possess some computational advantages like designing a suitable step size/changing the step size, stating the stopping criteria (prescribed tolerance level) and error control/minimization and help addressed the shortcomings stated above.

The block algorithm proposed in this study is based on interpolation and collocation. The continuous representation of the algorithm generates a main discrete collocation method to render the approximate solution Y_{n+i} to the solution of (Eq. 1) at points x_{n+i}, i = 1, ..., k as discussed by Akinfenwa *et al.* (2013).

MATERIALS AND METHODS

Derivation of the method: Following, Akinfenwa *et al.* (2013) in this study, the aim is to derive the principal block method of the form (Eq. 2). We move forward by seeking an approximation of the exact solution y(x) by assuming a continuous solution Y(x) of the form:

$$Y(x) = \sum_{i=0}^{q+k-1} m_i \vartheta_i(x) \tag{3}$$

such that $x \in [a, b]$, m_i are unknown coefficients and $\vartheta_i(x)$ are polynomial basis functions of degree q+k-1 where q is the number of interpolation point and the collocation points k are respectively chosen to satisfy $q = j \geq 3$ and $k > 1$. The integer $j \geq 1$ denotes the step number of the method. We thus construct a j-step block method with $\vartheta_i(x) = (x - x_i/h)^i$ by imposing the following conditions:

$$\sum_{i=0}^{q} m_i \left(\frac{x - x_i}{h}\right)^i = y_{n-i},\ i = 0, ..., q-1 \tag{4}$$

$$\sum_{i=0}^{q} m_i^i (i-1)(i-2) \left(\frac{x - x_i}{h}\right)^{i-3} = f_{n+i},\ i \in Z \tag{5}$$

where, y_{n+i} is he approximation for the exact solution $y(x_{n+i})$, $f_{n+i} = f(x_{n+i}, y_{n+i})$, n is the grid index and $x_{n+i} = x_n + ih$. It should be noted that (Eq. 4 and 5) leads to a system of q+1 equations of the AX = U where:

$$A = \left|\begin{matrix} x_n^0 & x_n^1 & x_n^2 & x_n^3 & . & . & . & x_n^q \\ x_{n-1}^0 & -x_{n-1}^1 & x_{n-1}^2 & -x_{n-1}^3 & . & . & . & x_{n-1}^q \\ & \cdots & \cdots & & & & & \\ x_{n-k}^0 & x_{n-k}^1 & x_{n-k}^2 & x_{n-k}^3 & . & . & . & x_{n-k}^q \\ 0 & 0 & 0 & k(k-1)(k-2)x_{n-k}^3 & . & . & . & k(k-1)(k-2)x_{n-k}^q \\ 0 & 0 & 0 & k(k-1)(k-2)x_{n+1}^3 & . & . & . & k(k-1)(k-2)x_{n+1}^q \\ & & & \cdots & & & & \\ 0 & 0 & 0 & k(k-1)(k-2)x_{n+k}^3 & . & . & . & k(k-1)(k-2)x_{n+k}^q \end{matrix}\right| \tag{6}$$

$$X = \left[X_0, X_1, X_2, X_3, ..., X_k\right]^T$$

$$U = \left[f_n, f_{n-1}, ..., f_{n-k-1}, f_{n+1}, f_{n+2}, ..., f_{n+k}, y_n, y_{n-1}, ..., y_{n-k-1}\right]^T$$

Solving Eq. 6 using Mathematica, we get the coefficients of m_i and substituting the values of m_i into Eq. 4 and after some algebraic computation, the implicit block multistep method is obtain as:

$$\sum\nolimits_{i=0}^{q-1} \alpha_i y_{n-i} = h^3\left[\sum\nolimits_{i=0}^{q-1} \beta_i f_{n-i} + \sum\nolimits_{i=0}^{q-1} \beta_i f_{n+i}\right] \tag{7}$$

where, α_i and β_i are continuous coefficients. Differentiating Eq. 7 once and twice, we have a block of first and second order derivatives which can be used to determine the derivative term in the initial value problem (Eq. 1) as presented by Ehigie *et al*. (2011):

$$\sum_{i=1}^{k} y'_{n+i} = \frac{1}{h}\sum\nolimits_{i=0}^{q-1} \alpha_i y_{n-i} + h^2\left[\sum\nolimits_{i=0}^{q-1} \beta_i f_{n-i} + \sum\nolimits_{i=0}^{q-1} \beta_i f_{n+i}\right] \tag{8}$$

$$\sum_{i=1}^{k} y''_{n+i} = \frac{1}{h^2}\sum\nolimits_{i=0}^{q-1} \alpha_i y_{n-i} + h\left[\sum\nolimits_{i=0}^{q-1} \beta_i f_{n-i} + \sum\nolimits_{i=0}^{q-1} \beta_i f_{n+i}\right] \tag{9}$$

Investigation of some theoretical properties

Order of accuracy: Adopting Lambert (1977), Akinfenwa *et al*. (2013) and Awoyemi *et al*. (2014), we define the associated linear multi-step method (Eq. 7) and the difference operator as:

$$L\left[y(x); h\right] = \sum_{i=0}^{j}\left[\alpha_i y(x+ih) + h^3\beta_i y'''(x+ih)\right] \tag{10}$$

Assuming that y(x) is sufficiently and continuously differentiable on an interval [a, b] and that y(x) has as many higher derivatives as needed then, we write the terms in Eq. 10 as a Taylor series expression of $y(x_{n+i})$ and $f(X_{n+i}) \equiv y'''(X_{n+i})$ as:

$$y(X_{n+i}) = \sum_{k=0}^{\infty} \frac{(ih)}{k!} y^{(k)}(X_n) \text{ and } y'''(X_{n+i}) = \sum_{k=0}^{\infty} \frac{(ih)}{k!} y^{(k+1)}(X_n) \tag{11}$$

Substituting Eq. 10 and 11 into Eq. 7 we obtain the following expression:

$$L[y(x); h] = c_0 y(x) + c_1 h y^{(1)}(x) + ... + c_{p+2} h^{p+2} y^{(p+2)}(x) + ..., \tag{12}$$

Agreeing with Lambert (1977) and Akinfenwa *et al*. (2013), we observed that the Implicit Block Multi-step Method of Eq. 7 has order p, if c_{p+2}, p = 0, 1, 2, ..., i = 1, 2, ..., j are given as follows:

$$\begin{aligned} c_0 &= \alpha_0 + \alpha_1 + \alpha_2 + ... + \alpha_k \\ c_1 &= \alpha_0 + 2\alpha_1 + ... + k\alpha_k \\ c_2 &= \frac{1}{2!}(\alpha_0 + \alpha_1 + \alpha_2 + ... + k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + ... + \beta_k) \\ c_q &= \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + ... + k^q\alpha_k) - \\ &\quad \frac{1}{(q-3)!}(\beta_0 + 2^{q-3}\beta_1 + \beta_2 + ... + k^{q-3}\beta_k), \; q = 4, 3, ... \end{aligned}$$

Thus, the method (Eq. 7) has order $p \geq 1$ and error constants given by the vector, $c_{p+3} \neq 0$. Agreeing with Lambert (1977), we say that the method (Eq. 2) has order p if:

$$\begin{aligned} &L[y(x); h] = O(h^{p+3}), \\ &C_0 = C_1 = ... = C_p = C_{p+1} = C_{p+2} = 0, \; C_{p+3} \neq 0 \end{aligned} \tag{13}$$

Therefore, C_{p+3} is the error constant and $C_{p+3} h^{p+3} y^{(p+3)}(x_n)$ is the principal local truncation error at the point x_n. Since, this definition stated above is true for first and second order ODEs according to Lambert (1977) then it is true for higher order ODEs.

Stability analysis: In order to analyze the method for stability (Eq. 7) is normalize and written as a block method given by the matrix finite difference equations as by Akinfenwa *et al*. (2013):

$$A^{(0)}Y_m = A^{(1)}Y_{m-1} + h^3(B^{(0)}F_m + B^{(1)}F_{m-1}) \quad (14)$$

Where:

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+1} \\ . \\ . \\ . \\ y_{n+r} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{n-r+1} \\ y_{n-r+2} \\ . \\ . \\ . \\ y_n \end{bmatrix}, F_m = \begin{bmatrix} f_{n+1} \\ f_{n+1} \\ . \\ . \\ . \\ f_{n+r} \end{bmatrix}, F_{m-1} = \begin{bmatrix} f_{n-r+1} \\ f_{n-r+2} \\ . \\ . \\ . \\ f_n \end{bmatrix}$$

The matrices $A^{(0)}$, $A^{(1)}$, $B^{(0)}$, $B^{(1)}$ are r by r matrices with real entries while Y_m, Y_{m-1}, F_m, F_{m-1} are r-vectors specified above.

Following Lambert (1977), we adopted the boundary locus method to determine the region of absolute stability of the block method and to obtain the roots of absolute stability. Substituting the test equation $y' = -\lambda y$ and $\bar{h} = h^3\lambda^3$ into the block (Eq. 14) to obtain:

$$\rho(r) = \det\left[r\left(A^{(0)} + B^{(0)}h^3\lambda^3\right) - \left(A^{(1)} - B^{(1)}h^3\lambda^3\right)\right] = 0 \quad (15)$$

Substituting h = 0 in Eq. 15, we obtain all the roots of the derived equation to be equal to be less than or equal to 1. Hence, according to Lambert (1977), the block method is absolutely stable. Thus as seen by Lambert (1977), the boundary of the region of absolute stability can be obtained by substituting Eq. 7 into:

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} \quad (16)$$

and let $r = e^{i\theta} = \cos\theta + i\sin\theta$ then after simplification together with evaluating Eq. 16 within [0°, 180°]. Therefore, the boundary of the region of absolute stability lies on the real axis (Fig. 1).

Implementation of the method: Adopting Lambert (1977), since this is implemented in the $P(EC)^m$ mode then it becomes beneficial if the predictor and the corrector are individually of the same order and this prerequisite makes it essential for the stepnumber of the predictor to be greater than that of the corrector. Subsequently, the mode $P(EC)^m$ can be formally determined as follows for m = 1, 2, ...:

$$P(EC)^m: y^{[0]}_{n+j} + \sum_{i=0}^{j-1} \alpha^{*}_i y^{[m]}_{n+i} = h^3 \sum_{i=0}^{j-1} \beta^{*}_i f^{[m]}_{n+i},\ f^{[s]}_{n+j} \equiv f(x_{n+j}, y^{[s]}_{n+j}),$$

$$y^{[s+1]}_{n+j} + \sum_{i=0}^{j-1} \alpha_i y^{[m]}_{n+i} = h^3\beta_j f^{[s]}_{n+j} + h^3 \sum_{i=0}^{j-1} \beta_i f^{[m-1]}_{n+i},\},\ s = 0, 1, ..., m-1 \quad (17)$$

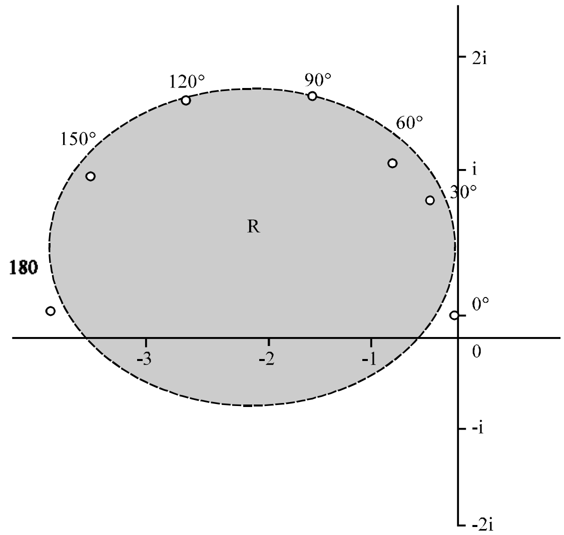

Fig. 1: Showing the region of absolute stability of the Implicit Block Multi-step Method if the root of the stability polynomial is less than or equal to one for non-stiff problems

Note that as $m \to \infty$, the consequence of calculating with the above mode will tend to those given by the mode of correcting to convergence.

Moreover, predictor-corrector pair based on Eq. 1 can be applied. The mode $P(EC)^m$ specified by Eq. 17 where h^3 is the step size. Since, the predictor and corrector both have the same order p, Milne's device is relevant.

According to Lambert (1977), Milne's device proposes that it possible to estimate the principal local truncation error of the predictor-corrector method without estimating higher derivatives of y(x). Assume that p = p* where p* and p represents the order of the predictor and corrector method with the same order. Now for a method of order p, the principal local truncation errors can be written as:

$$C^{*}_{p+3}h^{p+3}y^{(p+3)}(X_n) = y(X_{n+j}) - W_{n+j} + O(h^{p+3}) \quad (18)$$

Also:

$$C_{p+3}h^{p+3}y^{(p+3)}(X_n) = y(X_{n+j}) - C_{n+j} + O(h^{p+3}) \quad (19)$$

where, W_{n+j} and C_{n+j} are called the predicted and corrected approximations given by method of order p while C^{*}_{p+3} and C_{p+3} are independent of h.

Neglecting terms of degree p+3 and above, it is easy to make estimates of the principal local truncation error of the method as:

$$C_{p+3}h^{p+3}y^{(p+3)}(X_n)\frac{C_{p+3}}{C^{*}_{p+3}\ C_{p+3}}\left|W_{n+j} - C_{n+j}\right| \quad (20)$$

Noting the fact that $C_{p+3} \neq C^{*}_{p+3}$ and $W_{n+j} \neq C_{n+j}$. Moreover, the estimate of the principal local truncation error (Eq. 17) is used to decide whether to accept the results of the current step or to redo the step with a smaller step size. The step is accepted based on a test as described by Eq. 17 as in Uri and Linda. Equation 17 is the convergence test otherwise called Milne's estimate for correcting to convergence. Furthermore, equation 17 guarantees the convergence criterion of the method during the test evaluation.

RESULTS AND DISCUSSION

Numerical experiments: The performance of the Implicit Block Multi-step Method was carried out on non-stiff problem as discussed below.

Experiment 1: The first experiment to be discussed is given in Majid which was extracted from Awoyemi (2003). Moreover, Awoyemi (2003) developed a P-Stable Linear Multi-step Method for solving non-stiff third order ODEs using fixed step size. On the other hand, Majid constructed a two-point four step block method for solving stiff third order ODEs applying variable step size method. In addition, the newly proposed implicit block method is designed to evaluate non-stiff third order ODEs employing variable step size technique (Table 1). The problem is given as follows:

$$y'''(x) + 4y'(x) = x,\ y(0) = y'(0) = 0,\ y''(0) = 1,\ 0 \leq x \leq b$$

with theoretical solution:

$$y(x) = \frac{3}{16}(1\text{-}\cos 2x) + \frac{1}{8}X^2$$

Experiment 2: Secondly, experiment 5.2 is given by Olabode and Yusuph (2009) and later, Awoyemi *et al.* (2014). Olabode and Yusuph (2009) implemented experiment 2 on a new block method for special third order ODEs using fixed step size. While, Awoyemi *et al.* (2014) applied experiment 2 on five-step P-stable method for the numerical integration of third order ODEEs using the same fixed step size approach. Moreover, the newly proposed Implicit Block Multi-step Method is formulated to compute non-stiff third order ODEs employing variable step size technique (Table 2). The experiment is given as follows:

$$y'''(x) + y(x) = 0,\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 1,\ 0 \leq x \leq 1$$

with theoretical solution:

$$y(x) = e^{-x}$$

Table 1: Numerical results by Majid and NPM for solving problem 1

B	MAXE Majid	TOL	MAXE NPM
5.0	4.66 (-7)	10^{-6}	2.10218 (-7)
10.0	4.66 (-7)	10^{-6}	-
15.0	4.66 (-7)	10^{-6}	-
20.0	4.66 (-7)	10^{-6}	-
5.0	9.14 (-8)	10^{-8}	4.215148 (-8)
10.0	2.43 (-8)	10^{-8}	-
15.0	2.63 (-8)	10^{-8}	-
20.0	2.63 (-8)	10^{-8}	-
5.0	1.53 (-10)	10^{-10}	6.6431 (-11)
10.0	1.53 (-10)	10^{-10}	-
15.0	1.54 (-10)	10^{-10}	-
20.0	1.29 (-9)	10^{-10}	-

Table 2: Numerical results by Olabode and Yusuph (2009) and Awoyemi *et al.* (2014) compared with NMP for solving problem 2

MAXE (Olabode and Yusuph, 2009)	MAXE (Awoyemi *et al.*, 2014)	PTOL	MAXE NMP
1.36929 (-9)	2.1760 (-12)	10^{-10}	2.64623 (-11)
3.12273 (-8)	1.3935 (-11)	-	6.80603 (-11)
1.27694 (-7)	3.4443 (-11)	-	1.06455 (-10)
3.25196 (-7)	6.4477 (-11)	-	-
6.54730 (-7)	1.0316 (-10)	-	-
1.44406 (-6)	1.4979 (-10)	-	-
1.81781 (-6)	2.0486 (-10)	-	-
2.69774 (-6)	2.6756 (-10)	-	-
3.80241 (-6)	6.9382 (-10)	-	-
5.14755 (-6)	1.4224 (-10)	-	-
-	-	10^{-12}	1.03326 (-11)
-	-	-	3.24585 (-11)
-	-	-	1.90709 (-10)

Table 3: Numerical results of NPM at different prescribed tolerance levels of it's implementation for solving problem 3

PTOL	MAXE NPM
10^{-8}	5.15404 (-8)
-	1.88106 (-8)
-	2.4276 (-7)
10^{-10}	9.09765 (-11)
-	1.88106 (-10)
-	2.04415 (-10)
10^{-12}	1.55431 (-14)
-	5.10703 (-14)
-	2.70006 (-13)

Experiment 3 biomass transfer: Experiment 3 is extracted from www.math.edu/~gustafso/2250systems-de.pdf and solutions are provided using first order method for solving ODEs analytically. The newly proposed method transform the first order systems of first order ODEs into third order ODEs before they were successfully implemented using variable step size technique (Table 3). The experiment 3 is given as follows:

- w(x) = Biomass decayed into humus
- y(x) = Biomass of dead trees
- z(x) = Biomass of living trees
- x = Time in decades (decade = n years)

A typical biological model is:

$$W'(x) = -w(x) + 3y(x),$$
$$y'(x) = -3y(x) + 5z(x),\ z'(x) = -5z(x)$$

The method of substitution is employed in converting the above stated systems of first order ODEs into third order ODEs. The conversion is expressed below:

$$W'''(x) + 9W''(x) + 23W'(x) + 15w(x) = 0,$$
$$w(0) = W'(0) = 0,\ W''(0) = 1$$

with theoretical solution:

$$w(x) = \frac{e^{-x}}{8} - \frac{e^{-3x}}{4} + \frac{e^{-5x}}{8}$$

CONCLUSION

Majid constructed two-point four step block method for solving directly general third order ODEs. The scheme which is specifically designed to solve stiff ODEs but rather, solved problems were based on non-stiff ODEs extracted by Awoyemi (2003). However, the newly proposed implicit block multistep method is developed to solve directly general third order ODEs with particular interest on non-stiff problems. Hence, in comparison with the maximum errors at all tolerance levels of 10^{-6}, 10^{-8} and 10^{-10}, the newly proposed method performs better than Majid.

The results by Olabode and Yusuph (2009) as well as the newly proposed method. Olabode and Yusuph (2009) design a method which examines special third order ODEs using fixed step size while Awoyemi *et al.* (2014) constructed a five step P-Stable Method to numerically integrate third order ODEs applying fixed step size. Nevertheless, both results cannot be compared with the newly proposed method which is implemented employing variable step size technique. Moreover, comparing the results of both maximum errors with the newly proposed method, the newly proposed method is more efficient and perform better at all prescribed tolerance levels of 10^{-10} and 10^{-12}.

The prescribed tolerance levels of 10^{-8}, 10^{-10}, 10^{-12} and maximum error results. This shows that the newly proposed method which is specifically designed for non-stiff ODEs has shown to be more efficient and reliable employing variable step size technique in solving real life problem.

RECOMMENDATION

The Implicit Block Multi-step Method is executed on windows operating system and written in Mathematica language.

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NOMENCLATURE

- TOL: Tolerance Level
- PTOL: Prescribed Tolerance Level
- MAXE NPM: Magnitude of the Maximum Errors of the Newly Proposed Method
- MAXE: Magnitude of the Maximum Errors by Olabode and Yusuph (2009)
- MAXE: Magnitude of the Maximum Errors of Majid
- MAXE: Magnitude of the Maximum Errors by Awoyemi *et al.* (2014)

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