ISSN – 0974 – 195X

ICASTOR Journal of Mathematical Sciences
Vol. 7, No. 1 (2013) 65 - 78

SOME FIXED POINT RESULTS ON ORDERED G – PARTIAL METRIC SPACES

K. S. Eke
Department of Mathematics,
Covenant University,
Nigeria

AND

J. O. Olaleru
Department of Mathematics,
University of Lagos,
Nigeria

Correspondence: K. S. Eke, *Department of Mathematics Covenant University, Ota, Ogun State Nigeria E-mail: olaleru1@yahoo.co.uk*

ICASTOR Journal of Mathematical Sciences
Vol. 7, No. 1 (2013) 65 - 78

Some Fixed point results on ordered G-Partial metric spaces.

K. S. Eke
Department of Mathematics,
Covenant University,
Nigeria

J. O. Olaleru
Department of Mathematics,
University of Lagos,
Nigeria

Kanayo Stella Eke

ABSTRACT

We introduce a new concept of generalized metric space called a G- partial metric space and prove the fixed point of certain contraction maps defined on the ordered G-partial metric spaces. Our work extends some works in literature.

KEYWORDS: Fixed points, contraction maps, weak contraction maps, partially ordered set, G-partial metric

2000 AMS Subject Classification: 47H10,

INTRODUCTION

The notion of metric space was introduced as a result of the need to develop an axiomatic analysis system to provide an abstraction of different objects, studied by analysis, in a similar way to group theory which provide abstraction to algebraic system. Matthew [9] generalized metric spaces to partial metric spaces which he defined as follows:

1. Preliminary Definitions

Definition 1.1. [9]: A partial metric space is a pair $(X; p)$: $X \times X \rightarrow R$ such that:

(p1) $0 \leq p(x,x) \leq p(x,y)$
(p2) if $p(x,x) = p(x,y) = p(y,y)$, then x = y
(p3) $p(x,y) = p(y,x)$
(p4) $p(x,z) \leq p(x,y) + p(y,z)\ p(y,y)$.

He was able to established a relationship between partial metric spaces and the usual metric spaces with this example: $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$. In 1960, the notion of 2-metric space was introduced by Gahler as a generalization of the usual notion of metric space by replacing the triangular inequality with the tetrahedral inequality:

$$d(x,y,z) \leq d(x,y,a) + d(x,a,z) + d(a,y,z) \text{ for all } x,y,z,a \in X.$$

The example of a 2-metric was given with d(x, y, z) to be the area of a triangle with vertices at x,y,z in R^2. Gahler [6] claimed that a 2-metric is a generalization of the usual metric, but

Correspondence: **K. S. Eke,** Department of Mathematics Covenant University, Ota, Ogun State, Nigeria
E-mail: olaleru1@yahoo.co.uk

different authors e.g. see([7]) proved that there is no relationship between these two functions. For instance, the contraction mapping theorem in metric spaces defined as

$d(Tx, Ty) \leq kd(x, y)$ for all $k \in (0.1)$

is unrelated to the contraction of 2-metric which is of the form $d(Tx, Ty, a) \leq \cdots$ for any $a \in X$

Recently, Aliouche and Carlos [2], modified the original definition of 2- metric. Thereby making the contractivity condition $d(Tx, Ty, Tz) \leq kd(x, y, z)$ possible.

Dhage [5], introduced a new generalized metric space called D-metric spaces and gave an example of a D-metric to be the perimeter of the triangle with vertices at x, y *and* z in R^2 . He was able to establish a relationship between the D-metric and the usual metric. Furthermore, he attempted to develop topological structures in such spaces and then, claimed that D-metric provides a generalization of ordinary metric and then presented several fixed point results on D metric spaces..

But in 2003, Mustafa in collaboration with Brailey Sims, demonstrated that most of the claims concerning the fundamental topological structure of D-metric space are incorrect. For example, a D-metric need not be a continuous function of its variables. Also D-convergence of a sequence $\{x_n\}$ to x, in the sense of $D(x_m, x_n, x) \to 0 \; as \; n, m. \to \infty$ need not correspond to convergence in any topology. In view of this, Mustafa introduced a more acceptable and appropriate notion of generalized metric space which he defined as follows:

Definition 1.2. [12]: Let X be a nonempty set, and let $G: X \times X \times X \to R^+$ be a function satisfying:

(G1) $G(x, y, z) = 0 \; if \; x = y = z$,

(G2) $0 < G(x, x, y) for \; all \; x, y \in X \; with \; x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z) for \; all \; x, y, z \in X \; with \; y \neq z$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x)$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ *(rectangle inequality)*.

Then, the function G is called a generalized metric, or more specifically a G-metric on X, and the pair (X,G) is a G-metric space.

Mustafa[12] gave an example to show the relationship between G-metric spaces and ordinary metric spaces as: For any G-metric G on X, if $d_G(x, y) = G(x, y, y) + G(x, x, y)$, then d_G is a metric space.

In this paper, we combine the idea of the nonzero self distance of partial metric spaces and the rectangle inequality of G-metric spaces to develop a new generalized metric space which we define as follows:

Definition 1.3: Let X be a nonempty set, and let $G_p : X \times X \times X \to R^+$ be a function satisfying the following:

(G_p1) $G_p(x,y,z) \geq G_p(x,x,x) \geq 0$ for all $x,y,z, \in X$(small self distance),
(G_p2) $G_p(x,y,z) = Gp(x,x,y) = G_p(y,y,z) = G_p(z,z,x)\ iff\ x = y = z$, (equality),
(G_p3) $G_p(x,y,z) = G_p(z,x,y) = G_p(y,z,x)$ (symmetry in all three variables),
(G_p4) $G_p(x,y,z) \leq G_p(x,a,a) + G_p(a,y,z)\ G(a,a,a)$ (rectangle inequality).

The function G_p is called a G- partial metric and the pair (X, G_p) is called a G-partial metric spaces.

Definition 1.4: A G-partial metric space is said to be symmetric if $G_p(x,y,y) = G_p(y,x,x) for\ all\ x,y \in X.$

Example 1.5: Let a metric d_{Gp} be defined on a nonempty set X by $d_{Gp}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(y,y,y) - G_p(x,x,x). Then\ (X, d_{Gp})$ is a metric space

Example 1.6: Let $X = R^+$ and defined a $G - partial\ metric\ G_p : X \times X \times X \to R^+\ with\ G_p(x,y,z) = \max\{x,y,z\}$, then $(X, G_p)\ is\ a\ G -$ partial metric space.

The concept of Cauchy sequence, completeness of the space and contraction fixed point theorem have been studied in the partial metric space as well as in the G- metric spaces. We want to define these concepts in G-partial metric spaces. The next definitions generalize the notion of the properties in these metric spaces.

Definition 1.7: A sequence $\{x_n\}$ of points in a G-partial metric space (X, G_p) converges to some $a \in X$ if $lim_{n\to\infty} G_p(x_n, x_n, a) = lim_{n\to\infty} G_p(x_n, x_n, x_n) = G_p(a,a,a)$.

This means that whenever a sequence converges to a point, then the self- distances converge to the self distance of that point.

Definition 1.8: A sequence $\{x_n\}$ of points in a G-partial metric spaces (X, G_p) is Cauchy if the numbers $G_p(x_n, x_m, x_l,)$ converges to some $a \in X$ as n, m, l approach infinity.

The proof of the following result follows from definition.

Proposition 1.9: Let $\{x_n\}$ be a sequence in G -partial metric space X and $a \in X$. If $\{x_n\}$ converges to $a \in X$, then $\{x_n\}$ is a Cauchy sequence.

Definition 1.10: A G -partial metric space (X, G_p) is said to be complete if every Cauchy sequence in (X, G_p) converges to an element in (X, G_p).

The following lemma can be easily proved from the definition.

Lemma 1.11: Let (X, G_p) be a G-partial metric space.

(a) $\{x_n\}$ is a Cauchy sequence in (X, G_p) if and only if it is a Cauchy sequence in the metric space (X, d_{G_p});

(b) A G-partial metric space (X, G_p) is complete if and only if the metric space (X, d_{G_p}) is complete. Furthermore, $lim_{n\to\infty} d_{G_p}(x_n, x) = 0$ if and only if $G_p(x, x, x) = lim_{n\to\infty} G_p(x_n, x, x) = lim_{n,m\to\infty} G_p(x_n, x_m, x_m)$.

Definition 1.12 [4]: If $(X, \preceq)$ is a partially ordered set and $f : X \to X$, we say that f is monotone no decreasing if $x, y \in X, x \preceq y$ implies $fx \preceq fy$.

The most important Theorem which is the basis of all other contractive maps is the Banach contraction principle. The existence of the fixed point for Banach contraction and weak contraction maps in the context of G -metric spaces and partial metric spaces were proved in [1, 13].

Recently, Ran and Reurings [14] proved the existence of the fixed point of contraction maps satisfying certain conditions in a partially ordered set defined in a metric space. Altun et al. [3] gave similar result using generalized contractive maps in partial ordered metric spaces. Some authors (see: [4], [8]) worked on the existence of the fixed point for weak contractions in both partial ordered metric spaces and partially ordered G -metric spaces.

In this paper, we proved the existence and uniqueness of the fixed points for certain contraction mappings in ordered G –partial metric spaces which is an analogue of the results stated below.

Theorem 1.1 [14]: Let $(X, \preceq)$ be a partially ordered set such that every pair $x, y \in X$ has a lower bound and an upper bound. Furthermore, let d be a metric on X such that (X, d) is a complete metric space. If F is a continuous monotone (i.e, either order-preserving or order - reserving) map from X into X such that

$$\text{(i)} \; \exists\, 0 < c < 1: d(F(x), F(y)) \le cd(x, y) \forall x \ge y, \tag{1}$$

$$\text{(ii)} \; \exists x_0 \in X: x_0 \preceq F(x_0) \; or \; F(x_0) \preceq x_0,$$

then F has a unique fixed point x^-. Moreover, for every $x \in T, lim_{n\to\infty} F^n(x) = x^-$.

Theorem 1.2 [4]: Let $(X, \preceq)$ be a partially ordered set and let p be a partial metric on X such that (X, p) is complete. Let $f : X \to X$ be a non-decreasing map with respect to $\preceq$. Suppose that the following conditions hold: for $y \preceq x$, we have

$$\text{(i)} \quad p(fx, fy) \leq p(x, y) - \psi\,(p(x, y)) \tag{2}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that it is positive in $(0, \infty)$, $\psi\,(0) = 0$ and $lim_{n\to\infty}\, \psi\,(t) = \infty$;

(ii) $\exists x_0 \in X$ such that $x_0 \preceq f\, x_0$;

(iii) if a non-decreasing sequence $\{x_n\}$converges to $x \in X$, then $x_n \preceq x$ for all n .

Then f has a fixed point $u \in X$. Moreover, $p(u, u) = 0$.

2 MAIN RESULT

Theorem 2.1: Let $(X, \preceq)$ be a partially ordered set such that for $x;\ y \in X$ there exists a lower bound and an upper bound. Furthermore, let G_p be a G-partial metric on X such that (X, G_p) is a complete G -partial metric space. If T is a continuous and monotone (i.e, either order-preserving or order - reserving) map from X into X such that

$$\text{(i)} \ \exists\, 0 < k < 1 : G_p\,(Tx, Ty, Tz) \leq kG_p(x, y, z), \forall x \geq y \geq z, \tag{3}$$

$$\text{(ii)} \ \exists x_0 \in X : x_0 \preceq T(x_0) \ or\ T(x_0) \preceq x_0.$$

Then T has a unique fixed point p.

Proof: Let $x_0 \in X$ be arbitrary then $x_0 \preceq Tx_0$ or $T(x_0) \preceq x_0$. Suppose $x_0 = Tx_0$ then the fixed point exists. On the other hand, let $x_0 > Tx_0\ or\ x_0 < Tx_0$. Using the monotonicity of T we have that $T^n x_0 \leq T^{n+1}x_0\ or\ T^n x_0 \geq T^{n+1}x_0$ for $n = 0, 1, 2, \ldots$. Suppose that T satisfies condition (3) then we have for all

$$x, y \in X\ G_p\,(Tx, Ty, Ty) \leq kG_p(x, y, y), \forall x \geq y, \tag{4}$$

and

$$G_p\,(Ty, Tx, Tx) \leq kG_p(y, x, x);\ \forall x \geq y, \tag{5}$$

If G_p is symmetric, by adding (4) and (5) we get

$2\, G_p\,(Tx, Ty, Ty) \leq 2k\, G_p\,(x, y, y)$

which is equivalent to

$d\, G_p(Tx, Ty) \leq kd_{G_p}(x, y)$.

Since $T^n x_0 \preceq T^{n+1}x_0$ *or* $T^{n+1}x_0 \preceq T^n x_0$ *we have*

$dG_p(T^{n+1}x_0, T^n x_0) \leq kd_{G_p}(T^n x_0, T^{n-1}x_0)$.

By induction, we obtain,

$d_{G_p}(T^{n+1}x_0, T^n x_0) \leq k^n d_{G_p}(Tx_0, x_0)$.

For any $n, l > 1$, we have

$$\begin{aligned} d_{G_p}(T^n x_0, T^{n+l}x_0) &\leq d_{G_p}(T^n x_0, T^{n+1}x_0) + \cdots + d_{G_p}(T^{n+l-1}x_0, T^{n+l}x_0) \\ &\leq d_{G_p}(T^n x_0, T^{n+1}x_0).(1 + k + \cdots + k^{l-1}) \\ &= \frac{1-k^l}{1-k}\, d_{G_p}(T^n x_0, T^{n+1}x_0) \\ &\leq \frac{k^n}{1-k}\, d_{G_p}(x_0, Tx_0). \end{aligned}$$

This shows that $\{T^n x_0\}$ is Cauchy.

If G_p is not symmetric, we choose $x_n \in X$ such that $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$.

Then with (3) we have,

$G_p\,(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) \leq kG_p(T^{n-1}x_0; T^n x_0, T^n x_0)$.

Consequently,

$G_p\,(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) \leq k^n G_p(x_0, Tx_0, Tx_0)$.

Similarly,

$G_p(T^{n+1}x_0, T^n x_0, T^n x_0,) \leq k^n G_p(Tx_0, x_0, x_0)$

The last two inequalities show that T is either order- preserving or order- reserving.

For $m > n$, we get

$$\begin{aligned} G_p(T^n x_0, T^m x_0, T^m x_0) &\leq G_p(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) + G_p(T^{n+1}x_0, T^m x_0, T^m x_0) \\ &\quad - G_p\,(T^{n+1}x_0, T^{n+1}x_0, T^{n+1}x_0) \\ &\leq G_p\,(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) \\ &\quad + G_p\,(T^{n+1}x_0, T^{n+2}x_0, T^{n+2}x_0) \end{aligned}$$

$$+ G_p\,(T^{n+2}x_0, T^{n+3}x_0, T^{n+3}x_0)$$
$$+ \cdots + G_p\,(T^{m-1}x_0, T^{m}x_0, T^{m}x_0)$$
$$- G_p\,(T^{n+2}x_0; T^{n+2}x_0; T^{n+2}x_0)$$
$$- \ldots - G_p\,(T^{m-1}x_0, T^{m-1}x_0, T^{m-1}x_0)$$
$$\leq (k^n + k^{n+1} + \cdots + k^{m-1})\, G_p(x_0, Tx_0, Tx_0)$$
$$\leq \frac{k^n}{1-k}\, G_p(x_0, Tx_0, Tx_0,).$$

Let $m, n \rightarrow \infty$ then $G_p(T^n x_0, T^m x_0, T^m x_0) \rightarrow 0$. Hence $\{ T^n x_0 \}$ is a Cauchy sequence. Since X is complete and using lemma (1.11), the limit of T exist, say p.

Therefore $lim_{n\rightarrow\infty} T^n x_0 = p$ for some $p \in X$. But T is continuous, therefore p is the fixed point of T.

Next we show that the fixed point is unique, since there exist a lower bound or an upper bound for $y, z \in X$. In [16] it was proved that the condition for lower bound or upper bound is equivalent to:

for $y, z \in X$ there exist $x \in X$ which is comparable to y and z. Now, suppose that z and y in X are different fixed point of T, then $G_p(z, y, y) > 0$. Here, we consider two cases:
Case (i): If y and z are comparable, then $T^n y = y \; and \; T^n z = z$ are comparable for n = 0,1,2,...

Using condition (3) we have

$G_p(z, y, y) = G_p(T^n z, T^n y, T^n y) \leq kG_p(T^{n-1}z, T^{n-1}y, T^{n-1}y)$.
Letting $n \rightarrow \infty \; we \; obtain \; G_p(z, y, y) \leq kG_p(z, y, y)$.
Since $0 \leq k < 1$, then we get $z = y$.

Case (ii): If z and y are not comparable, then there exists $x \in X$ comparable to z and y. Since T is monotone, then $T^n x$ is comparable to $z = T^n z$ and $y = T^n y$ for $n = 0, 1, 2, \ldots$. Hence

$$\begin{aligned} G_p(z, T^n x, T^n x) &= G_p(T^n z, T^n x, T^n x) \\ &\leq kG_p(T^{n-1}z, T^{n-1}x; T^{n-1}x) \\ &\leq kG_p(z, T^{n-1}x, T^{n-1}x) < G_p(z, T^{n-1}x, T^{n-1}x). \end{aligned} \tag{6}$$

This shows that $\{T^n x\}$ is a nonnegative and non-increasing sequence and so has a limit say $\alpha \geq 0. Letting \; n \rightarrow \infty$ in (5) yields,

$G_p(z, \alpha, \alpha) < G_p(z, \alpha, \alpha)$.

This is a contradiction, hence $\alpha = 0$. Therefore $lim_{n\to\infty} G_p(z, T^n x, T^n x) = 0$. Similarly, using (5) we get,

$$\begin{aligned} G_p(T^n x, y, y) &= G_p(T^n x, T^n y, T^n y) \\ &\leq k G_p(T^{n-1}x, T^{n-1}y, T^{n-1}y) \\ &\leq k G_p(T^{n-1}x, y, y) < G_p(T^{n-1}x, y, y). \end{aligned} \tag{7}$$

This shows that $\{T^n x\}$ is nonnegative and nonincreasing sequence and so has a limit say $\beta \geq 0$. Letting $n \to \infty$ in (7) yields,

$G_p(\beta, y, y) < G_p(\beta, y, y)$.

This is a contradiction, hence $\beta = 0$.Therefore $lim_{n\to\infty} G_p(T^n x, y, y) = 0$. Finally,

$$\begin{aligned} G_p(z, y, y) &\leq G_p(z, T^n x, T^n x) + G_p(T^n x, y, y) - G_p(T^n x, T^n x, T^n x) \\ &\leq G_p(z, T^n x, T^n x) + G_p(T^n x, y, y) \end{aligned}$$

Taking $n \to \infty$ we obtain
$G_p(z, y, y) \leq 0$. But $G_p\ (z, y, y) \geq 0$. Hence $G_p\ (z, y, y) = 0$. Therefore z = y. The analogue for Theorem (2.1) was proved by Ran and Reurings [14] Theorem (2.1).
We now prove the above result for weak contraction maps satisfying certain conditions.

Theorem 2.2: Let $(X, \preceq)$ be a partially ordered set and let G_p be a G -partial metric on X such that (X, G_p) is complete. Let $T : X \to X$ be a non-decreasing map with respect to $\preceq$. Suppose that the following conditions hold, for y $\preceq x$, we have

$$(i) G_p\ (Tx, Ty, Tz) \leq G_p(x, y, z) - \psi\ (G_p(x;\ y;\ z)), \tag{8}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that it is positive in $[0, \infty)$, $\psi\,(0) = 0$ and $lim_{t\to\infty} \psi\,(t) = \infty$;

(ii) $\exists x_0 \in X$ *such that* $x_0 \preceq Tx_0$;

(iii) T is continuous in (X, G_p), or ;

(iv) a nondecreasing sequence $\{x_n\}$ converges to $x \in X$, implies $x_n \preceq x$ for all n.

Then T has a unique fixed point $u \in X$.

Proof: Let $x_0 \in X$ be arbitrary and $x_0 \preceq Tx_0$. Since T is nondecreasing map with respect to $\preceq$ we have

$x_0 \preceq Tx_0 \preceq T^2 x_0 \preceq T^3 x_0 \preceq \ldots \preceq T^n x_0 \preceq T^{n+1} \preceq \ldots.$

Suppose $x_0 = Tx_0$ then the fixed point exists. Let $x_0 \preceq Tx_0$ then we define $x_n = T^{n-1}x_0$ such that $x_{n-1} \preceq x_n$ for any $n \in N$, where N is the set of natural numbers. This means that x_n and x_{n-1} are comparable. Suppose $x_{n-1} = x_n = Tx_{n-1}$ then x_{n-1} is the fixed point of T. Now let $x_{n-1} \neq x_n$ for any $n \in N$. Let $y = z \in X$ in (8) then

$$G_p\ (Tx, Ty, Ty) \leq G_p(x,y,y) - \psi\ (G_p(x,y,y)), \tag{9}$$

And

$$G_p\ (Ty, Tx, Tx) \leq G_p(y,x,x) - \psi\ (G_p(y,x,x)). \tag{10}$$

Hence using (9) we have

$$G_p\ (x_n, x_{n+1}, x_{n+1}) \leq G_p(x_{n-1}, x_n, x_n) - \psi\ (G_p(x_{n-1}, x_n, x_n)), \tag{11}$$

By property of ψ we get,

$G_p\ (x_n, x_{n+1}, x_{n+1}) \leq G_p(x_{n-1}, x_n, x_n)$.

Let $G_p\ (x_n, x_{n+1}, x_{n+1}) = G_p$.

Similarly, using (10) gives

$$G_p(x_{n+1}, x_n, x_n) \leq G_p(x_n, x_{n-1}, x_{n-1}) - \psi\ (G_p\ (x_n, x_{n-1}, x_{n-1})) \tag{12}$$

Also the property of ψ yields

$G_p(x_{n+1}, x_n, x_n) \leq G_p(x_n, x_{n-1}, x_{n-1})$

By symmetry,

$G_p\ (x_n, x_{n+1}, x_{n+1}) = G_p\ (x_{n+1}, x_n, x_n) = G_n$.

This shows that $\{x_n\}$ is a nonnegative non-increasing sequence and hence converges to a point in X say $\alpha \geq 0$. Suppose $\alpha > 0$ then (11) and (12) yield $\alpha \leq \alpha - \psi\ (\alpha)$.

This is a contradiction, hence $\alpha = 0$. Therefore, $lim_{n\to\infty} G_n = 0$.

Next we shall show that $\{x_n\}$ is a Cauchy sequence. Given $\in > 0$, as $G_n = G_p\ (x_n, x_{n+1}, x_{n+1}) \to 0$, there exists $n_0 \in N$ such that

$$G_p\ (x_{no+1}, x_{no}, x_{no}) \leq min\left\{\frac{\in}{2}, \psi\ \left(\frac{\in}{2}\right)\right\}. \tag{13}$$

We claim that if $z \in X$ indicates that $G_p\ (z, x_{no}\ ; x_{no}) \leq \in$ and $x_{no} \preceq z$ then $G_p\ (T_z, x_{no}; x_{no}) \leq \in$. To show this we distinguish two cases:

Case (i):

Let $G_p\ (z, x_{no}, x_{no}) \leq \frac{\in}{2}$. Since z and x_{no} are comparable, we have

$$G_p\ (T_z, x_{no}, x_{no}) \leq G_p(T_z, Tx_{no}, Tx_{no}) + G_p\ (Tx_{no}, x_{no}, x_{no})$$
$$-G_p(Tx_{no}, Tx_{no}, Tx_{no})$$

$$\leq G_p(T_z, Tx_{no}, Tx_{no}) + G_p(Tx_{no}, x_{no}, x_{no})$$
$$= G_p(Tz, Tx_{no}, Tx_{no}) + G_p(x_{no+1}, x_{no}, x_{no})$$
$$\leq G_p(z, x_{no}, x_{no}) - \psi\ (G_p(z, x_{no}, x_{no}))$$
$$+ G_p(x_{no+1}, x_{no}, x_{no})$$
$$\leq G_p(z, x_{no}, x_{no}) + G_p(x_{no+1}, x_{no}, x_{no})$$
$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Case(ii):

$\frac{\epsilon}{2} \leq G_p(z, x_{no}, x_{no}) \leq \epsilon$. Here, ψ is a non-decreasing function,

$\psi\left(\frac{\epsilon}{2}\right) \leq \psi(G_p(z, x_{no}, x_{no})$ Therefore, from (13) we get

$$G_p(Tz, x_{no}, x_{no}) \leq G_p(Tz, Tx_{no}, Tx_{no}) + G_p(Tx_{no}, x_{no}, x_{no})$$
$$- G_p(Tx_{no}, Tx_{no}, Tx_{no})$$
$$\leq G_p(Tz, Tx_{no}, Tx_{no}) + G_p(Tx_{no}, x_{no}, x_{no})$$
$$= G_p(Tz, Tx_{no}, Tx_{no}) + G_p(x_{no+1}, x_{no}, x_{no})$$
$$\leq G_p(z, x_{no}, x_{no}) - \psi\ (G_p(Z, x_{no}, x_{no}))$$
$$+ G_p(x_{no+1}, x_{no}, x_{no})$$
$$\leq G_p(z, x_{no}, x_{no}) - \psi\left(\frac{\epsilon}{2}\right) + G_p(x_{no+1}, x_{no}, x_{no})$$
$$\leq G_p(z, x_{no}, x_{no}) - \psi\left(\frac{\epsilon}{2}\right) + \psi\left(\frac{\epsilon}{2}\right)$$
$$\leq G_p(z, x_{no}, x_{no}) \leq \epsilon.$$

As $x_{n_{0+1}}$ implies $G_p(x_{no+1}, x_{no}, x_{no}) \leq \epsilon$ and $x_{n_0} \preceq x_{no+1}$, the claim provides us with $x_{no+2} = Tx_{no+1}$ implies $G_p(x_{no+2}, x_{no}, x_{no}) \leq \epsilon$. Consequently

$G_p(x_n, x_{no}, x_{no}) \leq \epsilon$. for any $n \geq n_0$.

Hence for any $n, m \geq n_0$ we have,

$$G_p(x_n, x_m, x_m) \leq G_p(x_n, x_{no}, x_{no}) + G_p(x_{no}, x_m, x_m)$$
$$- G_p(x_{no}, x_{no}, x_{no})$$
$$\leq G_p(x_n, x_{no}, x_{no}) + G_p(x_{no}, x_m, x_m)$$

$$\leq \in + \in = 2 \in .$$

This shows that $\{x_n\}$ is a Cauchy sequence. Since X is complete then the limit exists say $x \in X$. i.e.

$$lim_{n\to\infty} G_p (x_n, x, x) = lim_{n\to\infty} G_p (x_n, x_m, x_m) = G_p (x, x, x). \qquad (14)$$

Next we show the existence of the fixed point using (iii) and (iv).

Case (i):

Since the mapping T is continuous, we have that $\{Tx_n\}$ converges to Tx whenever $\{x_n\}$ converges to x. By uniqueness of the limit point in X we get $Tx = x$. Hence x is the fixed point of T.

Case (ii):

Suppose the nondecreasing sequence $\{x_n\}$ converges to $x \in X$ with respect to $\preceq$, then it follows that $x_n \preceq x$. Let $x = x_n$ and $y = x$ in (9) we obtain ,

$$\begin{aligned} G_p (Tx_n, Tx, Tx) &\leq G_p(Tx_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, Tx, Tx) - G_p(x_{n+1}, x_{n+1}, x_{n+1}) \\ &\leq G_p (x_{n+1}, x_{n+1}, x_{n+1}) + G_p (x_{n+1}, Tx, Tx) \\ &= G_p (x_{n+1}, x_{n+1}, x_{n+1}) + G_p (Tx_n, Tx, Tx) \\ &\leq G_p (x_{n+1}, x_{n+1}, x_{n+1}) + G_p (x_n, x, x) (G_p (x_n, x, x)) \qquad (15) \end{aligned}$$

As $n \to \infty$ in (15), using (14) and the properties of ψ we obtain,

$G_p(x, Tx, Tx) < 0 - \psi(0) = 0.$

Hence $G_p (x, Tx, Tx) = 0,$ so $Tx = x.$

The following Theorem gives a sufficient condition for the uniqueness of the fixed point.

Theorem 2.3: Let all the hypotheses of Theorem 2.3 be satisfied and let the following condition be satisfied: for arbitrary two points $x, y \in X$ there exists $z \in X$ which is comparable to both x *and* y. Then the fixed point of T is unique.

Proof: Suppose there exist $x, y \in X$ which are different fixed points of T i.e, $y = T^n y$ and $x = T^n x$. We have two different cases.

Case (i):

Let x and y be comparable then, $T^n x$ = x is comparable to $y = T^n y$ for $n = 0; 1; 2; \ldots$. Using (9) we get,

G_p (x, y, y) = $G_p(T^n x, T^n y, T^n y)$

$$\leq G_p(T^{n-1}x, T^{n-1}y, T^{n-1}y) - \psi\left(G_p(T^{n-1}x, T^{n-1}y, T^{n-1}y)\right)$$
$$\leq G_p(x,y,y) - \psi\left(G_p(x,y,y)\right).$$

By property of ψ we have,

$G_p(x,y,y) \leq G_p(x,y,y)$.

This is a contradiction hence $G_p(x,y,y) = 0$ which gives $x = y$.

Case (ii):

If y is not comparable to x then there exist $z \in X$ which is comparable to both x and y. Due to the monotonicity of T we have that $T^n z$ is comparable to both $x = T^n x$ and $y = T^n y$ for $n = 0,1,2,\ldots$.

Using (9) we have ,

$$G_p(x, T^n z, T^n z) = G_p(T^n x\, T^n z, T^n z)$$
$$\leq G_p(T^{n-1}x, T^{n-1}z, T^{n-1}z) - \psi\left(G_p(T^{n-1}x, T^{n-1}z, T^{n-1}z)\right)$$
$$\leq G_p(T^{n-1}x, T^{n-1}z, T^{n-1}z). \quad (16)$$

Let $n \to \infty$ in (16) we have,

$G_p(x, T^n z, T^n z) \leq (G_p(x, T^{n-1}z, T^{n-1}z))$.

This shows that $\{G_p(x, T^n z, T^n z)\}$ is a nonnegative non-increasing sequence and have limit say $\beta \geq 0$. Suppose $\beta > 0$ then using (16) we obtain

$\beta \leq \beta - \psi(\beta)$.

This is a contradiction hence $\beta = 0$. Therefore $lim_{n\to\infty} G_p(\mathrm{x}, T^n z, T^n z) = 0$. Similarly using (10) we have,

$$G_p(T^n z, y, y) = G_p(T^n z, T^n y, T^n y)$$
$$\leq G_p(T^{n-1}z, T^{n-1}y, T^{n-1}y) - \psi\left(G_p(T^{n-1}z, T^{n-1}y, T^{n-1}y)\right)$$

This also shows that $\{G_p(x, T^n z, y, y)\}$ is a non-increasing sequence and has a limit say $\beta \geq 0$. Assuming $\beta > 0$ then using the last inequality we get $\beta \leq \beta - \psi(\beta)$. a contradiction. Hence $\beta = 0$ and $lim_{n\to\infty} G_p(\mathrm{x}, T^n z, \mathrm{y}, \mathrm{y}) = 0$.

Finally,

$$G_p(x,y,y) = G_p(x, T^n z, T^n z)$$
$$\leq G_p(x, T^n z, T^n z) + (G_p(T^n z, y, y)$$

Taking $n \to \infty$ we obtain,

$G_p(x, y, y) \leq 0$.

But $G_p(x, y, y) \geq 0$, hence $G_p(x, y, y) = 0$. Therefore x = y.

Example (see [4]): Let $X = [0, 1]$ be endowed with a G -partial metric G_p which is defined $G_p : X \times X \times X \to [0, \infty)$ with $G_p(x, y, z) = \max\{x, y, z\}$, then it is true that (X, G_p) is complete. We can define a partial order on X for $x; y \in X$ by $x \leq y \Leftrightarrow G_p(x, x, x) = G_p(x, y, y) \Leftrightarrow x = \max\{x, y, y\} \Leftarrow y \leq x$.

Therefore $(X; \preceq)$ is totally ordered. Again we define $Tx = \frac{x}{4}$.

The function T is continuous on (X, G_p). Let $\{x_n\}$ be a sequence converging to x in (X, G_p), then

$lim_{n\to\infty} \max\{x, y, y\} = lim_{n\to\infty} G_p(x_n, x, x) = G_p(x, x, x) = x.$

By the definition of T we have,

$lim_{n\to\infty} G_p(Tx_n, Tx, Tx) = lim_{n\to\infty} \max\{Tx_n, Tx, Tx\} = \frac{x}{4} = G_p(Tx, Tx, Tx)$. This means that $\{x_n\}$ converges to Tx in (X, G_p). Any two points $x, y \in X$ are comparable. e.g, let $x \preceq y$ then $G_p(x, x, x) = G_p(x, y, y)$ so $y \preceq x$. Since $T(y) \preceq T(x)$ we have $T(x) \preceq T(y)$ which yields that T is monotone non-decreasing with respect to $\preceq$. For any $x \preceq y$ we get,

$G_p(x, y, y) = x, G_p(Tx, Ty, Ty) = Tx = \frac{x}{4}.$

Lets define ψ: $[0, \infty) \to [0; \infty)$ by ψ (x) = $\frac{x}{8}$.We have for any $x \in X, \frac{x}{4} \leq x - \frac{x}{8}$.

Therefore

$G_p(Tx, Ty, Ty) \leq G_p(x, y, y) - \psi\,(G_p(x, y, y)),$

holds in (8). All the hypotheses of Theorem (2.3) are satisfied. Hence T has a unique fixed point in X, which is $x = 0$. Theorem (2.2, 2.3) is an analogue of the result of Aydi [4] Theorems (2.1, 2.3).

References:

1. Aage C.T. and Salunke J.N., Fixed points for weak contraction in G-metric spaces, Applied Mathematics E- Notes, 12(2011), pp. 23-28.
2. Aliouche A. and Carlos S., Fixed Points and Lines in 2-metric Spaces, Advances in Mathematics , vol 229, issues 1, (2012), pp. 668-690.

3. Altun I. and Erduran A., Fixed point theorems for monotone mappings on partial metric spaces, Fixed Point Theory and Applications, Vol. 2011, Article ID 508730, 10 pages.

4. Aydi H., Some _xed point results in ordered partial metric spaces, Journal of Nonlinear Science. Appl., 4(2011), no. 3, pp. 210-217.

5. Dhage B.C., Generalised metric space and mapping with _xed point, Bull. Cal. Math. Soc. 84, (1992) pp. 329-336.

6. Gahler S., 2-metriche raume und ihre topologische structure, Math. Nachr. 26, (1963) pp. 115-148.

7. HA. Et al., KIS , CHO,Y. J. and White A., Strictly convex and 2-convex 2-normed spaces, Math. Japonica, 33(3), (1988), pp. 375-384.

8. Harjani J. and Sadarangani K., Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Analysis, 71(2009) pp. 3403-3410.

9. Matthews S.G, Partial metric spaces, 8th British Colloquium for Theoretical Computer Science. In Research Report 212, Dept. of Computer Science, University of Warwick,(1992) pp. 708-718.

10. Matthews S.G., Partial metric topology, in Proceedings of the 11th Summer Conference on General Topology and Applications, Vol. 728,(1995) pp. 183-197, The New York Academy of Sciences, Gorham, Me, USA.

11. Mustafa Z. and Sims B., Some remarks concerning D-metric spaces, Proceedings of the International Conferences on Fixed Point Theory and Applications, Valencia (Spain), (2003) pp. 189-198.

12. Mustafa Z. and Sims B. , A new approach to generalised metric spaces, Journal Of Nonlinear And Convex Analysis, Vol. 7, Number 2,(2006) pp. 289-297.

13. Mustafa Z., Obiedat H. and Awawdeh F., Some _xed point theorm for mapping on complete G-metric space, Fixed Point Theory and Appl., Article ID 189870,(2009) 12 pages.

14. Ran A. C. M. and Reurings M. C. B., A _xed point theorem in partially ordered sets and some applications to matrix equations, Proceedings of the American Mathematical Society, Vol. 132, Number 5, (2003) pp. 1435-1443.

15. Rodriguez-Lopez R, and Nieto J. J., Contractive mapping theorems in partially ordered sets and applications to ordinary di_erential equation, A Journal on The Theory of Ordered Sets and its Applications - Order 22(2005) pp. 223-239.