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FIXED POINT THEOREMS FOR MULTIVALUED CONTRACTIVE MAPPINGS IN *G*-SYMMETRIC SPACES

KANAYO STELLA EKE

Department of Mathematics, Covenant University,

Canaanland, KM 10, Idiroko Road, P.M.B. 1023, Ota, Ogun State, Nigeria

Abstract. In this paper, we proved some fixed point theorems for multivalued contraction maps and the existence of common fixed points for a pair of occasionally weakly compatible maps satisfying some multivalued contractive conditions in G-symmetric spaces. The results extend and generalize some results in the literature.

Keywords: Common fixed points; Multivalued contractive mappings; Occasionally weakly compatible maps; G-symmetric spaces.

2000 AMS Subject Classification: 47H10.

1. Introduction

The fixed point theorem for multivalued contraction mappings which takes each point of a metric space (X,d) into a closed subset K of X has gained enormous applications in areas such as control theory, optimization, differential equations and economics. Nadler [18] proved the existence of the fixed points for multivalued contraction mapping in complete metric spaces. Dugundji and Granas [6] proved that a single-valued weakly contractive mapping of a complete metric space into itself has a unique fixed point. Kaneko [12] gave a partial generalization of

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the theorem of Dugundji and Granas to the multivalued mappings. Reich [19] proposed the following problem.

Conjecture. Let (X,d) be a complete metric space. Suppose that $T : X \to CB(X)$ satisfies $H(Tx,Ty) \leq k(d(x,y))d(x,y)$, for all $x,y \in X$, $x \neq y$, where $k : (0,\infty) \to [0,\infty)$ and $\limsup_{r \to t^+} k(r) < 1$, for all $0 < t < \infty$. Then T has a fixed point in X. This conjecture was proved by Mizoguchi and Takahashi [15]. Daffer and Kaneko [5] provided an alternative and somewhat more s-traight forward proof for the theorem of Mizoguchi and Takahashi. Abbas et. al.[1] proved fixed point theorems for multivalued mappings satisfying generalized contractive conditions in ordered generalized metric spaces.

The notion of metric spaces has been generalized by many authors. In particular, Mustafa and Sims [17] generalized the notion of metric spaces. Eke and Olaleru [7] introduced the concept of G-partial metric spaces which generalized the notion of G-metric spaces. Cartan [3] also generalized the notion of metric spaces by omitting the triangle inequality axiom of the metric spaces. Recently, Eke and Olaleru [8] extend the concept of symmetric spaces to G-symmetric spaces by omitting the rectangle inequality axiom of the G-metric spaces. In the same reference, the existence of common fixed points for singled-valued occasionally weakly compatible mappings in G-symmetric spaces is proved. Also, Eke and Olaleru [9] proved existence of common fixed point for pair of hybrid contractive mappings in G-symmetric spaces. In this work, some fixed point theorems for multivalued contraction mappings in G-symmetric spaces are proved.

2. Preliminaries

The following definitions and results are needed in the sequel.

Definition 2.1. [17] Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying:

(G1) $G(x,y,z) = 0$ if $x = y = z$,

(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$,

(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$,

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x)$ (symmetry in all three variables), (G5) $G(x,y,z) \leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then, the function G is called a generalised metric, or more specifically a G-metric on X, and the pair (X,G) is a G-metric space.

Definition 2.2. [3] A symmetric on a set X is a real valued function d on $X \times X$ such that

(i) $d(x,y) \geq 0$ and $d(x,x) = 0$ if and only if $x = y$; and

(ii) $d(x,y) = d(y,x)$.

Wilson [21] also gave two more axioms of a symmetric d on X as:

(W_1) Given $\{x_n\}$, x and y in X, $d(x_n,x) \to 0$ and $d(x_n,y) \to 0$ imply that $x = y$;

(W_2) Given $\{x_n\}$, $\{y_n\}$ and $x \in X$, $d(x_n,x) \to 0$ and $d(x_n,y_n) \to 0$ imply that $d(y_n,x) \to 0$.

Definition 2.3. [13] A mapping $T : X \to 2^X$ is called a multivalued mapping. A point $x \in X$ is called a fixed point of T if $x \in Tx$.

Definition 2.4. [11] Let X be a given nonempty set. Assume that $g : X \to X$ and $T : X \to 2^X$. If $w = gx \in Tx$ for some $x \in X$ then x is called a coincidence point of g and T and w is a point of coincidence of g and T.

Definition 2.5. [11] Maps $g : X \to X$ and $T : X \to 2^X$ are said to be weakly compatible if $gx \in Tx$ for each $x \in X$ implies $gTx \subseteq Tgx$.

Definition 2.6. [11] Maps $g : X \to X$ and $T : X \to 2^X$ are said to be occasionally weakly compatible mappings if and only if there exists some point x in X such that $gx \in Tx$ and $gTx \subseteq Tgx$. An occasionally weakly compatible map is weakly compatible but not vice - verse.

The following definitions and motivations are found in [8].

Definition 2.7. A G-symmetric on a set X is a function $G_d : X \times X \times X \to R^+$ such that for all $x,y,z \in X$ the following conditions are satisfied:

$G_d(1)$ $G_d(x,y,z) \geq 0$ and $G_d(x,y,z) = 0$, if $x = y = z$;

$G_d(2) 0 < G_d(x,x,y)$ for all $x,y \in X$ with $x \neq y$,

$G_d(3)$ $G_d(x,x,y) \leq G_d(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$,

$G_d(4)$ $G_d(x,y,z) = G_d(y,z,x) = G_d(z,x,y) =,...,$ (symmetry in all three variables).

It should be observed that our notion of a G-symmetric space is the same as that of G-metric space (Definition 2.1) without the rectangular property (G_5).

Example 2.8. Let $X = [0,1]$ equipped with a G-symmetric defined by $G_d(x,y,z) = (x-y)^2 + (y-z)^2 + (z-x)^2$ for all $x,y,z \in X$. Then (X, G_d) is a G-symmetric space.

This does not satisfy the rectangle inequality property of a G-metric space, hence it is not a G-metric space. The analogue of axioms of Wilson [21] in G-symmetric space is as follows:
(W_3) Given $\{x_n\}$, x and y in X; $G_d(x_n,x,x) \to 0$ and $G_d(x_n,y,y) \to 0$ imply that $x = y$.
(W_4) Given $\{x_n\}$, $\{y_n\}$ and an x in X; $G_d(x_n,x,x) \to 0$ and $G_d(x_n,y_n,y_n) \to 0$ imply that $G_d(y_n,x,x) \to 0$.
(W_5) Let (X, G_d) be a complete G-symmetric space. For any sequence $\{x_n\}$ in X,
$\lim_{n,m\to\infty} G_d(x_n,x_m,x_m) = 0$ iff $\lim_{n\to\infty} G_d(x_n,x_{n+1},x_{n+1}) = 0$.

Definition 2.9. Let (X, G_d) be a G-symmetric space.
(i) (X, G_d) is G_d- complete if for every G_d- Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n\to\,\infty} G_d(x_n,x,x) = 0$.
(ii) $f : X \to X$ is G_d-continuous if $\lim_{n\to\,\infty} G_d(x_n,x,x) = 0$ implies $\lim_{n\to\,\infty} G_d(fx_n, fx, fx) = 0$.
Let (X, G_d) be a G-symmetric space. We denote the family of all nonempty closed and bounded subsets of X by CB(X).

Definition 2.10. Two mappings $T,S : X \to CB(X)$ are said to be weak-contraction if there exists $0 < \alpha < 1$ such that
$H_{G_d}(Tx,Sy,Sy) \leq \alpha M(x,y,y)$
and
$H_{G_d}(Tx,Tx,Sy) \leq \alpha M(x,x,y)$
for all $x,y \in X$, where
$M(x,y,y) = max\{G_d(x,y,y), G_d(x,Tx,Tx), G_d(y,Sy,Sy), G_d(x,Sy,Sy), G_d(y,Tx,Tx)\}$
and
$M(x,x,y) = max\{G_d(x,x,y), G_d(x,x,Tx), G_d(y,y,Sy), G_d(x,x,Sy), G_d(y,y,Tx)\}$.

Definition 2.11. A mapping $T : X \to CB(X)$ are said to be weak-contraction if there exists $0 < \alpha < 1$ such that

$H_{G_d}(Tx,Ty,Ty) \leq \alpha N(x,y,y)$

and

$H_{G_d}(Tx,Tx,Ty) \leq \alpha N(x,x,y)$

for all $x,y \in X$, where

$N(x,y,y) = max\{G_d(x,y,y), G_d(x,Tx,Tx), G_d(y,Ty,Ty), G_d(x,Ty,Ty), G_d(y,Tx,Tx)\}$

and

$N(x,x,y) = max\{G_d(x,x,y), G_d(x,x,Tx), G_d(y,y,Ty), G_d(x,x,Ty), G_d(y,y,Tx)\}$

where H_{G_d} denotes the Hausdorff G-symmetric on CB(X) induced by G_d, that is

$H_{G_d}(A,B,B) = max\{Sup_{x\in A} G_d(x,B,B), Sup_{y\in B} G_d(y,A,A)\}$,

for all $A,B \in CB(X)$.

Moutawakil [16] gave a generalization of the well-known Nadler multivalued contraction fixed point to the setting of symmetric spaces. Chandra and Bhatt [4] proved a fixed point theorem for generalized contractions under restrictive conditions in symmetric spaces. Joshi et al. [10] proved some fixed point theorems for multivalued contractive mappings in symmetric spaces. Bhatt et al. [2] established some fixed point theorems for occasionally weakly compatible mappings in symmetric spaces. Jungck and Rhoades [11] proved some fixed point theorems for occasionally weakly compatible maps satisfying certain contractive conditions in symmetric spaces. Rouhani and Moradi [20] established the existence and uniqueness of fixed and coincidence points of multivalued generalized φ-weak contractive mappings on complete metric spaces. The aim of this paper, is to prove some existence and uniqueness of fixed point of multivalued contraction maps and common fixed theorems for occasionally weakly compatible mappings satisfying multivalued weak - contractive conditions in G-symmetric spaces.

3. Main results

Theorem 3.1. *Let (X,G_d) be a G_d-complete G-symmetric space which satisfies W_4 and W_5 such that*

(i) the map $f : X \to R$ defined by $f(x) = G_d(x,Tx,Tx)$, $x \in X$, is lower semi-continuous;

(ii) $T : X \to C(X)$ a multivalued mapping such that :

$$D(Tx,Ty,Ty) \le kG_d(x,y,y),\, k \in [0,1), \tag{3.1}$$

for all $x,y \in X$. Then there exists $u \in X$ such that $u \in Tu$.

Proof. Let $x_1 \in X$ and $\alpha \in (k,1)$. Since Tx_1 is nonempty, there exists $x_2 \in Tx_1$ such that $G_d(x_1,x_2,x_2) = G_d(x_1,Tx_1,Tx_1)$. In view of (3.1) we obtain,

$$\begin{aligned} G_d(x_2,Tx_2,Tx_2) &\le D(Tx_1,Tx_2,Tx_2) \\ &\le kG_d(x_1,x_2,x_2) \\ &< \alpha G_d(x_1,x_2,x_2). \end{aligned}$$

Similarly there exists $x_3 \in Tx_2$ such that $G_d(x_2,x_3,x_3) = G_d(x_2,Tx_2,Tx_2)$ and

$$\begin{aligned} G_d(x_3,Tx_3,Tx_3) &\le D_{G_d}(Tx_2,Tx_3,Tx_3) \\ &\le kG_d(x_2,x_3,x_3) \\ &< \alpha G_d(x_2,x_3,x_3). \end{aligned}$$

Consequently, we obtain a sequence $\{x_n\}_{\ge 1}^{\infty}$ in X satisfying $x_{n+1} \in Tx_n$ such that $G_d(x_n,x_{n+1},x_{n+1}) = G_d(x_n,Tx_n,Tx_n)$ and have

$$\begin{aligned} G_d(x_n,Tx_n,Tx_n) &\le D_{G_d}(Tx_{n-1},Tx_n,Tx_n) \\ &\le kG_d(x_{n-1},x_n,x_n) \\ &< \alpha G_d(x_{n-1},x_n,x_n) \\ &< (\alpha)^{n-1} G_d(x_1,x_2,x_2). \end{aligned}$$

This implies that $\lim_{n\to\infty} G_d(x_n,Tx_n,Tx_n) = 0$. Since $G_d(x_n,x_{n+1},x_{n+1}) = G_d(x_n,Tx_n,Tx_n)$ then we have $\lim_{n\to\infty} G_d(x_n,x_{n+1},x_{n+1}) = 0$ and with (W_5), $\{x_n\}$ is a Cauchy sequence. Hence there exists $u \in X$ such that $x_n \to u$. Also by (i) where $f(u) = G_d(u,Tu,Tu)$ we obtain $0 \le G_d(u,Tu,Tu) \le \lim_{n\to\infty} G_d(x_n,Tx_n,Tx_n) = 0$. Since T is closed and (X,G_d) is complete then $u \in Tu$. Thus u is a fixed point of T.

Remark 3.2. *Theorem 2.1 is an extension of Theorem 2.2 in [16] in the G-symmetric spaces.*

Corollary 3.3. *Let (X,G_d) be a G_d-bounded and G_d-complete G-symmetric space which satisfies W_4 and $T : X \to C(X)$ be a multivalued mapping such that:*

$$G_d(Tx,Ty,Ty) \leq kG_d(x,y,y), \tag{3.2}$$

for all $x,y \in X$ *and* $k \in [0,1)$*. Then there exist* $u \in X$ *such that* $u \in Tu$*.*

The proof of corollary 3.2 follows from Theorem 2.1. The following Lemma is needed in the proof of the next theorem and we refer to Nadler [18] for its proof.

Lemma 3.4. *If* $A,B \in CB(X)$ *and* $a \in A$*, then for each* $\varepsilon > 0$*, there exists* $b \in B$ *such that* $G_d(a,b,b) \leq H_{G_d}(A,B,B)+\varepsilon$*.*

Theorem 3.5. *Let* T *and* S *be multivalued weak-contraction mapping of a G-symmetric space* X *such that the pair* $\{T,S\}$ *are occasionally weakly compatible. If for all* $x,y \in X$

$$H_{G_d}(Tx,Sy,Sy) \leq \alpha M(x,y,y) \tag{3.3}$$

and

$$H_{G_d}(Tx,Tx,Sy) \leq \alpha M(x,x,y) \tag{3.4}$$

where $0 \leq \alpha < 1$ *and* $\beta = \alpha+\varepsilon < 1$*. Then there exists a point* $x \in X$ *such that* $x \in Tx$ *and* $x \in Sx$ *(i.e.* T *and* S *have a common fixed point in* X*).*

Proof. Since $\{T,S\}$ are occasionally weakly compatible then there exist $x \in X$ such that $Tx \in Sx$. Suppose there exist x_1, $x_2 \in X$ such that $x_1 \in Tx_1$ and $x_2 \in Sx_2$ with $TSx_2 \subseteq STx_1$ then using (3.3) and Lemma 3.4 we show that $x_1 = x_2$. On the contrary we assume that $x_1 \neq x_2$ and have,

$$\begin{aligned}
G_d(x_1,x_2,x_2) &\leq H_{G_d}(Tx_1,Sx_2,Sx_2)+\varepsilon M(x_1,x_2,x_2) \\
&\leq \alpha M(x_1,x_2,x_2)+\varepsilon M(x_1,x_2,x_2) \\
&= \beta\ M(x_1,x_2,x_2) \\
&= \beta\ max\{G_d(x_1,x_2,x_2), G_d(x_1,Tx_1,Tx_1), G_d(x_2,Sx_2,Sx_2), G_d(x_1,Sx_2,Sx_2), \\
&\quad G_d(x_2,Tx_1,Tx_1)\} \\
&\leq \beta\ max\{G_d(x_1,x_2,x_2), G_d(x_1,x_1,x_1), G_d(x_2,x_2,x_2), G_d(x_1,x_2,x_2), \\
&\quad G_d(x_2,x_1,x_1)\} \\
&\leq \beta\ max\{G_d(x_1,x_2,x_2), G_d(x_2,x_1,x_1)\}.
\end{aligned}$$

Case (i)

Suppose $\max\{G_d(x_1,x_2,x_2), G_d(x_2,x_1,x_1)\} = G_d(x_1,x_2,x_2)$. Then $G_d(x_1,x_2,x_2) \leq \beta\ G_d(x_1,x_2,x_2)$.

Case (ii)

If $\max\{G_d(x_1,x_2,x_2),G_d(x_2,x_1,x_1)\} = G_d(x_2,x_1,x_1)$. Then

$$G_d(x_1,x_2,x_2) \leq \beta\ G_d(x_2,x_1,x_1) \tag{3.5}$$

Similarly, using (3.4), we obtain

$$\begin{aligned}
&G_d(x_1,x_1,x_2)\\
&\leq H_{G_d}(Tx_1,Tx_1,Sx_2)+\varepsilon M(x_1,x_1,x_2)\\
&\leq \alpha M(x_1,x_1,x_2)+\varepsilon M(x_1,x_1,x_2)\\
&= \beta\ M(x_1,x_1,x_2)\\
&= \beta\ max\{G_d(x_1,x_1,x_2),G_d(x_1,x_1,Tx_1),G_d(x_2,x_2,Sx_2),G_d(x_1,x_1,Sx_2),\quad G_d(x_2,x_2,Tx_1)\}\\
&\leq \beta\ max\{G_d(x_1,x_1,x_2),G_d(x_1,x_1,x_1),G_d(x_2,x_2,x_2),G_d(x_1,x_1,x_2),\\
&G_d(x_2,x_2,x_1)\}\\
&\leq \beta\ max\{G_d(x_1,x_1,x_2),G_d(x_2,x_2,x_1)\}
\end{aligned}$$

Case (i)

Suppose max $\{G_d(x_1,x_1,x_2),G_d(x_2,x_2,x_1)\} = G_d(x_1,x_1,x_2)$ then

$G_d(x_1,x_1,x_2) \leq \beta G_d(x_1,x_1,x_2)$

Case(ii)

If max $\{G_d(x_1,x_1,x_2),G_d(x_2,x_2,x_1)\} = G_d(x_2,x_2,x_1)$ then

$$G_d(x_1,x_1,x_2) \leq \beta\ G_d(x_2,x_2,x_1) \tag{3.6}$$

Combining (3.5) and (3.6) with $G_d(4)$ yields

$G_d(x_1,x_2,x_2) \leq \beta G_d(x_1,x_2,x_2)$

Since $\beta < 1$ then $x_1 = x_2$. This implies that $x_1 \in Tx_1$ and $x_1 \in Sx_1$. Hence T and S have a common fixed point.

Remarks 3.6. *Theorem 2.5 is an analogy result of Theorem 3.1 in [20] in the setting of G-symmetric spaces.*

If $T = S$ in Theorem 2.5, then we have the following corollary.

Corollary 3.7. *Let (X, G_d) be a G_d-complete G-symmetric space and let $T : X \to CB(X)$ be a multivalued weak-contraction mapping such that for all $x, y \in X$*

$$H_{G_d}(Tx, Ty, Ty) \leq \alpha N(x, y, y), \tag{3.7}$$

where $0 \leq \alpha < 1$. Then there exists a point $x \in X$ such that $x \in Tx$.

Example 3.8. *Let $X = [0, 1]$ be endowed with the G-symmetric . Let $S, T : X \to CB(X)$ be defined by $Tx = [0, \frac{x}{6}]$ and $Sy = [\frac{y}{6}]$.*

$$\begin{aligned} H_{G_d}(Tx, Sy, Sy) &= max\{(\frac{y}{6} - \frac{x}{6})^2, (\frac{y}{6})^2\} \\ &\leq \frac{1}{3} max\{(y-x)^2, (y - \frac{y}{6})^2\} \\ &\leq \frac{1}{3} max\{G_d(x, y, y), G_d(y, Sy, Sy)\} \\ &\leq \frac{1}{3} M(x, y, y). \end{aligned}$$

So, T and S have a common fixed point $(x = 0)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] M. Abbas, A. R. Khan and T. Nazir, Common fixed point of multivalued mappings in ordered generalized metric spaces, Filomat 26 (2012), 1045-1053.

[2] A. Bhatt, H. Chandra and D. R. Sahu, Common fixed point theorems for occasionally weakly compatible mappings under relaxed conditions, Nonlinear Anal. 73 (2010),176-182.

[3] E. Cartan, Sur la determination d'un systeme orthogonal complet dans un espace de Riemman symetrique clos, Rend. Ciric. Mat. Palermo 53 (1929), 217-252.

[4] H. Chandra and A. Bhatt, Some fixed point theorems for set valued maps in symmetric spaces, Int. J. Math. Anal. 3 (2009), 839-846.

[5] P. Z. Daffer, Fixed points of generalised contractive multi-valued mappings, J. Math. Anal. Appl. 192 (1995), 655-666.

[6] J.Dugundji and A. Granas, Weakly contractive maps and elementary domain invariance theorem, Bull. Greek Math. Soc. 19 (1978), 141-151.

[7] K. S. Eke and J. O. Olaleru, Some fixed point results on ordered G-partial metric spaces, ICASTOR J. Mathe. Sci. 7 (2013), 65-78 .

[8] K. S. Eke and J. O. Olaleru, Common fixed point results for occasionally weakly compatible maps in G-symmetric spaces, Appl. Math. 5 (2014), 744-752.

[9] K. S. Eke and J. O. Olaleru, Common fixed point results for pairs of hybrid maps in G-symmetric spaces, Int. J. Pure Appl. Math. 101 (2015), 117-132.

[10] L. K. Joshi, M. C. Joshi and A. Kumar, Fixed point theorems for multivalued mappings in symmetric spaces, Adv. Inequal. Appl. 2014 (2014), Article ID 31.

[11] G. Jungck and B. E. Rhoades, Fixed point theorems for occasionally weakly compatible mappings, Fixed Point Theory, 7 (2006), 286-296.

[12] H. Kaneko, Generalized contractive multivalued mappings and their fixed points, Math. Japon. 33 (1988), 57-64.

[13] M. S. Khan, Common fixed point theorems for multivalued mappings, Pacific J. Math. 95 (1981), 337-347.

[14] D. Klim and D. Wardowski, Fixed point theorem for set valued contractions in complete metric spaces, J. Math. Anal. Appl. 334 (2007), 132-139.

[15] N. Mizoguchi and W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl. 141 (1989), 177-188.

[16] D. El Moutawakil, A fixed point theorem for multivalued maps in symmetric spaces, Appl. Math. E-Notes 4 (2004), 26-32.

[17] Z. Mustafa and B. Sims, A New Approach To Generalised Metric Spaces, J. Nonlinear Convex Anal. 7 (2006), 289-297.

[18] S. B. Nadler, Multivalued contraction mappings, Pacific J. Math. 1969.

[19] S. Reich, Some problems and results in fixed point theory, Contemp. Math. 21 (1983), 179-187.

[20] B. D. Rouhani and S. Moradi, Common fixed point of multivalued generalized ψ-weak contractive mappings, Fixed Point Theory Appl. 2010 (2010), Article ID 708984.

[21] W. A. Wilson, On semi-metric spaces, Amer. J. Math. 53 (1931), 361-373.