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Variational Stability for Kurzweil Equations associated with Quantum Stochastic Differential Equations}

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Abstract: In this work, The Lyapunov's method is used to establish all kinds of variational stability of solution of quantum stochastic differential equations associated with the Kurzweil equations. The results here generalize analogous results for classical initial value problems to the noncummutative quantum setting involving unbounded linear operators on a Hilbert space. The theory of Kurzweil equations associated with quantum stochastic differential equations provides a basis for subsequent application of the technique of topological dynamics to the study of quantum stochastic differential equations as in the classical cases.

Key words: Non classical ODE; Kurzweil equations; Noncommutative stochastic processes; Variational stability; Asymptotic variational stability.

INTRODUCTION

Most differential equations, deterministic or stochastic, cannot be solved explicitly as given in the following references, Ekhaguere (1992), Milman and Myskis (1960), Ayoola (2001(a)). Nevertheless we can often deduce a lot of useful information by qualitative analysis about the behavior of their solutions from the functional form of their coefficients. The long term asymptotic behaviour and sensitivity of the solutions to small changes is of great interest. This is very important especially in measurement errors, initial values and many more. Variational stability is a generalized concept which is suitable for the class of generalized nonclassical ordinary differential equations studied here because of the local finiteness of the variation of a solution.

The theory of qualitative properties of solutions of ordinary differential equations and generalized ordinary differential equations such as stability, convergence, boundedness, etc. have received series of attention in recent years Bacciotti and Rosier (2005), Luis (1949), Pandit (1977), Ping and Yuan (2001), Rama and Hari (1977), Samojilenko (1977, 1981) and Schwabik (1984).

It is worth noting that this is the first time variational stability of quantum stochastic differential equation (QSDE) is considered here. Existence and uniqueness of solution of the Kurzweil equation associated with QSDEs have been established in Bishop (2012) under a more general Lipschitz condition.

In this paper, we investigate all kinds of variational stability of solution of Kurzweil equations associated with QSDE introduced by Hudson and Parthasarathy Hudson and Parthasarathy (1984). We employ the Kurzweil equation associated with this class of QSDE to establish results on stability. This research is strongly motivated by the need to create a framework for the application of the techniques of topological dynamics to the study of quantum stochastic differential equations as obtained in the case of ordinary differential equations Artstein (1977), Henstock (1972), Schwabik (1989, 1992). The space of Kurzweil equations (1.4) associated with QSDEs will then be a completion of the space of the equivalent non classical first order ordinary differential equations (1.3) as observed by Ayoola (2001(b)).

The rest of this paper is organized as follows. Section 2 will be devoted to some fundamental concepts, notations, definitions and structures that will be employed in subsequent sections. In section 3, we shall discuss the concept of variational stability of the Kurzweil equation associated with QSDE. Here, results on variational stability, variational attracting, relationship between variational attracting and asymptotic variational stability will be established using their definitions and the converse method. Some preliminary results which will be used to establish the major results will be established in section 4. Our major results will be established in section 5. This section will be devoted to variational stability and asymptotic variational stability using the method of Lyapunov.

In what follows, as in Ayoola (2001(b)), Bishop (2012) and Ekhaguere (1992), we employ the locally convex topological state space $Ad(\tilde{A})$, $Ad(\tilde{A})_{wac}$, $L^{p}_{loc}(\tilde{A})$, $L^{\infty}_{loc}(\mathbb{R}_+)$, $BV(\tilde{A})$ and the integrator processes Λ_{Π}, A^{+}_{f}, A_g for $f, g \in L^{\infty}_{\gamma, loc}(\mathbb{R}_+)$, $\pi \in L^{\infty}_{B(\gamma), loc}(\mathbb{R}_+)$, and E, F, G, H lying in $Loc^{2}_{loc}(I \times \tilde{A})$. We consider the quantum stochastic differential equation given by

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$$dX(t) = E\big(t, X(t)\big)d\Lambda_\pi(t) + F\big(t, X(t)\big)dA_f^+(t) + G\big(t, X(t)\big)dA_g(t) + H\big(t, X(t)\big)dt$$
$$X(t_0) = X_0, \qquad t \in [t_0, T], \qquad (1.1)$$

In equation (1.1), the coefficients E, F, G, and H lie in a certain class of stochastic processes for which quantum stochastic integrals against the gauge, creation, annihilation processes Λ_π, A_f^+, A_g and the Lebesgue measure t are defined in Ekhaguere (1992). Equation (1.1) is understood in integral form as

$$X(t) = X_0 + \int_0^t (E(t, X(s))d\Lambda_\pi(s) + F\big(s, X(s)\big)dA_f^+(s) + G(s, X(s))dA_g(s) + H(s, X(s))ds), \quad t \,\epsilon\, [0, T] \qquad (1.2)$$

In the work of Ekhaguere (1992), the Hudson and Parthasarathy (1984) quantum stochastic calculus was employed to establish the equivalent form of quantum stochastic differential equation (1.1) given by

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle = P(X(t), t)(\eta, \xi)$$
$$X(t_0) = X_0\,, \quad t \,\epsilon\, [t_0, T]\ , \qquad (1.3)$$

where η, ξ lie in some dense subspaces of some Hilbert spaces which has been defined in Ekhaguere (1992). For the explicit form of the map $P(x,t) \to P(x,t)(\,\eta, \xi\,)$ appearing in equation (1.3), see Bishop (2012). Equation (1.3) is a first order non-classical ordinary differential equation with a sesquilinear form valued map P as the right hand side.

In Ayoola (2001(b)), the equivalence of the non-classical ordinary differential equation (1.3) with the associated Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = DF(X(\tau), t)(\eta, \xi)\,, \qquad t \,\epsilon\, [t_0, T] \qquad (1.4)$$

was established along with some numerical approximations. The map F in (1.4) is given by

$$F(x,t)(\eta, \xi) = \int_0^t P\,(x, s)(\eta, \xi)ds \qquad (1.5)$$

Fundamental Concepts and Notations:

We shall employ certain spaces of maps (introduced above) whose values are sesquilinear forms on $(\mathbb{D} \underline{\otimes} \mathbb{E}\,)$.

2.1 Definition: A member $z \in L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E}\,)$ is:

i. absolutely continuous if the map t→ $z(t)(\eta, \xi)$ is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$
ii. of bounded variation if over all partition $\{t_j\}_{j=0}^n$ of I,
$\mathrm{Sup}_I(\sum_{j=1}^n |Z(t_j)(\eta, \xi) - Z(t_{j-1})(\eta, \xi)|) < \infty.$
iii. of essentially bounded variation if z is equal almost everywhere to some member of $L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E}\,)$ of bounded variation.
iv. A stochastic process $X\ : [t_0, T] \longrightarrow \tilde{\mathcal{A}}$ is of bounded variation if
$\mathrm{Sup}\ (\sum_{j=1}^n |\langle \eta, X(t_j)\xi\rangle - \langle \eta, X(t_{j-1})\xi\rangle|) < \infty.$
for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and where supremum is taken over all partitions $\{t_j\}_{j=0}^n$ of I.

1.2 ***Notation:*** We denote by BV(Ã) the set of all stochastic processes of bounded variation on I.

2.3 Definition: For x ∈ BV(Ã), define for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$,

$$\mathrm{Var}_{[a,b]}X_{\eta\xi} = \mathrm{Sup}_\tau(\textstyle\sum_{j=1}^n \|X(t_j) - X(t_{j-1})\|_{\eta\xi}).$$

where τ is the collection of all partitions of the interval [a, b] ⊂ I. If [a, b] = I, we simply write $\mathrm{Var}_I X_{\eta\xi} = \mathrm{Var}X_{\eta\xi}$. Then $\{\mathrm{Var}X_{\eta\xi}, \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\,\}$ is a family of seminorms which generates a locally convex topology on BV(Ã).

2.4 Notation:

i. We denote by $\overline{BV}$(Ã) the completion of BV(Ã) in the said topology.

ii. For any member Z of $L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ of bounded variation, we write $\mathrm{Var}Z_{\eta\xi}$ for its variation on [a, b] ⊆ I.
iii. for any arbitrary complex valued map f : I ⟶ ℂ of bounded variation we write

$$\mathrm{Var}_{[a,b]}Z = \mathrm{Sup}_{\tau}(\sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|)$$

For [a, b] ⊆ I , where τ is the collection of all partitions of [a, b].

2.5 Definition: For each $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ let $h_{\eta\xi}: [t_0, T] \longrightarrow \mathbb{R}$ be a family of non decreasing function defined on [t_o, T] and $W: [0, \infty) \longrightarrow \mathbb{R}$ be a continuous and increasing function such that $W(0) = 0$. Then we say that the map $F : \hat{\mathcal{A}} \times [t_0, T] \longrightarrow sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$ belongs to the class $\mathcal{F}(\hat{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ for each $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ if for all
$x, y \in \hat{\mathcal{A}},\ t_2 t_1 \in [t_0, T]$
(i). $|P(x, t_2)(\eta, \xi) - P(x, t_1)(\eta, \xi)| \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ (2.1)
(ii). $|P(x, t_2)(\eta, \xi) - P(x, t_1)(\eta, \xi) + P(y, t_2)(\eta, \xi) - P(y, t_1)(\eta, \xi)|$
$\leq W(\|x - y\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ (2.2)

Next we introduce the concept of variational stability of quantum stochastic differential equations (1.1). In Ayoola (2001(b)), it has been shown that the map (x, t) ⟶ F(x, t)(η, ξ) is of class $\mathcal{F}(\hat{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ and the map (x, t) ⟶ P(x, t)(η, ξ) is of class $C(\hat{\mathcal{A}} \times [t_0, T], W)$. Existence of solution of equation (1.3) and the associated Kurzweil equation (1.4) has been established in Ayoola (2001(b)) and Bishop (2012). This takes care of both the Lipschitz case and the general case respectively. Consequently, existence results enable one to investigate variational stability of solutions. This investigation is made possible since solutions of equation (1.3) are quantum stochastic processes of bounded variations. Variational stability deals with the measurement of the distance between solutions in the space of stochastic processes of bounded variation using the seminorms defined by their bounded variations.

Since the solution x ∈ $Ad(\tilde{A})_{wac}$ of equation (1.3) are stochastic processes of bounded variations, we introduce and study the issue of variational stability of (1.3) in analogy to the case of generalized ordinary differential equation of classical type Schwabik (1992). In addition to other assumptions, assume that the map (x, t) ⟶ F(x, t)(η, ξ) satisfies

$$F(0, t_2)(\eta, \xi) - F(0, t_1)(\eta, \xi) = 0 \qquad (2.3)$$

For every $t_1, t_2 \in [0, T]$ and for arbitrary $, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. This assumption evidently implies that

$$\int_{s_1}^{s_2} DF(0, s)(\eta, \xi) = F(0, s_2)(\eta, \xi) - F(0, s_1)(\eta, \xi) =$$

$$= \int_{s_1}^{s_2} P(0, s)(\eta, \xi) ds = 0, s_1, s_2 \in [0, T]$$

and therefore the trivial process given by x(s) ≡ 0, for s ∈ [0,T] is a solution of the Kurzweil equation (1.4).

Next we introduce some concepts of stability of the trivial solution x(s) ≡ 0, s ∈ [0, T] of equation (1.4). All through the remaining sections we take $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ to be arbitrary.

2.6 Definition: The trivial solution x ≡ 0 of equation (1.4) is said to be variationally stable if for every ε > 0, there exists $\delta(\eta, \xi, \varepsilon) := \delta_{\eta\xi} > 0$ such that if y : [0, T] ⟶ Ã is a stochastic process lying in $Ad(\tilde{A})_{wac} \cap BV(\tilde{A})$ with

$$\|Y(0)\|_{\eta\xi} < \delta_{\eta\xi}$$

and

$$Var(\langle\eta, y(s)\xi\rangle - \int_0^s DF(y(\tau), t)(\eta, \xi)) < \delta_{\eta\xi}$$

then we have

$$\|y(t)\|_{\eta\xi} < \varepsilon$$

For all t ∈ [0, T] and for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

2.7 Definition: The trivial solution x ≡ 0 of equation (1.4) is said to be variationally attracting if there exists $\delta_0 >$ 0 and for every ε > 0, there exists A = A(ε) ≥ 0,
0 ≤ A(ε) < T and B(η, ξ, ε) = B > 0 such that if y ∈ $Ad(\tilde{A})_{wac} \cap BV(\tilde{A})$ with $\|y(0)\|_{\eta\xi} < \delta_0$ and

$$Var(\langle\eta, y(s)\xi\rangle - \int_0^s DF(y(\tau), t)(\eta, \xi)) < B$$

Then

$$\|y(t)\|_{\eta\xi} < \varepsilon \quad \text{for all } t \in [A, T].$$

2.8 Definition: The trivial solution x ≡ 0 of equation (1.4) is called variationally asymptotically stable if it is variationally stable and variationally attracting.

Together with (1.1) we consider the perturbed QSDE

$$dx(t) = E(x(t), t)d\,\Lambda_{\Pi}(t) + F(x(t), t)dA_f^{+}(t) + G(x(t), t)dA_g(t) + (H(x(t), t) + p(t))dt$$

$$x(t_0) = x_0\,,\ t \in [0, T] \qquad (2.4)$$

where p ∈ Ad(Ã)$_{wac}$ ∩ BV (Ã). The perturbed equivalent form of (2.4) is given by

$$\frac{d}{dt}\langle\eta, x(t)\xi\rangle = P(x,t)(\eta,\xi) + \langle\eta, p(t)\xi\rangle \qquad (2.5)$$

The Kurzweil equation associated with the perturbed QSDE (2.5) then becomes

$$\frac{d}{d\tau}\langle\eta, x(\tau)\xi\rangle = D[F(x,t)(\eta,\xi) + Q(t)(\eta,\xi)] \qquad (2.6)$$

Where Q : [0,T] → Ã belongs to Ad(Ã)$_{wac}$ ∩ BV (Ã) as well.

We remark here that the map P given by equation (2.5) is of class C(Ã×[a, b], W) while (1.5) becomes

$$F(x, t)(\eta, \xi) + Q(t)(\eta, \xi) = \int_0^t [P(x,s)(\eta,\xi) + \langle\eta, p(s)\xi\rangle]ds \qquad (2.7)$$

where the left hand side $F(x,t)(\eta,\xi) + Q(t)(\eta,\xi)$ of (2.7) is of class $\mathcal{F}$(Ã × [a, b], $*\,h_{\eta\xi}$, W) , $\langle\eta, p(s)\xi\rangle := Q(t)(\eta,\xi)$ and

$$*\,h_{\eta\xi}(t) = h_{\eta\xi}(t) + Var_{[0,T]}Q(t)(\eta,\xi) \qquad (2.8)$$

In [4], it has been shown that the map $F(x,t)(\eta,\xi)$ in equation (1.5) is of class $\mathcal{F}$(Ã × [a, b], $h_{\eta\xi}$, W) where $h_{\eta\xi}(t) = \int_0^t M_{\eta\xi}(s)ds + K_{\eta\xi}^{p}(s)ds$. Hence in similar manner, it can be shown that (2.8) is class $\mathcal{F}$(Ã × [a, b], $*\,h_{\eta\xi}$, W) and all fundamental results in [4, 5] (e.g. the existence of solution) hold for equation (2.5) and hence (2.6).

2.9 Definition: The trivial solution x ≡ 0 of equation (1.4) is said to be variationally stable with respect to perturbations if for every ε > 0 there exists a $\delta = \delta_{\eta\xi} > 0$ such that if $||y_0||_{\eta\xi} < \delta_{\eta\xi}$, y_0 ∈ Ã and the stochastic process Q belongs to the set Ad(Ã)$_{wac}$ ∩ BV(Ã) such that

$$\text{Var}(Q(t)(\eta,\xi)) < \delta\eta\xi, \text{ then } ||y(t)||_{\eta\xi} < \varepsilon$$

for t ∈ [0, T] where y(t) is a solution of (2.6) with y(0) = y_0.

2.10 Definition: The solution x ≡ 0 of (1.4) is called attracting with respect to perturbations if there exists δ_0 > 0 and for every ε > 0, there is a A = A(ε) ≥ 0 and B(η, ξ, ε) = B > 0 such that if

$$||y_0||_{\eta\xi} < \delta_0\,,\ y_0 \in \tilde{A}$$

and Q ∈ Ad(Ã)$_{wac}$ ∩ BV(Ã), satisfying Var($Q(t)(\eta,\xi)$) < B, then

$$||y(t)||_{\eta\xi} < \varepsilon\,,$$

for all t ∈ [A, T], where y(t) is a solution of (2.6).

2. 11 Definition: The trivial solution x ≡ 0 of equation (1.3) is called asymptotically stable with respect to perturbations if it is stable and attracting with respect to perturbations.

2.12 Notation: Denote by Ad(Ã)$_{wac}$ ∩ BV(Ã) := Å the set of all adapted stochastic processes φ : [0; T]→ Ã that are weakly absolutely continuous and of bounded variation on [t_0, T].

In the next section, we show the equivalence of the concepts of stability defined above. All through the remaining sections, except otherwise stated take $\eta,\xi \in \mathbb{D}\,\underline{\otimes}\,\mathbb{E}$ to be arbitrary.

3.Equivalence of the Concepts of Stabilities:

3.1 Theorem:

(a) The trivial solution x ≡ 0 of the Kurzweil equation (1.4) associated with the equivalent form (1.3) of QSDE (1.1) is variationally stable if and only if it is stable with respect to perturbation.

(b) The trivial solution x ≡ 0 of (1.4) is variationally attracting if and only if it is attracting with respect to perturbations.

Proof (a)(i) Assume that the trivial solution of x ≡ 0 (1.4) is variationally stable. For a given ε > 0, let $\delta_{\eta\xi} = \delta_{\eta\xi}(\varepsilon) > 0$ be given by Definition 2.6. Assume that Y_0 ∈ Ã, $||Y_0||_{\eta\xi} < \delta_{\eta\xi}$ and $\text{Var}_{[0,T]}(Q(t)(\eta,\xi) < \delta_{\eta\xi}$ ∀ $\eta,\xi \in \mathbb{D}\,\underline{\otimes}\,\mathbb{E}$ and Y(t), t ∈ [0,T] is a solution of (2.6) satisfying Y(0) = Y_0 . Since Y is a solution of (2.5) and hence of (2.6), then Y∈ Å and satisfies for any $s_1, s_2 \in [0,T]$

$$\langle\eta, Y(s_2)\xi\rangle - \langle\eta, Y(s_1)\xi\rangle = \int_{s_1}^{s_2} DF(Y(\tau),t)\,(\eta,\xi) + Q(s_2)(\eta,\xi) - Q(s_1)(\eta,\xi)$$

Hence, by the additivity of the Kurzweil integrals

$$\langle \eta, Y(s_2)\xi\rangle - \int_{s_0}^{s_2} DF(Y(\tau),t)\,(\eta,\xi) - \langle \eta, Y(s_1)\xi\rangle$$
$$+\int_{s_0}^{s_1} DF(Y(\tau),t)\,(\eta,\xi) = Q(s_2)(\eta,\xi) - Q(s_1)(\eta,\xi)$$

for any $s_0, s_1 \in [0,T]$.
Consequently,

$$Var_{[0,T]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)\,(\eta,\xi)\right) = Var_{[0,T]}(Q(\tau)(\eta,\xi)) < \delta_{\eta\xi}$$

By the assumption of variational stability of the trivial solution, we have
$$\|Y(t)\|_{\eta\xi} < \varepsilon,\ t \in [0,T].$$
This implies that the trivial solution x ≡ 0 of (1.4) is stable with respect to perturbations.
(ii) Assume that the trivial solution of (1.4) is stable with respect to perturbations. For ε > 0, let δ > 0 be given by definition (2.9). Suppose that the process Y : [0,T]↦ Ã lying in the set Å, is a solution of (2.6) such that $\|Y(0)\|_{\eta\xi} < \delta_{\eta\xi}$ and

$$Var_{[0,T]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)\,(\eta,\xi)\right) < \delta_{\eta\xi}$$

for $s_1, s_2 \in [0,T]$, we have

$$\langle \eta, Y(s_2)\xi\rangle - \langle \eta, Y(s_1)\xi\rangle = \int_{s_1}^{s_2} DF(Y(\tau),t)\,(\eta,\xi) + \langle \eta, Y(s_2)\xi\rangle -$$
$$-\int_0^{s_2} DF(Y(\tau),t)\,(\eta,\xi) + \langle \eta, Y(s_1)\xi\rangle + \int_0^{s_1} DF(Y(\tau),t)\,(\eta,\xi)$$
$$= \int_{s_1}^{s_2} DF(Y(\tau),t)\,(\eta,\xi) + Q(s_2)(\eta,\xi) - Q(s_1)(\eta,\xi) \qquad (3.1)$$

where

$$Q(s)(\eta,\xi) = \langle \eta, Y(s)\xi\rangle - \int_{s_0}^{s} DF(Y(\tau),t)\,(\eta,\xi), \text{for } s \in [0,T]$$

Since Q ∈ Å and (3.1) shows that the stochastic process Y is a solution of equation (2.6) on [0,T] with this Q and $\|Y(0)\|_{\eta\xi} < \delta_{\eta\xi}$. Moreover

$$Var_{[0,T]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)\,(\eta,\xi)\right) < \delta_{\eta\xi}$$

Hence by the assumption of stability with respect to perturbations we get $\|Y(t)\|_{\eta\xi} < \varepsilon$ for t ∈ [0, T] and x ≡ 0 is variationally stable.
(b)(i) Assume that the trivial solution of (1.4) is variationally attracting. Then there exists a $\delta_0 > 0$ and for a given ε > 0 also A > 0 and B > 0, by the Definition 2.7. If now Y_0 ∈ Ã is such that $\|Y(0)\|_{\eta\xi} < \delta_0$, Q belong to the set Å where $Var_{[0,T]}(Q(\tau)(\eta,\xi)) < \delta_{\eta\xi}$ and Y(t) is a solution of (1.4) on [0,T] then

$$Var_{[0,T]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)\,(\eta,\xi)\right) = Var(Q(t)(\eta,\xi)) < B$$

Hence by Definition 2.7 we have $\|Y(t)\|_{\eta\xi} < \varepsilon$ for all t ∈ [A, T] and x ≡ 0 is variationally attracting with respect to perturbations.
(ii) If x ≡ 0 is attracting with respect to perturbations, for ε > 0, let δ_0 > 0, A = A(ε) ≥ 0,
B = B(ε) > 0 be given by Definition 2.10 such that $\|Y(0)\|_{\eta\xi} < \delta_0$. Assume that
Y : [0,T] → Ã lies in the set Å such that $\|Y(0)\|_{\eta\xi} < \delta_0$, Y_0 ∈ Ã and

$$Var_{[0,T]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)\,(\eta,\xi)\right) < B,$$

s ∈ [0,T], then for $s_1, s_2 \in [0,T]$, we have

$$\langle \eta, Y(s_2)\xi\rangle - \langle \eta, Y(s_1)\xi\rangle = \int_{s_1}^{s_2} DF(Y(\tau),t)\,(\eta,\xi)$$
$$+ \langle \eta, Y(s_2)\xi\rangle - \int_0^{s_2} DF(Y(\tau),t)\,(\eta,\xi)$$
$$-\langle \eta, Y(s_1)\xi\rangle + \int_0^{s_1} DF(Y(\tau),t)\,(\eta,\xi)$$

$$= \int_0^{s_1} DF(Y(\tau),t)\,(\eta,\xi) + Q(s_2)(\eta,\xi) - Q(s_1)(\eta,\xi)$$

Hence, we can set

$$Q(s)(\eta,\xi) = \langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)\,(\eta,\xi) \qquad (3.2)$$

for s ∈ [0, T]. So that from Definition 2.10, since Y is a solution of (2.6) on [0,T] with this Q and $\|Y(0)\|_{\eta\xi} < \delta_0$ such that $Var_{[0,T]}\big(Q(s)(\eta,\xi)\big) < B$, then from (3.2) we get

$$Var_{[0,T]}\big(Q(s)(\eta,\xi)\big) < B \Rightarrow Var_{[0,T]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)\,(\eta,\xi)\right) < B$$

and by the assumption of attracting with respect to perturbation we get

$$\|Y(t)\|_{\eta\xi} < \varepsilon, \forall\, t \in [A,T]$$

and x ≡ 0 is variationally attracting.

The following result is a consequence of Theorem 3.1, Definitions 2.8 and 2.11.

3.2 Theorem: The trivial solution x ≡ 0 of equation (1.4) is variationally asymptotically stable if and only if it is asymptotically stable with respect to perturbations.

4. Auxiliary Results:

The following auxiliary results will be used to establish the major results in the next section.

4.1 Proposition: Assume that [a, b] ⊂ [0, T] and that there exists family of functions $f_{\eta\xi}, g_{\eta\xi} : [a,b] \to \mathbb{R}$ defined and continuous on [a, b]. If for every $\sigma \in [a,b]$ there exists $\partial(\sigma) > 0$ such that for every $\beta \in (0,\partial(\sigma))$ the inequality

$$f_{\eta\xi}(\sigma+\beta) - f_{\eta\xi}(\sigma) \le g_{\eta\xi}(\sigma+\beta) - g_{\eta\xi}(\sigma)$$

holds, then

$$f_{\eta\xi}(s) - f_{\eta\xi}(a) \le g_{\eta\xi}(s) - g_{\eta\xi}(a)$$

for all s ∈ [a, b].

Proof: Let us denote

$M_{\eta\xi} = \{s \in [a,b];\ f_{\eta\xi}(\sigma) - f_{\eta\xi}(a) \le g_{\eta\xi}(\sigma) - g_{\eta\xi}(a),\ \sigma \in [a,s] \subset [0,T]\}$

and set $\zeta = Sup M_{\eta\xi}$. Since

$$f_{\eta\xi}(a+\beta) - f_{\eta\xi}(a) \le g_{\eta\xi}(a+\beta) - g_{\eta\xi}(a)$$

for $\beta \in \big(0,\partial(a)\big)$ and $\partial(a) > 0$, the set $M_{\eta\xi}$ is non-empty, $\zeta > a$ and

$$f_{\eta\xi}(s) - f_{\eta\xi}(a) \le g_{\eta\xi}(s) - g_{\eta\xi}(a) \text{ for every } s < \zeta.$$

Using the continuity of $f_{\eta\xi}$ and $g_{\eta\xi}$ we have also that

$$f_{\eta\xi}(\zeta) - f_{\eta\xi}(a) \le g_{\eta\xi}(\zeta) - g_{\eta\xi}(a).$$

If $\zeta < b$ then by assumption we have

$$f_{\eta\xi}(\zeta+\beta) - f_{\eta\xi}(\zeta) \le g_{\eta\xi}(\zeta+\beta) - g_{\eta\xi}(\zeta).$$

for every $\beta \in \big(0,\partial(\zeta)\big)$ and $\partial(\zeta) > 0$ and therefore also

$$f_{\eta\xi}(\zeta+\beta) - f_{\eta\xi}(a) = f_{\eta\xi}(\zeta+\beta) - f_{\eta\xi}(\zeta) + f_{\eta\xi}(\zeta) - f_{\eta\xi}(a)$$
$$\le g_{\eta\xi}(\zeta+\beta) - g_{\eta\xi}(\zeta) + g_{\eta\xi}(\zeta) - g_{\eta\xi}(a) = g_{\eta\xi}(\zeta+\beta) - g_{\eta\xi}(a).$$

This implies that $\zeta+\beta \in M_{\eta\xi}$ for $\beta \in \big(0,\partial(\zeta)\big)$, i.e. $\zeta < Sup M_{\eta\xi}$ and this contradiction yields $\zeta = b$ and $M_{\eta\xi} = [a,b]$ and the proof is complete.

4.2 Lemma: Since $\mathbb{C} \cong \mathbb{R}^2$ we assume the following:

(i) the map $(x,t) \to V(x,t)(\eta,\xi)$ is real-valued such that for every $x \in \tilde{A}$, the real-valued map $t \to V(x,t)(\eta,\xi)$ is continuous on [0,T].

(ii) $|V(x,t)(\eta,\xi) - V(y,t)(\eta,\xi)| \le K\|x-y\|_{\eta\xi}$ (4.1)

for every x, y Ã, t ∈ [0,T] with a constant $K_{\eta\xi} := K > 0$.

(iii) there is a real valued map $\Phi_{\eta\xi} : \tilde{A} \to \mathbb{R}$ such that for every solution x : [0,T] → Ã of equation (1.4), we have

$$\lim_{\beta\to 0} Sup \frac{V(x(t+\beta),t+\beta)(\eta,\xi) - V(x(t),t)(\eta,\xi)}{\beta} < \Phi_{\eta\xi}(x(t)) \qquad (4.2)$$

for t ∈ [0,T].

(iv) If $Y : [0,t_1] \to \tilde{A}$, $[0,t_1] \subset [0,T]$ belongs to Å then the inequality

$$V(x(t_1),t_1)(\eta,\xi) \le V(x(0),0)(\eta,\xi) K Var_{[0,t_1]}\left(\langle \eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)\,(\eta,\xi)\right) +$$
$$+ M_{\eta\xi}(t_1 - 0) \qquad (4.3)$$

holds, where $M_{\eta\xi} = Sup_{t\in[0,t_1]}\Phi_{\eta\xi}(\mathrm{Y(t)})$.

Proof: Let $Y:[0,t_1] \to \tilde{A}$ belong to Å be given and let $\sigma \in [0,t_1] \subset [0,T]$ be an arbitrary point. Hence the real-valued function $V(y(t),t)(\eta,\xi)$ is continuous on $[0,t_1]$.
Assume that $x:[\sigma,\sigma+\beta_1(\sigma)] \subset [0,T] \to \tilde{A}$ is a solution of (1.4) on the interval $[\sigma,\sigma+\beta_1(\sigma)]$, $\beta_1(\sigma) > 0$ with the initial condition $x(\sigma) = y(\sigma)$. the existence of such a solution is guaranteed by the existence results established in Ayoola (2001(b)) and Bishop (2012). By the assumption (4.1) we then have

$$V(y(\sigma+\beta),\sigma+\beta)(\eta,\xi) - V(x(\sigma+\beta),\sigma+\beta)(\eta,\xi)$$
$$\leq K\|y(\sigma+\beta)-x(\sigma+\beta)\|_{\eta\xi}$$
$$= K\left|\langle\eta,y(\sigma+\beta)\xi\rangle - \langle\eta,Y(\sigma)\xi\rangle - \int_\sigma^{\sigma+\beta} DF(Y(\tau),t)\,(\eta,\xi)\right| \qquad (4.4)$$

for every $\beta \in [0,\beta_1(\sigma)]$.

Remark: the last inequality is obtained from the following

$$K\left|\langle\eta,y(\sigma+\beta)\xi\rangle - \langle\eta,Y(\sigma)\xi\rangle - \int_\sigma^{\sigma+\beta} DF(Y(\tau),t)\,(\eta,\xi)\right|$$
$$K\|y(\sigma+\beta)-y(\sigma)-x(\sigma+\beta)+\ x(\sigma)\|_{\eta\xi}$$

Where $y(\sigma) = x(\sigma)$ and

$$\int_\sigma^{\sigma+\beta} DF(Y(\tau),t)\,(\eta,\xi) =\ x(\sigma+\beta) -\ x(\sigma)$$

By the inequality (4.4) and by (4.2) we obtain

$$V(y(\sigma+\beta),\sigma+\beta)(\eta,\xi) - V(x(\sigma),\sigma)(\eta,\xi)$$
$$= V(y(\sigma+\beta),\sigma+\beta)(\eta,\xi) - V(x(\sigma+\beta),\sigma+\beta)(\eta,\xi)$$
$$+V(x(\sigma+\beta),\sigma+\beta)(\eta,\xi) - V(x(\sigma),\sigma)(\eta,\xi) \leq$$
$$\leq K\left|\langle\eta,y(\sigma+\beta)\xi\rangle - \langle\eta,Y(\sigma)\xi\rangle - \int_\sigma^{\sigma+\beta} DF(Y(\tau),t)\,(\eta,\xi)\right| + \beta\Phi_{\eta\xi}(y(\beta))$$
$$\leq K\left|\langle\eta,y(\sigma+\beta)\xi\rangle - \langle\eta,Y(\sigma)\xi\rangle - \int_\sigma^{\sigma+\beta} DF(Y(\tau),t)\,(\eta,\xi)\right| + \beta\mathrm{M}_{\eta\xi} + \beta\varepsilon$$

where $\varepsilon > 0$ is arbitrary and $\beta \in (0,\beta_2(\sigma))$ with $\beta_2(\sigma) \leq \beta_1(\sigma)$, $\beta_2(\sigma) > 0$ is sufficiently small. Setting

$$\langle\eta,Q(\sigma)\xi\rangle = \langle\eta,p(\sigma)\xi\rangle = \langle\eta,y(s)\xi\rangle - \int_0^s DF(Y(\tau),t)\,(\eta,\xi)$$

for $s \in [0,t_1]$.
As $(\eta,\xi) \longrightarrow Q(s)(\eta,\xi)$ is a sesquilinear form, there exists $Q:[0,t_1] \longrightarrow \tilde{A}$ lying in Å such that $Q(s)(\eta,\xi) = \langle\eta,Q(\sigma)\xi\rangle$. The last inequality can be used to derive the following estimates

$$V(y(\sigma+\beta),\sigma+\beta)(\eta,\xi) - V(x(\sigma),\sigma)(\eta,\xi)$$
$$\leq K\left|\langle\eta,y(\sigma+\beta)\xi\rangle - \langle\eta,Y(\sigma)\xi\rangle - \int_\sigma^{\sigma+\beta} DF(Y(\tau),t)\,(\eta,\xi)\right| +$$
$$+K\left|\int_\sigma^{\sigma+\beta} D[F(y(\tau),t)\,(\eta,\xi) - F(x(\tau),t)(\eta,\xi)]\right| + \beta\mathrm{M}_{\eta\xi} + \beta\varepsilon \leq$$
$$\leq K|Q(\sigma+\beta)(\eta,\xi) - Q(\sigma)(\eta,\xi)| + \beta\mathrm{M}_{\eta\xi} + \beta\varepsilon +$$
$$+K\left|\int_\sigma^{\sigma+\beta} D[F(y(\tau),t)\,(\eta,\xi) - F(x(\tau),t)(\eta,\xi)]\right| \leq$$
$$\leq K\left(Var_{[0,\sigma+\beta]}Q(t)(\eta,\xi) -\ Var_{[0\,,\sigma]}Q(t)(\eta,\xi)\right) + \beta\mathrm{M}_{\eta\xi} + \beta\varepsilon +$$
$$+K\left|\int_\sigma^{\sigma+\beta} D[F(y(\tau),t)\,(\eta,\xi) - F(x(\tau),t)(\eta,\xi)]\right| \qquad (4.5)$$

for every $\beta \in (0,\beta_2(\sigma))$. Considering the last term in (4.5), since the map $(x,t) \to F(x,t)(\eta,\xi)$ is of class $\mathcal{F}(\hat{\mathcal{A}} \times [t_0,T],h_{\eta\xi},W)$, we obtain by Theorems 1.9.4 and 1.9.5 (iii) in Bishop (2012), the estimate

$$K\left|\int_\sigma^{\sigma+\beta} D[F(y(\tau),t)\,(\eta,\xi) - F(x(\tau),t)(\eta,\xi)]\right|$$
$$\leq \int_\sigma^{\sigma+\beta} W\|x(\tau)-y(\tau)\|_{\eta\xi}d\,h_{\eta\xi}(\tau)$$

$$= \lim_{\alpha\to 0}\left[\int_{\sigma}^{\sigma+\alpha} W\|x(\tau)-y(\tau)\|_{\eta\xi}\, dh_{\eta\xi}(\tau) + \int_{\sigma+\alpha}^{\sigma+\beta} W\|x(\tau)-y(\tau)\|_{\eta\xi}\, dh_{\eta\xi}(\tau)\right]$$
$$= W\big(\|x(\sigma)-y(\sigma)\|_{\eta\xi}\big)\left(h_{\eta\xi}(\sigma_1)-h_{\eta\xi}(\sigma)\right) + \lim_{\alpha\to 0}\int_{\sigma+\alpha}^{\sigma+\beta} W\|x(\tau)-y(\tau)\|_{\eta\xi}\, dh_{\eta\xi}(\tau)$$
$$= \lim_{\alpha\to 0}\int_{\sigma+\alpha}^{\sigma+\beta} W\|x(\tau)-y(\tau)\|_{\eta\xi}\, dh_{\eta\xi}(\tau)$$
$$\le Sup_{s\in[\sigma,\sigma+\beta]} W(\|x(s)-y(s)\|_{\eta\xi}) \lim_{\alpha\to 0}(h_{\eta\xi}(\sigma+\beta) - h_{\square\xi}(\sigma+\alpha))$$
$$= Sup_{s\in[\sigma,\sigma+\beta]} W(\|x(s)-y(s)\|_{\eta\xi}) \left(h_{\eta\xi}(\sigma+\beta)-h_{\eta\xi}(\sigma)\right), \qquad (4.6)$$

because $x(\sigma)=y(\sigma)$ and hence $W(\|x(\sigma)-y(\sigma)\|_{\eta\xi})=0$.
For $s\in[\sigma,\sigma+\beta_2(\sigma)]$ we have

$$\langle\eta, y(s)\xi\rangle - \langle\eta, x(s)\xi\rangle = \langle\eta, y(s)\xi\rangle - \langle\eta, y(\sigma)\xi\rangle - \int_{\sigma}^{s} DF(x(\tau),t)\,(\eta,\xi)$$

and therefore

$$\lim_{s\to\sigma_1}(\langle\eta, y(s)\xi\rangle - \langle\eta, x(s)\xi\rangle) = \langle\eta, y(\sigma_1)\xi\rangle - \langle\eta, y(\sigma)\xi\rangle -$$
$$-\lim_{s\to\sigma_1}(F(x(\sigma),s)(\eta,\xi) - F(x(\sigma),\sigma)(\eta,\xi))$$
$$= \langle\eta, y(\sigma_1)\xi\rangle - \langle\eta, y(\sigma)\xi\rangle - (F(x(\sigma),\sigma_1)(\eta,\xi) - F(x(\sigma),\sigma)(\eta,\xi))$$
$$= \langle\eta, Q(\sigma_1)\xi\rangle - \langle\eta, Q(\sigma)\xi\rangle,\ \ \sigma_1>\sigma$$

and also

$$\lim_{s\to\sigma_1}\|x(s)-y(s)\|_{\eta\xi} = |Q(\sigma_1)(\eta,\xi) - Q(\sigma)(\eta,\xi)| \qquad (4.7)$$

For every $\varepsilon>0$ we define

$$\alpha = \frac{\varepsilon}{K(h_{\eta\xi}(t_1)-h_{\eta\xi}(0)+1)} > 0 \qquad (4.8)$$

and assume that $r=r(\alpha)>=0$ is such that $W(r)<\alpha$. Further, we choose $\gamma\in\left[0,\frac{r}{2}\right]\subset[0,T]$. Since (4.7) holds, there is an $\beta_3(\sigma)\in(0,\beta_2(\sigma))$ such that

$$\|y(s)-x(s)\|_{\eta\xi} \le |Q(\sigma_1)(\eta,\xi) - Q(\sigma)(\eta,\xi)| + \gamma \qquad (4.9)$$

for $s\in(\sigma,\sigma+\beta_3(\sigma))$ and also

$$W(\|y(s)-x(s)\|_{\eta\xi}) \le W(|Q(\sigma_1)(\eta,\xi) - Q(\sigma)(\eta,\xi)| + \gamma) \quad (4.10)$$

Setting:

$$N(\alpha) := N(\alpha,\eta,\xi) = \{\sigma_1\in[0,T];\, |Q(\sigma_1)(\eta,\xi) - Q(\sigma)(\eta,\xi)| \ge \tfrac{r}{2}\}$$

since Q lies in Å, the set $N(\alpha)$ is finite and we denote by $l(\alpha)$ the number of elements of $N(\alpha)$. If $\sigma\in[0,T]\backslash N(\alpha)$ and $s\in(\sigma,\sigma+\beta_3(\sigma))$ then by (4.10) we have

$$\|y(s)-x(s)\|_{\eta\xi} \le W\left(\tfrac{r}{2}+\gamma\right) < W(\tfrac{r}{2}+\tfrac{r}{2})$$
$$= W(r) < \alpha$$

and by (4.6) also

$$\left|\int_{\sigma}^{\sigma+\beta} D[F(y(\tau),t)\,(\eta,\xi) - F(x(\tau),t)(\eta,\xi)]\right| \le \alpha\left(h_{\eta\xi}(\sigma+\beta)-h_{\eta\xi}(\sigma)\right) \qquad (4.11)$$

whenever $\beta\in(0,\beta_3(\sigma))$.
If $\sigma\in[0,T]\cap N(\alpha)$ then there exists $\beta_4(\sigma)\in(0,\beta_3(\sigma))$ such that for $\beta\in(0,\beta_4(\sigma))$ we set

$$|h_{\eta\xi}(\sigma+\beta)-h_{\eta\xi}(\sigma)| <$$
$$< \frac{\alpha}{(l(\alpha)+1)W(|Q(\sigma_1)(\eta,\xi) - Q(\sigma)(\eta,\xi)| + \gamma)}$$

$\sigma_1\in[0,T]$, $\sigma_1>\sigma>0$. Hence (4.6) and (4.10) yield

$$\left|\int_{\sigma}^{\sigma+\beta} D[F(y(\tau),t)\,(\eta,\xi) - F(x(\tau),t)(\eta,\xi)]\right| \le$$
$$\le W(|Q(\sigma_1)(\eta,\xi) - Q(\sigma)(\eta,\xi)| + \gamma)\frac{\alpha}{(l(\alpha)+1)W(|Q(\sigma_1)(\eta,\xi) - Q(\sigma)(\eta,\xi)| + \gamma)}$$
$$= \frac{\alpha}{l(\alpha)+1} \qquad (4.12)$$

for every $\beta\in[\beta,\sigma+\beta_4(\sigma)]$. Since the function $h_{\eta\xi,\alpha}:[0,T]\to\mathbb{R}$ is non-decreasing and continuous on [0,T], we set

$$Var_{[t_1,t_2]} h_{\eta\xi,\alpha}(t) = h_{\eta\xi,\alpha}(t_2) - h_{\eta\xi,\alpha}(t_1) = \frac{\alpha}{l(\alpha)+1} l(\alpha) < \alpha \qquad (4.13)$$

for every $t_1,t_2\in[0,T]$ and from (4.8) and (4.13) we have

$$h_{\square\xi,\alpha}(t_2) - h_{\eta\xi,\alpha}(t_1) < \alpha\left[h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1) + 1\right] = \frac{\varepsilon}{K} \qquad (4.14)$$

and by (4.11), (4.12) and by the definition of $h_{\eta\xi,\alpha}$ we obtain the inequality

$$\left|\int_{\sigma}^{\sigma+\beta} D[F(y(\tau),t)\,(\eta,\xi) - F(x(\tau),t)(\eta,\xi)]\right| \leq$$
$$\leq |h_{\eta\xi,\alpha}(\sigma+\beta) - h_{\eta\xi,\alpha}(\sigma)|$$

for $\beta \in [0, \partial(\sigma)]$ and (4.5) gives

$$V(y(\sigma+\beta),\sigma+\beta)(\eta,\xi) - V(x(\sigma),\sigma)(\eta,\xi)$$
$$\leq K\left(Var_{[t_0,\sigma+\beta]}Q(t)(\eta,\xi) - Var_{[t_0,\sigma]}Q(t)(\eta,\xi)\right) + \beta \mathrm{M}_{\eta\xi} + \beta\varepsilon +$$
$$+K(h_{\eta\xi,\alpha}(\sigma+\beta) - h_{\eta\xi,\alpha}(\sigma)) = g_{\eta\xi}(\sigma+\beta) - g_{\eta\xi}(\sigma) \qquad (4.15)$$

for all $\sigma \in [0,T]$ and $\beta \in [0, \partial(\sigma)]$ where

$$g_{\eta\xi}(t) = K\left(Var_{[0,T]}Q(t)(\eta,\xi)\right) + \mathrm{M}_{\eta\xi}(t) + \varepsilon(t) + Kh_{\eta\xi,\alpha}(t), t \in [0,T].$$

The function $g_{\eta\xi}$ is of bounded variation on [0, T] and continuous on [0, T].
From (4.14) and Proposition 4.1 we obtain by (4.15) the inequality

$$V(y(t_2),t_2)(\eta,\xi) - V(x(t_1),t_1)(\eta,\xi) \leq g_{\eta\xi}(t_2) - g_{\eta\xi}(t_1) =$$
$$= KVar_{[t_1,t_2]}Q(t)(\eta,\xi) + \mathrm{M}_{\eta\xi}(t_2-t_1) + \varepsilon(t_2-t_1) + +K(h_{\eta\xi,\alpha}(t_2) - h_{\eta\xi,\alpha}(t_1))$$
$$\leq KVar_{[t_1,t_2]}Q(t)(\eta,\xi) + \mathrm{M}_{\eta\xi}(t_2-t_1) + \varepsilon(t_2-t_1) + \varepsilon$$

for t_1, $t_2 \in [0,T]$, since $\varepsilon > 0$ can be arbitrary, we obtain from this inequality the result in (4.3) and the proof is completed.
The following definition will be useful in establishing our major results in the next section.

4.1 Definition: The real valued map $(x,t) \to V(x,t)(\eta,\xi)$ is said to be positive definite if
(i) There exists a continuous nondecreasing function $b : [0,\infty) \to \mathbb{R}$ such that b(0) = 0 and
(ii) $V(x,t)(\eta,\xi) \leq b(\|x\|_{\eta\xi})$ for all $(x,t) \in \tilde{A} \times [0,T]$
(iii) $V(0,t)(\eta,\xi) = 0$ for all $(x,t) \in \tilde{A} \times [0,T]$

Next we establish the major results on the variational stability of solution of equation (1.4) using Lyapunov's method applied in Schwabik (1992).

5. Variational Stability and Asymptotic Variational Stability:

Theorem 5.1:
(i) the real valued map $t \to V(x,t)(\eta,\xi)$ is continuous on [0,T] for every $x \in \tilde{A}$.
(ii) the map $(x,t) \to V(x,t)(\eta,\xi)$is positive definite in the sense of definition (4.1) above.
(iii) $V(0,t)(\eta,\xi) = 0$ and $|V(x,t)(\eta,\xi) - V(y,t)(\eta,\xi)| \leq K\|x-y\|_{\eta\xi}$ for all $x, y \in \tilde{A}$, $K_{\eta\xi} := K > 0$ being a constant.
(iv) the map $(x,t) \to V(x,t)(\eta,\xi)$ is non-increasing along every solution x(t) of equation (1.4). Then the trivial solution x ≡ 0 of (1.4) is variationally stable.

Proof: Since we assumed that the map $(x,t) \to V(x,t)(\eta,\xi)$ is non-increasing whenever $x : [0,T] \to \tilde{A}$, is a solution of (1.4) we have from equation (4.2) in Lemma 4.2

$$\mathrm{limsup}_{\beta\to 0}\frac{V(x(t+\beta),t+\beta)(\eta,\xi) - V(x(t),t)(\eta,\xi)}{\beta} \leq 0\,, t \in [0,T] \quad (5.1)$$

To establish the theorem, we show that the conditions of variational stability according to definition (2.6) are fulfilled under these assumptions.
by lemma 4.2 the map $(x,t) \to V(x,t)(\eta,\xi)$, satisfies the following;
(i) Let $\varepsilon > 0$ and let y: $[0,t_1] \to \tilde{A}$ lie in Å be given. Then we have

$$\mathrm{limsup}_{\beta\to 0}\frac{V(x(t+\beta),t+\beta)(\eta,\xi) - V(x(t),t)(\eta,\xi)}{\beta} \leq 0$$

for every t ∈ [0,T], by replacing $\Phi_{\eta\xi}x(t)$ in (4.2) with $\Phi_{\eta\xi}x(t) \equiv 0$.
(ii) Again since the map $(x,t) \to V(x,t)(\eta,\xi)$ is continuous, we obtain the relation

$$|V(x,t)(\eta,\xi) - V(y,t)(\eta,\xi) \leq K\|x-y\|_{\eta\xi}$$

for every x, y ∈ Ã, t ∈ [0,T] with a constant K > 0. Hence we obtain by (4.3) in Lemma 4.2, (iii) in definition (4.1) and hypothesis (iii) above the inequality

$$V(y(r),r)(\eta,\xi) \leq V(y(0),0)(\eta,\xi) + K\mathrm{Var}_{[0,r]}(\langle\eta, y(s)\xi\rangle - \int_0^s DF(x(\tau),t)\,(\eta,\xi))$$
$$K\|\mathrm{y}(0)\|_{\eta\xi} + K\mathrm{Var}_{[0,r]}\left(\langle\eta, y(s)\xi\rangle - \int_0^s DF(x(\tau),t)\,(\eta,\xi)\right) \qquad (5.2)$$

which holds for every $r, s \in [0, t_1] \subset [0, T]$. Setting $\alpha(\varepsilon) = inf_{r \leq \varepsilon} b(r)$. Then $\alpha(\varepsilon) > 0$ for $\varepsilon > 0$ and $\lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0$. Further, choose $\delta_{\eta\xi} > 0$ such that $2K\delta_{\eta\xi} < \alpha(\varepsilon)$.
If in this situation the function y is such that $\|y(0)\|_{\eta\xi} < \delta_{\eta\xi}$ and

$$\mathrm{Var}_{[0,t_1]}\left(\langle \eta, y(s)\xi \rangle - \int_0^s DF(x(\tau), t)\,(\eta, \xi)\right) < \delta_{\eta\xi}$$

then by (5.2) we obtain the inequality

$$V(y(r), r)(\eta, \xi) \leq 2K\delta_{\eta\xi} \qquad (5.3)$$

provided $r \in [0, t_1]$.
If there exists a $\hat{t} \in [0, t_1]$ such that $\|y(\hat{t})\|_{\eta\xi} \geq \varepsilon$ then by (ii) of definition (4.1) we get the inequality

$$V(y(\hat{t}), \hat{t})(\eta, \xi) \geq b(\|y(\hat{t})\|_{\eta\xi}) \geq inf_{r \leq \varepsilon} b(r) = \alpha(\varepsilon)$$

which contradicts (5.3). Hence $\|y(t)\|_{\eta\xi} < \varepsilon$ for all $t \in [0, t_1]$ and by Definition (2.6) the solution x ≡ 0 of equation (1.4) is variationally stable.

5.2 Theorem: Suppose that the following conditions hold:
(i) The map $(x, t) \to V(x, t)(\eta, \xi)$ satisfies the hypothesis of Theorem 5.1.
(ii)

$$\mathrm{limsup}_{\beta \to 0} \frac{V(x(t+\beta), t+\beta)(\eta, \xi) - V(x(t), t)(\eta, \xi)}{\beta} \leq \Phi_{\eta\xi}(x(t))$$

holds for every solution $x \in \tilde{A}$ of equation (1.4)
(iii) $\Phi_{\eta\xi} : \tilde{A} \to \mathbb{R}$ is continuous with $\Phi_{\eta\xi}(0) = 0, \Phi_{\eta\xi}(x) > 0$ for $x \neq 0$.
Then the trivial solution x ≡ 0 of (1.4) is variationally asymptotically stable.

Proof: From hypothesis (ii) above, the map $V(x, t)(\eta, \xi)$ is non-increasing along every solution x(t) of (1.4) and therefore by Theorem 5.1 the trivial solution x ≡ 0 of (1.4) is variationally stable. By Definition (2.8) it remains to show that the solution x ≡ 0 of equation (1.4) is variationally attracting in the sense of Definition (2.7). From the variational stability of the trivial solution x ≡ 0 of equation (1.4) there is a $\delta_0 > 0$ such that if $y : [0, T] \to \tilde{A} \in$ Å and such that $\|y(0)\|_{\eta\xi} < \delta_0$,

$$\mathrm{Var}_{[0,T]}\left(\langle \eta, y(s)\xi \rangle - \int_0^s DF(x(\tau), t)\,(\eta, \xi)\right) < \delta_0\,,$$

then set $\|y(t)\|_{\eta\xi} < a,\ a > 0$ for t ∈ [0,T], i.e. $y : [0, T] \to \tilde{A}$ is continuous on [0,T]. Let $\varepsilon > 0$ be arbitrary. From the variational stability of the trivial solution we obtain that there is a $\delta_{\eta\xi}(\varepsilon) > 0$ such that for every $y : [0, T] \to \tilde{A} \subset$ Å on [0, T] and such that

$$\|y(0)\|_{\eta\xi} < \delta_{\eta\xi}(\varepsilon) \qquad (5.4)$$

and

$$\mathrm{Var}_{[0,T]}\left(\langle \eta, y(s)\xi \rangle - \int_0^s DF(x(\tau), t)\,(\eta, \xi)\right) < \delta(\varepsilon) \qquad (5.5)$$

we have

$$\|y(t)\|_{\eta\xi} < \varepsilon, \text{ for } t \in [0,T]\,. \qquad (5.6)$$

Again set $B(\varepsilon) = \min\,(\delta_{\eta\xi}(0), \delta_{\eta\xi}(\varepsilon))$ and

$$A(\varepsilon) = -K\frac{\delta_0 + B(\varepsilon)}{M_{\eta\xi}} > 0$$

where

$$M_{\eta\xi} = Sup\{-\Phi_{\eta\xi}(x);\ B(\varepsilon) \leq \|x\|_{\eta\xi} < \varepsilon\} = -\inf\{\Phi_{\eta\xi}(x);\ B(\varepsilon) \leq \|x\|_{\eta\xi} < \varepsilon\} < 0$$

Assume that $y : [0, T] \to \tilde{A} \subset$ Å such that

$$\|y(t)\|_{\eta\xi} < \delta_0,$$

$$\mathrm{Var}_{[0,T]}\left(\langle \eta, y(s)\xi \rangle - \int_0^s DF(y(\tau), t)\,(\eta, \xi)\right) < B(\varepsilon) \qquad (5.7)$$

Assume that 0 < A(ε) < T. We show that there exists a $t^* \in [0, A] \subset [0, T]$ such that

$$\|y(t^*)\|_{\eta\xi} < B(\varepsilon).$$

Assume the contrary i.e., $\|y(s)\|_{\eta\xi} \geq B(\varepsilon)$ for every s ∈ [0, A]. Lemma 4.2 yields

$$V(y(A), A)(\eta, \xi) - V(y(0), 0)(\eta, \xi) \leq$$

$$\leq K\mathrm{Var}_{[0,A]}\left(\langle \eta, y(s)\xi \rangle - \int_0^s DF(x(\tau), t)\,(\eta, \xi)\right) + M_{\eta\xi}A(\varepsilon) <$$

$$< KB(\varepsilon) + M_{\eta\xi}\frac{-K(\delta_0 + B(\varepsilon))}{M_{\eta\xi}} = -K\delta_0\,.$$

Hence,

$$V(y(A),A)(\eta,\xi) \leq V(y(0),0)(\eta,\xi) - K\delta_0 \leq K\|y(0)\|_{\eta\xi} - K\delta_0 < K\delta_0 - K\delta_0 = 0$$

and this contradicts the inequality

$$V(y(A),A)(\eta,\xi) \geq b\|y(A)\|_{\eta\xi} \geq b\big(B(\varepsilon)\big) > 0.$$

Hence necessarily there is a $t^* \in [0,A]$ such that

$$\|y(t^*)\|_{\eta\xi} < B(\varepsilon)$$

and by (5.7) we have $\|y(t^*)\|_{\eta\xi} < \varepsilon$ for $t \in [t^*,T] \subset [0,T]$ because (5.4) and (5.5) hold in view of the choice of B(ε) and (5.6) is satisfied for the case $t^* = 0$. Consequently, also $\|y(t)\|_{\eta\xi} < \varepsilon$ for t ∈ [0,T], T > A, because $t^* \in [0,A]$ and therefore the trivial solution x≡ 0 is a variationally attracting solution of (1.4). Therefore, by Definition 2.8, the trivial solution of (1.4) is variationally asymptotically stable and thus the result is established.

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