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Some Fixed Point Results of Ciric-Type Contraction Mappings on Ordered *G*-Partial Metric Spaces

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Abstract

We introduce the concept of generalized quasi-contraction mappings in *G*-partial metric spaces and prove some fixed point results in ordered *G*-partial metric spaces. The results generalize and extend some recent results in literature.

Keywords

Fixed Points, Generalized Quasi-Contraction Maps, Bounded Orbit, Partially Ordered Set, *G*-Partial Metric Spaces

1. Introduction and Preliminary Definitions

The Banach contraction principle has been generalized and extended in many directions for some decades. Of all the generalizations, Ciric [1] [2] generalizations seem outstanding. Cho Song Wong [3] dealt with a pair of operators in which the control functions in the generalized contraction maps are upper semi-continuous, while Ciric considered a single operator and took the control function to be a constant. If the control function is an upper semi-continuous, then the result of Ciric [1] is invalid. In Kiany and Amini-Harandi [4], a condition is imposed on the control function and the mapping is termed a Ciric generalized quasi-contraction mapping. In this work, we introduce the concept of generalized quasi-contraction mappings in the new framework of *G*-partial metric spaces.

Rodriguez-Lopez and Nieto [5], Ran and Reuring [6] presented some new results for the existence of the fixed point for some mappings in partially ordered metric spaces. The main idea in [5] [6] involves combing the ideas of an iterative technique in the contraction mapping principle with those in the monotone technique. In this

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work, the existence of a unique fixed point for generalized contraction mappings in ordered G-partial metric spaces is proved.

Matthew [7] generalized the notion of metric spaces by introducing the concept of nonzero self-distance and thus, defined a generalized metric space known as partial metric space, as follows:

Definition 1.1. [7]. A partial metric space is a pair (X, p), where X is a nonempty set and $p: X \times X \to \mathbb{R}$ such that:

(p1) $0 \le p(x,x) \le p(x,y)$

(p2) if $p(x,x) = p(x,y) = p(y,y)$, then $x = y$

(p3) $p(x,y) = p(y,x)$

(p4) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$.

He was able to establish a relationship between partial metric spaces and the usual metric spaces when

$$dp(x,y) = 2p(x,y) - p(x,x) - p(y,y).$$

Mustafa and Sims [8] also extended the concepts of metric to G-metric by assigning a positive real number to every triplet of an arbitrary set as follows:

Definition 1.2. [8]. Let X be a nonempty set, and let $G: X \times X \times X \to \mathbb{R}_+$ be a function satisfying:

(G1) $G(x,y,z) = 0$ if $x = y = z$,

(G2) $0 < G(x,x,y)$ for all $x, y \in X$ with $x \ne y$,

(G3) $G(x,x,y) \le G(x,y,z)$ for all $x, y, z \in X$ with $y \ne z$,

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x)$ (symmetry in all three variables),

(G5) $G(x,y,z) \le G(x,a,a) + G(a,y,z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a generalized metric, or more specifically, a G-metric on X, and the pair (X,G) is a G-metric space.

Mustafa [8] gave an example to show the relationship between G-metric spaces and ordinary metric spaces as: For any G-metric G on X, if $d_G(x,y) = G(x,y,y) + G(x,x,y)$, then (X,d_G) is a metric space.

In this work, the idea of the nonzero self-distance of partial metric spaces and the rectangle inequality of G-metric spaces are combined to develop a new generalized metric space which is defined as the following:

Definition 1.3. Let X be a nonempty set, and let $G_p: X \times X \times X \to \mathbb{R}_+$ be a function satisfying the following:

(Gp1) $G_p(x,y,z) \ge G_p(x,x,x) \ge 0, \forall x, y, z \in X$ (small self-distance),

(Gp2) $G_p(x,y,z) = G_p(x,x,y) = G_p(y,y,z) = G_p(z,z,x)$ iff $x = y = z$, (equality),

(Gp3) $G_p(x,y,z) = G_p(z,x,y) = G_p(y,z,x)$ (symmetry in all three variables),

(Gp4) $G_p(x,y,z) \le G_p(x,a,a) + G_p(a,y,z) - G_p(a,a,a)$ (Rectangle inequality).

The function G_p is called a G-partial metric and the pair (X,G_p) is called a G-partial metric space.

Definition 1.4. A G-partial metric space is said to be symmetric if $G_p(x,y,y) = G_p(y,x,x)$ for all $x, y \in X$.

In this work, we will assume that (X,G_p) is symmetric. The following proposition establishes the relation between G-partial metric spaces and (partial) metric spaces.

Definition 1.5. Let (X,G_p) be a G-partial metric space. Define the functions $p: X \times X \to \mathbb{R}_+$ and $d: X \times X \to \mathbb{R}_+$ by $p(x,y) = G_p(x,y,y) + G_p(y,x,x)$ and $d(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(y,y,y) - G_p(x,x,x)$. Then

1) (X, p) is a partial metric space.

2) (X, d) is a metric space.

Proof

1) From (Gp1), we have that for all $x, y \in X$,

$$p(x,y) = Gp(x,y,y) + Gp(y,x,x) \ge Gp(x,x,x) + Gp(x,x,x) = p(x,x) \ge 0,$$

hence (p1) is satisfied.

If $p(x,x) = p(y,y) = p(y,y)$, then

$$Gp(x,x,x) + Gp(x,x,x) = Gp(x,y,y) + Gp(y,x,x) = Gp(y,y,y) + Gp(y,y,y).$$

By (Gp1), it must follow that $Gp(x,y,y) = Gp(y,x,x) = Gp(y,y,y) = Gp(x,x,x)$.

From the symmetry of G_p and by (Gp2), $x = y$, hence (p2) is satisfied.

(p3) follows from (Gp3) and the triangle inequality (p4) is easily verifiable using (Gp4).

2) Since (X, p) is a partial metric space, then

$$p^s = 2p(x,y) - p(x,x) - p(y,y) = 2\left[Gp(x,y,y) + Gp(y,x,x) - Gp(y,y,y) - Gp(x,x,x)\right]$$

defines a metric on X and so $d(x,y) = \frac{1}{2}p^s(x,y)$ also defines a metric on X.

Example 1.6. Let $X = \mathbb{R}_+$ and define the function $Gp: X \times X \times X \to R_+$ as $Gp(x, y, z) = \max\{x, y, z\}$. Then (X, G_p) is a G-partial metric space.

We state the following definitions and motivations.

Definition 1.7. A sequence $\{x_n\}$ of points in a G-partial metric space (X, G_p) converges to some $a \in X$ if $\lim_{n\to\infty} Gp(x_n, x_n, a) = \lim_{n\to\infty} Gp(x_n, x_n, x_n) = Gp(a,a,a)$.

Definition 1.8. A sequence $\{x_n\}$ of points in a G-partial metric spaces (X, G_p) is Cauchy if the numbers $G_p(x_n, x_m, x_l)$ converges to some $a \in X$ as n, m, l approach infinity.

The proof of the following result follows from the above definitions:

Proposition 1.9. Let $\{x_n\}$ be a sequence in G-partial metric space X and $a \in X$. If $\{x_n\}$ converges to $a \in X$ then $\{x_n\}$ is a Cauchy sequence.

Definition 1.10. A G-partial metric space (X, G_p) is said to be complete if every Cauchy sequence in (X, G_p) converges to an element in (X, G_p).

Definition 1.11. [6]. If $(X, \prec)$ is a partially ordered set and T: $X \to X$, then T is monotone non-decreasing if for every $x, y \in X$, $x \prec y$ implies $Tx \prec Ty$.

Definition 1.12. Let $(X, \prec)$ be a partially ordered set. Then two elements $x, y \in X$ are said to be totally ordered or ordered if they are comparable, *i.e.* $x \prec y$ or $y \prec x$.

Gordji *et al.* [9] proved the existence of a unique fixed point for contraction type maps in partially ordered metric spaces using a control function. Kiany and Amini-Harandi [4] proved the existence of a unique fixed point for a generalized Ciric quasi-contraction mapping in what they tagged a generalized metric space. The map they considered extend that of Gordji *et al.*, albeit the space they considered was not endowed with an order. Saadati *et al.* [10] considered the concept of Omega-distances on a complete partially ordered G-metric space and proved some fixed point theorems. Turkoglu *et al.* [11] and Sastry *et al.* [12] proved some fixed point theorems for generalized contraction mappings in cone metric spaces and metric spaces respectively.

In this work, the existence of unique fixed points of the two generalized contraction mappings below is proved in ordered G-partial metric spaces, extending thus the results in [2] [4] [9] [11].

Definition 1.13. Let (X, G_p) be a G-partial metric space. The self-map T: $X \to X$ is said to be a generalized Ciric quasi-contraction if

$$G_p(Tx,Ty,Ty) \le \alpha(x,y,y)\max\left\{G_p(x,y,y), G_p(x,Tx,Tx), G_p(y,Ty,Ty), G_p(x,Ty,Ty), G_p(y,Tx,Tx)\right\} \quad (1)$$

for any $x, y \in X$, where $\alpha: [0,\infty) \to [0,1)$ is a mapping.

Definition 1.14. Let (X, G_p) be a G-partial metric space. The self-map T: $X \to X$ is said to be a generalized G-contraction if for all $x, y \in X$,

$$\begin{aligned} G_p(Tx,Ty,Ty) &\le \alpha(x,y,y)G_p(x,y,y) + \beta(x,y,y)G_p(x,Tx,Tx) \\ &+ \gamma(x,y,y)G_p(y,Ty,Ty) + \delta(x,y,y)\left[G_p(x,Ty,Ty) + G_p(y,Tx,Tx)\right], \end{aligned} \quad (2)$$

where $\alpha, \beta, \gamma, \delta: X \times X \to [0,1)$ are functions such that

$$\lambda = \sup\left\{\alpha(x,y,y) + \beta(x,y,y) + \gamma(x,y,y) + 2\delta(x,y,y) : x, y \in X\right\} < 1.$$

2. Main Results

Theorem 2.1. Let $(X, \prec)$ be a partially ordered set and suppose there exists a G-partial metric G_p in X such that (X, G_p) is a complete G-partial metric space. Let $T: X \to X$ be a self-mapping in X such that for each

$x, y \in X$ satisfying $x \prec y$,

$$G_p(Tx,Ty,Ty) \leq \alpha(x,y,y)G_p(x,y,y) + \beta(x,y,y)G_p(x,Tx,Tx) \\ +\gamma(x,y,y)G_p(y,Ty,Ty) + \delta(x,y,y)\left[G_p(x,Ty,Ty) + G_p(y,Tx,Tx)\right], \tag{3}$$

where $\alpha,\beta,\gamma,\delta : X \times X \to [0,1)$ are functions such that

$$\lambda = \sup\left\{\alpha(x,y,y) + \beta(x,y,y) + \gamma(x,y,y) + 2\delta(x,y,y) : x, y \in X\right\} < 1. \tag{4}$$

Suppose T is a non-decreasing map such that there exists an $x_0 \in X$ with $x_0 \prec Tx_0$. Also suppose that X is such that for any non-decreasing sequence $\{x_n\}$ converging to x, $x_n \prec x$, $\forall n \in \mathbb{N}$. $x_n \prec x$ for all $n \in \mathbb{N}$.

Then T has a fixed point. Moreover, if for each $u, v \in X$, there exists $z \in X$ which is comparable to u and v, then T has a unique fixed point.

Proof. Fix $x_0 \in X$ Let $\{x_n\}$ be defined by $x_1 = Tx_0$, $x_2 = Tx_1$, $\cdots$, $x_{n+1} = Tx_n$. Since $x_0 \prec Tx_0$ and T is non-decreasing, then $x_0 \prec Tx_0 \prec Tx_1 \prec Tx_2 \prec \cdots \prec Tx_n \prec \cdots$

This implies that $x_n \prec Tx_n$ for each $n \geq 1$.

Since $x_n \prec x_{n+1}$ for each $n \in \mathbb{N}$ then by (3) we have

$$\begin{aligned}
&G_p(x_n,x_{n+1},x_{n+1}) = G_p(Tx_{n-1},Tx_n,Tx_n)\\
&\leq \alpha(x_{n-1},x_n,x_n)G_p(x_{n-1},x_n,x_n) + \beta(x_{n-1},x_n,x_n)G_p(x_{n-1},Tx_{n-1},Tx_{n-1}) + \gamma(x_{n-1},x_n,x_n)G_p(x_n,Tx_n,Tx_n)\\
&\quad + \delta(x_{n-1},x_n,x_n)\left[G_p(x_{n-1},Tx_n,Tx_n) + G_p(x_n,Tx_{n-1},Tx_{n-1})\right]\\
&\leq \alpha(x_{n-1},x_n,x_n)G_p(x_{n-1},x_n,x_n) + \beta(x_{n-1},x_n,x_n)G_p(x_{n-1},x_n,x_n) + \gamma(x_{n-1},x_n,x_n)G_p(x_n,x_{n+1},x_{n+1})\\
&\quad + \delta(x_{n-1},x_n,x_n)\left[G_p(x_{n-1},x_{n+1},x_{n+1}) + G_p(x_n,x_n,x_n)\right]\\
&\leq \alpha(x_{n-1},x_n,x_n)G_p(x_{n-1},x_n,x_n) + \beta(x_{n-1},x_n,x_n)G_p(x_{n-1},x_n,x_n) + \gamma(x_{n-1},x_n,x_n)G_p(x_n,x_{n+1},x_{n+1})\\
&\quad + \delta(x_{n-1},x_n,x_n)\left[G_p(x_{n-1},x_n,x_n) + G_p(x_n,x_{n+1},x_{n+1}) - G_p(x_n,x_n,x_n) + G_p(x_n,x_n,x_n)\right]\\
&\leq \alpha(x_{n-1},x_n,x_n)G_p(x_{n-1},x_n,x_n) + \beta(x_{n-1},x_n,x_n)G_p(x_{n-1},x_n,x_n) + \gamma(x_{n-1},x_n,x_n)G_p(x_n,x_{n+1},x_{n+1})\\
&\quad + \delta(x_{n-1},x_n,x_n)\left[G_p(x_{n-1},x_n,x_n) + G_p(x_n,x_{n+1},x_{n+1})\right].
\end{aligned}$$

Thus, with $\alpha,\beta,\gamma,\delta$ evaluated at (x_{n-1},x_n,x_n), we have

$$\begin{aligned}
G_p(x_n,x_{n+1},x_{n+1}) &\leq \alpha G_p(x_{n-1},x_n,x_n) + \beta G_p(x_{n-1},x_n,x_n) + \gamma G_p(x_n,x_{n+1},x_{n+1})\\
&\quad + \delta\left[G_p(x_{n-1},x_n,x_n) + G_p(x_n,x_{n+1},x_{n+1})\right]\\
&\leq (\alpha+\beta+\gamma)\max\left\{G_p(x_{n-1},x_n,x_n), G_p(x_n,x_{n+1},x_{n+1})\right\}\\
&\quad + 2\delta\max\left\{G_p(x_{n-1},x_n,x_n), G_p(x_n,x_{n+1},x_{n+1})\right\}\\
&\leq (\alpha+\beta+\gamma+2\delta)\max\left\{G_p(x_{n-1},x_n,x_n), G_p(x_n,x_{n+1},x_{n+1})\right\}\\
&\leq \lambda \max\left\{G_p(x_{n-1},x_n,x_n), G_p(x_n,x_{n+1},x_{n+1})\right\}.
\end{aligned} \tag{5}$$

Since $\lambda < 1$, then (5) becomes $G_p(x_n,x_{n+1},x_{n+1}) \leq \lambda G_p(x_{n-1},x_n,x_n)$.

Consequently, $G_p(x_n,x_{n+1},x_{n+1}) \leq \lambda^n G_p(x_0,x_1,x_1)$.

For $m > n$ we get,

$$\begin{aligned}
&G_p(x_n,x_m,x_m) \leq G_p(x_n,x_{m-1},x_{m-1}) + G_p(x_{m-1},x_m,x_m) - G_p(x_{m-1},x_{m-1},x_{m-1})\\
&\leq G_p(x_n,x_{m-2},x_{m-2}) + G_p(x_{m-2},x_{m-1},x_{m-1}) - G_p(x_{m-2},x_{m-2},x_{m-2}) + G_p(x_{m-1},x_m,x_m) - G_p(x_{m-1},x_{m-1},x_{m-1})\\
&\leq \cdots \leq G_p(x_n,x_{n+1},x_{n+1}) + G_p(x_{n+1},x_{n+2},x_{n+2}) - G_p(x_{n+1},x_{n+1},x_{n+1})\\
&\quad + G_p(x_{n+2},x_{n+3},x_{n+3}) - G_p(x_{n+2},x_{n+2},x_{n+2}) + \cdots + G_p(x_{m-1},x_m,x_m) - G_p(x_{m-1},x_{m-1},x_{m-1})\\
&\leq \lambda^n\left(1+\lambda+\cdots+\lambda^{m-n}\right)G_p(x_0,x_1,x_1) \leq \lambda^n(1+\lambda+\cdots)G_p(x_0,x_1,x_1) \leq \frac{\lambda^n}{1-\lambda}G_p(x_0,x_1,x_1).
\end{aligned} \tag{6}$$

Take the limit as $n\to\infty$ in (6) yields $\lim_{n,m\to\infty} G_p(x_n,x_m,x_m)=0$ which implies that $\{x_n\}$ is a Cauchy sequence. Since X is a complete space then there exists $x\in X$ such that $\{x_n\}$ converges to x and

$$\lim_{n\to\infty} G_p(x_n,x,x)=\lim_{n,m\to\infty} G_p(x_n,x_n,x_n)=G_p(x,x,x)=\lim_{n,m\to\infty} G_p(x_n,x_m,x_m)=0.$$

Next we prove that x is the fixed point of T. From (3) and (4), since $x_n\prec x$, for all $n\in\mathbb{N}$,

$$\begin{aligned}G_p(x,Tx,Tx)&\le G_p(x,x_n,x_n)+G_p(x_n,Tx,Tx)-G_p(x_n,x_n,x_n)\le G_p(x,x_n,x_n)+G_p(Tx_{n-1},Tx,Tx)\\&\le G_p(x,x_n,x_n)+\alpha G_p(x_{n-1},x,x)+\beta G_p(x_{n-1},Tx_{n-1},Tx_{n-1})+\gamma G_p(x,Tx,Tx)\\&\quad+\delta\left[G_p(x_{n-1},Tx,Tx)+G_p(x,Tx_{n-1},Tx_{n-1})\right],\end{aligned}$$

where $\alpha,\beta,\gamma,\delta$ are evaluated at (x,x_{n-1},x_{n-1}).

Take limit as $n\to\infty$ yields

$$G_p(x,Tx,Tx)\le\gamma G_p(x,Tx,Tx)+\delta G_p(x,Tx,Tx)\le(\gamma+\delta)G_p(x,Tx,Tx)\le\lambda G_p(x,Tx,Tx).$$

Since $\lambda<1$, then $G_p(x,Tx,Tx)=0$. Hence $x=Tx$.

For uniqueness, suppose $u,v\in X$ and $u\ne v$ are two fixed points of T, and there exists $z\in X$ which is comparable to u and v. Monotonicity of T implies that Tz_n is comparable to $T^nu=u$ and $T^nv=v$ for $n=0,1,2,\cdots$.

Moreover

$$\begin{aligned}G_p(T^nz,T^nu,T^nu)&\le\alpha G_p(T^{n-1}z,u,u)+\beta G_p(T^{n-1}z,T^nz,T^nz)+\gamma G_p(u,T^nu,T^nu)\\&\quad+\delta\left[G_p(T^{n-1}z,T^nu,T^nu)+G_p(u,T^nz,T^nz)\right]\\&\le\alpha G_p(T^{n-1}z,u,u)+\beta G_p(T^{n-1}z,T^nz,T^nz)+\gamma G_p(u,T^nu,T^nu)\\&\quad+\delta\left[G_p(T^{n-1}z,T^nu,T^nu)+G_p(u,T^{n-1}z,T^{n-1}z)+G_p(T^{n-1}z,T^nz,T^nz)-G_p((T^{n-1}z,T^{n-1}z,T^{n-1}z)\right]\end{aligned}$$

where $\alpha,\beta,\gamma,\delta$ are evaluated at $(T^{n-1}z,T^{n-1}u,T^{n-1}u)$.

Taking the limit as $n\to\infty$ and by symmetry we get,

$$G_p(T^nz,u,u)\le\alpha G_p(T^{n-1}z,u,u)+2\delta G_p(T^{n-1}z,u,u)\le(\alpha+2\delta)G_p(T^{n-1}z,u,u)\le\lambda G_p(T^{n-1}z,u,u)\tag{7}$$

Consequently, $G_p(T^nz,u,u)\le\lambda^nG_p(Tz_0,u,u)$.

Similarly, $G_p(T^nz,v,v)\le\lambda^nG_p(Tz_0,v,v)$.

Finally for all $n\in\mathbb{N}$ with $n\ge\tau$ where $\tau\in\mathbb{N}$ we have,

$$\begin{aligned}G_p(u,v,v)&\le G_p(u,T^{n-1}z,T^{n-1}z)+G_p(T^{n-1}z,v,v)-G_p(T^{n-1}z,T^{n-1}z,T^{n-1}z)\\&\le\lambda^{n-\tau}G_p(u,T^\tau z_0,T^\tau z_0)+\lambda^{n-\tau}G_p(T^\tau z_0,v,v).\end{aligned}$$

Letting $n\to\infty$ yields $G_p(u,v,v)=0$. Hence $u=v$.

Theorem 2.1 can be viewed as an extension of results of Turkoglu *et al.* ([11], Theorem 2.1) to the setting of G-partial metric spaces endowed with an order. The following corollary can be obtained:

Corollary 2.2. Let $(X,\prec)$ be a partially ordered set and let there exist a G-partial metric G_p in X such that (X,G_p) is a complete G-partial metric space. Let $T:X\to X$ be a self-mapping in X such that for each $x,y\in X$ satisfying $x\prec y$,

$$G_p(Tx,Ty,Ty)\le k\max\left\{G_p(x,y,y),G_p(x,Tx,Tx),G_p(y,Ty,Ty),\frac{1}{2}\left[G_p(x,Ty,Ty)+G_p(y,Tx,Tx)\right]\right\},$$

where $k\in[0,1)$.

Suppose T is a non-decreasing map such that there exists an $x_0\in X$ with $x_0\prec Tx_0$. Also suppose that X is such that for any non-decreasing sequence $\{x_n\}$ converging to x, $x_n\prec x$ for all $n\in\mathbb{N}$. Then T has a fixed point. Moreover, if for each $u,v\in X$, there exists $z\in X$ which is comparable to u and v, then T has a unique fixed point.

Proof: Observe that

$$\begin{aligned}&k\max\left\{G_p(x,y,y),G_p(x,Tx,Tx),G_p(y,Ty,Ty),\frac{1}{2}\left[G_p(x,Ty,Ty)+G_p(y,Tx,Tx)\right]\right\}\\&=\alpha(x,y,y)G_p(x,y,y)+\beta(x,y,y)G_p(x,Tx,Tx)+\gamma(x,y,y)G_p(y,Ty,Ty)\\&\quad+\delta(x,y,y)\left[G_p(x,Ty,Ty)+G_p(y,Tx,Tx)\right]\end{aligned}$$

where $\alpha,\beta,\gamma:X\times X\to\{0,k\}$ and $\delta:X\times X\to\left\{0,\frac{k}{2}\right\}$ are chosen such that for any $(x,y)\in X\times X$, one and only one of $\alpha(x,y,y),\beta(x,y,y),\gamma(x,y,y),\delta(x,y,y)$ is non-null. In such case,

$$\alpha(x,y,y)+\beta(x,y,y)+\gamma(x,y,y)+2\delta(x,y,y)=k<1.$$

Thus, the proof of the corollary follows from Theorem 2.1.

Theorem 2.3. Let $(X,\prec)$ be a partially ordered set and suppose there exists a G-partial metric G_p in X such that (X,G_p) is a complete G-partial metric space. Let $T:X\to X$ be a generalized Ciric quasi-contraction map such that α satisfies $\limsup_{t\to a}\alpha(t)<1$ for each $a\in[0,\infty)$, for any $x,y\in X$ with $x\prec y$.

Assume that there exists an $x_0\in X$ with the bounded orbit, that is the sequence $\{x_n\}$, defined by $x_{n+1}=Tx_n$ for all n, is bounded. Furthermore, if T is an increasing map such that there exists an $x_0\in X$ with $x_0\prec Tx_0$, and if any non-decreasing sequence $x_n\to x$ satisfies $x_n\prec x$ for all n, then T has a fixed point. Moreover, if for each $x,y\in X$, there exists $z\in X$ which is comparable to x and y, then T has a unique fixed point.

Proof. Starting with $x_0\in X$ such that $x_0\prec Tx_0$, and with T non-decreasing, we have

$$x_0\prec Tx_0\prec T^2x_0\prec T^3x_0\prec\cdots\prec T^nx_0\prec\cdots.$$

We prove that there exists $0<c<1$ such that

$$\alpha\left(G_p\left(x_n,x_{n+1},x_{n+1}\right)\right)<c,\ \text{for each } n\geq 0. \tag{8}$$

On the contrary, assume that

$$\lim_{k\to\infty}\alpha\left(G_p\left(x_{n_k},x_{n_k+1},x_{n_k+1}\right)\right)=1,$$

for some subsequence $\left\{\alpha\left(G_p\left(x_{n_k},x_{n_k+1},x_{n_k+1}\right)\right)\right\}$ of $\left\{\alpha\left(G_p\left(x_n,x_{n+1},x_{n+1}\right)\right)\right\}$. Since by our assumption the sequence $\left\{G_p\left(x_n,x_{n+1},x_{n+1}\right)\right\}$ is bounded, then the subsequence $\left\{G_p\left(x_{n_k},x_{n_k+1},x_{n_k+1}\right)\right\}$ is bounded too. Since the sequence is monotonic and bounded then it converges. Let $a=\lim_{k\to\infty}G_p\left(x_{n_k},x_{n_k+1},x_{n_k+1}\right)$. From our assumption, $\limsup_{t\to a}\alpha(t)=1$, a contradiction. Thus (8) holds.

Now, we show that $\{x_n\}$ is a Cauchy sequence. To prove the claim, we show by induction that for each $n\geq 2$,

$$G_p\left(x_{n-1},x_n,x_n\right)\leq Kc^{n-1}, \tag{9}$$

where K is a bound for the bounded sequence $\left\{G_p\left(x_0,x_n,x_n\right)\right\}$. When $n=2$,

$$\begin{aligned}G_p\left(Tx_0,T^2x_0,T^2x_0\right)\leq{}&\alpha\left(G_p\left(x_0,Tx_0,Tx_0\right)\right)\max\left\{G_p\left(x_0,Tx_0,Tx_0\right),G_p\left(x_0,Tx_0,Tx_0\right),\right.\\&\left.G_p\left(Tx_0,T^2x_0,T^2x_0\right),G_p\left(x_0,T^2x_0,T^2x_0\right),G_p\left(Tx_0,Tx_0,Tx_0\right)\right\}.\end{aligned}$$

From the axiom ($Gp1$), $G_p\left(Tx_0,Tx_0,Tx_0\right)\leq G_p\left(x_0,Tx_0,Tx_0\right)$. Thus

$$\begin{aligned}G_p\left(x_1,x_2,x_2\right)&\leq\alpha\left(G_p\left(x_0,x_1,x_1\right)\right)\max\left\{G_p\left(x_0,x_1,x_1\right),G_p\left(x_1,x_2,x_2\right),G_p\left(x_0,x_2,x_2\right)\right\}\\&\leq\alpha\left(G_p\left(x_0,x_1,x_1\right)\right)\max\left\{G_p\left(x_0,x_1,x_1\right),G_p\left(x_0,x_2,x_2\right)\right\}\leq Kc.\end{aligned}$$

Thus (9) holds for $n=2$.

Suppose that (9) holds for each $k<n$; let us show that it holds for $k=n$. Since T is a generalized Ciric quasi-contraction map,

$$\begin{aligned} G_p\left(x_{n-1},x_n,x_n\right) \le \alpha\left(G_p\left(x_{n-2},x_{n-1},x_{n-1}\right)\right)\max\{&G_p\left(x_{n-2},x_{n-1},x_{n-1}\right),G_p\left(x_{n-2},x_{n-1},x_{n-1}\right),\\ &G_p\left(x_{n-1},x_n,x_n\right),G_p\left(x_{n-2},x_n,x_n\right),G_p\left(x_{n-1},x_{n-1},x_{n-1}\right)\}. \end{aligned} \tag{10}$$

From axiom ($Gp1$), $G_p\left(x_{n-1},x_{n-1},x_{n-1}\right)\le G_p\left(x_{n-1},x_n,x_n\right)$.

Hence (10) becomes

$$\begin{aligned} G_p\left(x_{n-1},x_n,x_n\right) &\le \alpha\left(G_p\left(x_{n-2},x_{n-1},x_{n-1}\right)\right)\max\left\{G_p\left(x_{n-2},x_{n-1},x_{n-1}\right),G_p\left(x_{n-1},x_n,x_n\right),G_p\left(x_{n-2},x_n,x_n\right)\right\}\\ &\le \alpha\left(G_p\left(x_{n-2},x_{n-1},x_{n-1}\right)\right)\max\left\{G_p\left(x_{n-2},x_{n-1},x_{n-1}\right),G_p\left(x_{n-2},x_n,x_n\right)\right\}\\ &\le c\max\left\{G_p\left(x_{n-2},x_{n-1},x_{n-1}\right),G_p\left(x_{n-2},x_n,x_n\right)\right\}. \end{aligned}$$

From the induction hypothesis, $G_p\left(x_{n-2},x_{n-1},x_{n-1}\right)\le Kc^{n-2}$. Thus,

$$G_p\left(x_{n-1},x_n,x_n\right)\le c\max\left\{Kc^{n-2},G_p\left(x_{n-2},x_n,x_n\right)\right\}\le\max\left\{Kc^{n-1},cG_p\left(x_{n-2},x_n,x_n\right)\right\}. \tag{11}$$

We also have from the definition of T and the induction hypothesis,

$$\begin{aligned} G_p\left(x_{n-2},x_n,x_n\right) \le \alpha\left(G_p\left(x_{n-3},x_{n-1},x_{n-1}\right)\right)\max\{&G_p\left(x_{n-3},x_{n-1},x_{n-1}\right),G_p\left(x_{n-3},x_{n-2},x_{n-2}\right),\\ &G_p\left(x_{n-1},x_n,x_n\right),G_p\left(x_{n-3},x_n,x_n\right),G_p\left(x_{n-1},x_{n-2},x_{n-2}\right)\}\\ \le c\max\{&G_p\left(x_{n-3},x_{n-1},x_{n-1}\right),Kc^{n-3},G_p\left(x_{n-1},x_n,x_n\right),G_p\left(x_{n-3},x_n,x_n\right),Kc^{n-2}\}\\ \le c\max\{&Kc^{n-3},G_p\left(x_{n-3},x_{n-1},x_{n-1}\right),G_p\left(x_{n-3},x_n,x_n\right),G_p\left(x_{n-1},x_n,x_n\right)\}\\ \le \max\{&Kc^{n-2},cG_p\left(x_{n-3},x_{n-1},x_{n-1}\right),cG_p\left(x_{n-3},x_n,x_n\right),cG_p\left(x_{n-1},x_n,x_n\right)\}. \end{aligned}$$

The inequality (11) becomes

$$\begin{aligned} G_p\left(x_{n-1},x_n,x_n\right) &\le \max\left\{Kc^{n-1},c^2G_p\left(x_{n-3},x_{n-1},x_{n-1}\right),c^2G_p\left(x_{n-3},x_n,x_n\right),c^2G_p\left(x_{n-1},x_n,x_n\right)\right\}\\ &\le \max\left\{Kc^{n-1},c^2G_p\left(x_{n-3},x_{n-1},x_{n-1}\right),c^2G_p\left(x_{n-3},x_n,x_n\right)\right\}. \end{aligned} \tag{12}$$

Repeating the same process,

$$\begin{aligned} G_p\left(x_{n-1},x_n,x_n\right) &\le \max\left\{Kc^{n-1},c^3G_p\left(x_{n-4},x_{n-2},x_{n-2}\right),c^3G_p\left(x_{n-4},x_{n-1},x_{n-1}\right),c^3G_p\left(x_{n-4},x_n,x_n\right)\right\}\\ &\le\cdots\le\max\left\{Kc^{n-1},c^{n-1}G_p\left(x_0,x_1,x_1\right),\cdots,c^{n-1}G_p\left(x_0,x_n,x_n\right)\right\}\le Kc^{n-1}. \end{aligned}$$

Thus (9) holds for each $n\ge 2$. From (9) we deduce that $\{x_n\}$ is a Cauchy sequence.

Since X is complete then there exists $q\in X$ such that $\lim_{n\to\infty}x_n=q$ and

$$\lim_{n,m\to\infty}G_p\left(x_n,x_m,x_m\right)=\lim_{n\to\infty}G_p\left(x_n,q,q\right)=\lim_{n\to\infty}G_p\left(x_n,x_n,x_n\right)=G_p\left(q,q,q\right)=0.$$

Now we prove that q is the fixed point of T. To show that, we claim that there exists $0<b<1$ such that $\alpha\left(G_p\left(q,x_n,x_n\right)\right)<b$.

On the contrary, we assume $\lim_{k\to\infty}\alpha\left(G_p\left(q,x_{n_k},x_{n_k}\right)\right)=1$ for some subsequences $\{x_{n_k}\}$. Since $\lim_{k\to\infty}G_p\left(q,x_{n_k},x_{n_k}\right)=0$, then $\limsup_{t\to 0}\alpha(t)=1$, a contradiction.

Since T is a generalized quasi-contraction mapping we have

$$\begin{aligned} G_p\left(Tq,Tx_n,Tx_n\right)\le\alpha\left(G_p\left(q,x_n,x_n\right)\right)\max\{&G_p\left(q,x_n,x_n\right),G_p\left(q,Tq,Tq\right),\\ &G_p\left(x_n,x_{n+1},x_{n+1}\right),G_p\left(q,x_{n+1},x_{n+1}\right),G_p\left(x_n,Tq,Tq\right)\}. \end{aligned}$$

Letting $n \to \infty$, we have, $G_p(Tq,q,q) \le bG_p(q,Tq,Tq)$.
Also $G_p(q,Tq,Tq) \le bG_p(Tq,q,q)$. Hence $G_p(Tq,q,q) \le b^2 G_p(Tq,q,q)$. Since $b < 1$, $q = Tq$.
The uniqueness of the fixed point follows from the quasicontractive condition.
Theorem 2.3 is an extension of Theorem 2.3 of Gordji *et al.* [4] to G-partial metric space in the sense that, if

$$\max\left\{G_p(x,y,y), G_p(x,Tx,Tx), G_p(y,Ty,Ty), G_p(x,Ty,Ty), G_p(y,Tx,Tx)\right\} = G_p(x,y,y),$$

in (1), then we get

$$G_p(Tx,Ty,Ty) \le \alpha\left(G_p(x,y,y)\right)G_p(x,y,y),$$

which is the G-partial metric version of the map of Gordji [9].

The proof of Corollary 2.4 follows from Theorem 2.3.

Corollary 2.4. Let $(X,\prec)$ be a partially ordered set such that there exists a G-partial metric on X such that (X,G_p) is a complete G-partial metric space. Let $T: X \to X$ be an increasing mapping such that there exists $x_0 \in X$ with $x_0 \prec Tx_0$. Suppose that there exists $\alpha: \mathbb{R}^+ \to [0,1)$ such that

$$G_p(Tx,Ty,Ty) \le \alpha\left(G_p(x,y,y)\right)G_p(x,y,y),$$

for all comparable $x,y \in X$. If T is continuous and if for each $x,y \in X$, there exists $z \in X$ which is comparable to x and y. Then T has a unique fixed point.

Example 2.5. Let $X = \mathbb{R}^+$ and a G-partial metric defined by $G_p(x,y,z) = \max\{x,y,z\}$ for all $x,y,z \in \mathbb{R}$. On the set X, we consider the usual ordering $\le$. Clearly, (X,G_p) is a complete G-partial metric space and $(X,\le)$ is a partially ordered set. Define a function $T:\mathbb{R}^+ \to \mathbb{R}^+$ as follows: $Tx = \frac{x}{2}$ for all $x \in \mathbb{R}$. Define $\alpha:[0,\infty) \to [0,1)$ by $\alpha(t) = \frac{t}{1+2t}$ for each $t \in [0,\infty)$. Then we have,

$$G_p(x,y,y) \le \alpha \max\left\{G_p(x,y,y), G_p(x,Tx,Tx), G_p(y,Ty,Ty), G_p(x,Ty,Ty), G_p(y,Tx,Tx)\right\}$$

for each $x,y \in X$. Thus, all of the hypotheses of Theorem 2.3 are satisfied and so T has a unique fixed point (0 is the unique fixed point of T).

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