Common fixed point theorems for weakly compatible non-self mappings in metric spaces of hyperbolic type

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Abstract

In this paper, we establish common fixed point theorems for a pair of weakly compatible nonself mappings satisfying generalized contractive conditions in metric space of hyperbolic type. The results generalize and extend some results in literature.

Keywords: common fixed points, generalized contractive mapping, metric space of hyperbolic type, nonself mappings, weakly compatible mappings.

1. Introduction

In literature, fixed point theory has diverse results on fixed point theorems for self-mappings in metric and Banach spaces. However, an area that seems not broadly investigated is the fixed point theorems for non-self mappings. Kirk [1] extended the metric space to metric space of hyperbolic type by replacing Krasnoselskii’s result with the framework of convex metric space. The study of fixed point theorems for multivalued non-self mappings in a metric space \( (X, d) \) was initiated by Assad [2] and Assad and Kirk [3]. Many authors have studied the existence and uniqueness of fixed and common fixed points result for nonself contraction mappings in cone metric spaces [see: 4, 5, 6, 7]. Some authors studied common fixed point theorems for non-self mappings in metric spaces of hyperbolic type [See: 8, 9]. Motivated by Jankovic et al. [7], we prove some common fixed point theorems for a pair of weakly compatible non-self mappings satisfying a generalized contraction condition in the setting of metric space of hyperbolic type. Throughout our consideration, we suppose that \( (X, d) \) is a metric space which contains a family \( L \) of metric segments (isometric images of real line segment) such that

a) each two points \( x, y \in X \) are endpoints of exactly one number \( \text{seg}[x, y] \) of \( L \), and
b) If \( u, x, y \in X \) and if \( z \in \text{seg}[x, y] \) satisfies \( d(x, z) = \lambda d(x, y) \) for \( \lambda \in [0, 1] \) then

\[
d(u, z) \leq (1 - \lambda)d(u, x) + \lambda d(u, y) \tag{1.1}
\]

A space of this type is called metric space of hyperbolic type.

The following definition was introduced by Jungck et al. [4] in the setting of cone metric spaces.

Definition 1.1 Let \( (X, d) \) be a complete cone metric space, let \( C \) be a non empty closed subset of X, and let \( f, g : C \to X \) be non-self mappings. Denote for \( x, y \in C \)

\[
M^{fg} = \{d(gy, gx), d(fy, gx), d(fy, gy), \frac{d(fy, gx) + d(fy, gy)}{2}\} \tag{1.2}
\]

Then \( f \) is called a generalized \( ^{fg}R \)-contractive mapping in \( C \) into \( X \) if, for some \( \lambda \in (0, \sqrt{2} - 1) \), there exists \( U(x, y) \in M^{fg} \) such that for all \( x, y \in C \),

\[
d(fx, fy) \leq \lambda U(x, y)
\]

2. Main results

Jankovic et al. [7] proved the following fixed point theorem for a pair of non-self mappings defined on a nonempty closed subset of complete metrically convex cone metric spaces with new contractive conditions.

Theorem 2.1: Let \( (X, d) \) be a complete cone metric space, let \( K \) be a non empty closed subset of \( X \) such that for each \( x \in C \) and \( y \notin C \) there exists a point \( z \in \delta K \) (the boundary of \( K \)) such that

\[
d(x, z) + d(z, y) = d(x, y)
\]

Suppose that \( f, g : C \to X \) are such that \( f \) is a generalized \( ^{fg}R \)-contractive mapping of \( C \) into \( X \) and

(i) \( \delta C \subseteq gC, fC \cap C \subseteq gC \),
(ii) \( gx \in \delta C \iff fx \in C \),
(iii) \( gC \) is closed in \( X \).

Then the pair \( (f, g) \) has a coincidence point. Moreover, if \( (f, g) \) are coincidentally commuting, then \( f \) and \( g \) have a unique common fixed point.

In this paper, we extend the above theorem to fixed point theorem of weakly compatible non- self mappings in metric space of hyperbolic type.

We state and prove our main result as follows.
Theorem 2.2: Let $X$ be a metric space of hyperbolic type. K a non-empty closed subset of $X$ and $\delta K$ the boundary of K. Let $\delta K$ be nonempty and let $T: K \to X$ and $f: K \cap (T(K)) \to X$ be two non-self mappings satisfying the following conditions:

$$d(Tx,Ty), d(Tx,fx), d(Ty, fy), d(Tx, fy)+d(Ty, fx)/2 \leq \lambda \mu$$

where $\mu \in \{d(Tx, Ty), d(Tx, fx), d(Ty, fy), d(Tx, fy)+d(Ty, fx)/2 \}$

(2.1)

for all $x,y \in C, 0 < \lambda < 1$. If

(i) $\delta K \subset TK, fK \cap K \subset TK$,

(ii) $Tx \in \delta K \implies fK \in K$,

(iii) $fK \cap K$ is complete.

Then $f$ and $T$ have a coincidence point $z$ in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $z$ is the unique common fixed point of $f$ and $T$.

Proof: Let $x \in \delta K$ be arbitrary. We construct three sequences, $\{x_n\}$ and $\{z_n\}$ in $K$ and a sequence $\{y_n\}$ in $fK \subset X$ as follows. Choose $z_0 = x$. Since $z_0 \in \delta K$ then there exists $x_0 \in K$ such that $z_0 = T x_0 \in \delta K$. By (iii) $x_0 \in K$. Now choose $y_1 = f x_0$ with $y_1 \in f \cap K \subset X$. This implies that $y_1 \in fK \subset \delta K \subset TK$. Set $y_1 = f x_0$, we choose $x_1 \in K$ such that $T x_1 = x_0$. Hence $z_1 = T x_1 = f x_0 = y_1$. This gives $y_2 = f x_1$. Since $y_2 \in fK \cap K$ then $y_2 \in TK$ by (ii). Let $x_1 \in K$ with $z_1 = T x_1 \in \delta K$ such that $y_2 = T x_2 = f x_2 = y_2$. If $x_2 = y_2 \in \delta K$, then there exists $z_2 \in \delta K(z_2 \notin \delta K$ such that $z_2 \in \delta K(y_1, y_2)$. Since $x_2 \in K$, then by (i) we have $x_2 = z_2$. Hence $z_2 \in \delta K \cap \{y_1, y_2\}$.

We can choose $x_3 \in f K \cap K$, and by (ii), $y_3 \in TK$ and let $z_2 \in K$ such that $T x_3 = y_3 = f x_2$. Continuing in the process, we construct three sequences $\{x_n\} \subset K, \{z_n\} \subset K$ and $\{y_n\} \subset fK \subset X$ such that

(a) $y_n = f x_{n-1}$
(b) $z_n = T x_n$
(c) $z_n \in y_n$ if and only if $y_n \in K$
(d) $z_n \notin y_n$ whenever $y_n \notin K$ and $z_n \in \delta K$ such that $z_n \in \delta K \cap \{y_{n-2}, f x_{n-1}\}$.

This proves that $f$ and $T$ are non-self mappings.

Remark 2.3: By (d) if $z_n \notin y_n$, then $z_n \in \delta K$ and combining (b), (ii) and (a) we have $z_{n+1} = y_{n+1}$. Likewise $z_{n-1} = y_{n-1} \in K$. If $z_{n+1} \in \delta K$, then it implies $y_{n+1} \in y_n \subset K$. Next, we show that $x_n \neq y_{n+1}$ for all $n$. From (a), (b), (c) and (d) we can establish three possibilities.

(1) $z_n = y_n \in K$ and $z_{n+1} = y_{n+1}$
(2) $z_n = y_n \in K$ but $z_{n+1} \notin y_{n+1}$
(3) $z_n \notin y_n \subset K$ in which case $z_n \in \delta K \cap \{y_{n-2}, f x_{n-1}\}$.

Case (1)

Let $z_n = y_n \in K$ and $z_{n+1} = y_{n+1}$. Using (2.1) we obtain

$$d(z_n, z_{n+1}) = d(y_n, y_{n+1}) + d(y_{n+1}, f x_n)/2 \leq \lambda \mu_n$$

where $\mu_n \in \{d(TX_{n-1}, TX_n), d(TX_{n-1}, f x_n), d(Tx_n, f x_n), d(Tx_n, f x_n)/2 + d(Tx_{n-1}, f x_{n-1})/2 \}$

$$= d(z_{n-1}, z_n) + d(z_{n-1}, y_n) + d(z_{n-1}, y_n) + d(z_{n-1}, y_n)/2 + d(z_{n-1}, y_n)/2$$

$$= \{d(z_{n-1}, z_n), d(z_{n-1}, z_n), d(z_{n-1}, z_n), d(z_{n-1}, z_n)/2 + d(z_{n-1}, z_n)/2 \}$$

Obviously, there are infinite many $n$ such that at least one of the following cases holds:

I: $d(z_n, z_{n+1}) \leq \lambda d(z_{n-1}, z_n)$

II: $d(z_{n+1}, z_{n+2}) \leq \lambda, d(z_{n-1}, z_n)$

III: $d(z_n, z_{n+1}) \leq \lambda, d(z_{n-1}, z_{n+1})$. A contradiction.

IV: $d(z_n, z_{n+1}) \leq \lambda, d(z_{n+1}, z_{n+1})/2 + d(z_{n+1}, z_{n+1})/2$.

$$\leq \lambda d(z_{n+1}, z_{n+1}) \leq \lambda d(z_{n+1}, z_{n+1})$$

From I, II, III, IV it follows that

$$d(z_n, z_{n+1}) \leq \lambda d(z_{n-1}, z_n)$$

(2.2)

Case 2

Let $z_n = y_n \in K$ but $z_{n+1} \neq y_{n+1}$. Then $z_{n+1} \in \delta K \cap \{y_{n+1}, y_{n+1}\}$. From (1.1) with $u = y$, we obtain

$$d(y, z) \leq (1 - \lambda) d(y, x)$$

Therefore

$$d(x, y) \leq d(x, z) + d(z, y) \leq (1 - \lambda) d(x, y) + (1 - \lambda) d(x, y)$$

Hence

$$d(x, y) \leq (1 - \lambda) d(x, y)$$

Since $z_{n+1} \in \delta K \cap \{y_{n+1}, y_{n+1}\}$, we have

$$d(z_n, z_{n+1}) = d(y_n, z_{n+1}) = d(y_n, y_{n+1}) - d(z_{n+1}, y_{n+1})$$

In view of (1), we obtain

$$d(y_{n+1}, y_n) \leq \lambda d(z_{n+1}, z_{n+1})$$

This implies that $d(z_n, z_{n+1}) \leq \lambda, d(z_{n+1}, z_{n})$.

Case (3)

$z_n \neq y_n$. Then $z_n \in \delta K \cap \{f x_{n-2}, f x_{n-1}\}$. i.e. $z_n \in \delta K \cap \{f x_{n-2}, f x_{n-1}\}$.

By remark (2.3) we have $z_{n+1} = y_{n+1}$ and $z_{n-1} = y_{n-1}$. This implies that

$$d(z_n, z_{n+1}) = d(y_n, y_{n+1})$$

$$\leq d(z_n, y_{n+1}) + d(y_n, y_{n+1})$$

$$= d(z_n, y_{n+1}) - d(z_{n+1}, y_{n+1}) + d(y_n, y_{n+1})$$

(2.3)

We shall find $d(y_n, y_{n+1})$ and $d(y_{n+1}, y_{n+1})$. Since $z_{n-1} = y_{n-1}$ then we can conclude that

$$d(y_{n-1}, y_n) \leq \lambda d(z_{n-2}, z_{n-1})$$

(2.4)

with respect to (2).

Now

$$d(y_{n+1}, y_{n+1}) = d(f x_{n-1}, f x_n) \leq \lambda, \mu_n$$

where $\mu_n \in \{d(TX_{n-1}, TX_n), d(TX_{n-1}, f x_n), d(Tx_n, f x_n), d(Tx_n, f x_n)/2 + d(Tx_{n-1}, f x_{n-1})/2 \}$

$$= \{d(z_{n-1}, z_n), d(z_{n-1}, y_n), d(z_{n-1}, y_n), d(z_{n-1}, y_n)/2 + d(z_{n-1}, y_n)/2 \}$$

$$= \{d(z_{n-1}, z_n), d(z_{n-1}, z_n), d(z_{n-1}, z_n), d(z_{n-1}, z_n)/2 + d(z_{n-1}, z_n)/2 \}$$

Clearly, there are infinite many $n$ such that at least one of the following cases holds:
I: \( d(y_n, y_{n+1}) \leq \lambda d(z_{n-1}, z_n) \)

II: \( d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n) \leq \lambda^2 d(z_{n-2}, z_n) \)

III: \( d(y_n, y_{n+1}) \leq \lambda d(z_{n-1}, z_n) \)

IV: \( d(y_n, y_{n+1}) \leq \lambda d(z_{n-1}, z_n) + \frac{d(z_{n-2}, z_{n-1})}{2} d(z_n, z_{n+1}) \)

Substituting I, II, III, IV in (2.4) yields

\[
d(z_n, z_{n+1}) \leq \lambda d(z_{n-2}, z_{n-1}) - d(z_n, z_{n-1}) + \lambda \mu_n
\]

from which we have four cases:

V: \( d(z_n, z_{n+1}) \leq \lambda d(z_{n-2}, z_{n-1}) - d(z_n, z_{n-1}) + \lambda d(z_{n-1}, z_n) \)

\[
\leq \lambda d(z_{n-2}, z_{n-1}) - (1 - \lambda) d(z_n, z_{n-1})
\]

\[
\leq \lambda d(z_{n-2}, z_{n-1})
\]

VI: \( d(z_n, z_{n+1}) \leq \lambda d(z_{n-2}, z_{n-1}) - d(z_n, z_{n-1}) + \lambda^2 d(z_{n-2}, z_{n-1}) \)

\[
\leq (\lambda + \lambda^2) d(z_{n-2}, z_{n-1}) - d(z_n, z_{n-1})
\]

\[
\leq (\lambda + \lambda^2) d(z_{n-2}, z_{n-1})
\]

VII: \( d(z_n, z_{n+1}) \leq \lambda d(z_{n-2}, z_{n-1}) - d(z_n, z_{n-1}) + \lambda d(z_{n-1}, z_n) \)

\[
\leq \frac{1}{1-\lambda} d(z_{n-2}, z_{n-1}) - \frac{1}{1-\lambda} d(z_n, z_{n-1})
\]

\[
\leq \frac{1}{1-\lambda} d(z_{n-2}, z_{n-1})
\]

VIII: \( d(z_n, z_{n+1}) \leq \lambda d(z_{n-2}, z_{n-1}) - d(z_n, z_{n-1}) + \lambda d(z_{n-1}, z_n) + \frac{d(z_{n-2}, z_{n-1})}{2} d(z_n, z_{n+1}) \)

\[
\leq \lambda d(z_{n-2}, z_{n-1}) - (1 - \lambda) d(z_n, z_{n-1}) + \frac{d(z_{n-2}, z_{n-1})}{2} d(z_n, z_{n+1})
\]

\[
\leq \frac{\lambda}{2(1-\lambda)} d(z_{n-2}, z_{n-1}) - \frac{2(1-\lambda)}{2(1-\lambda)^2} d(z_n, z_{n-1})
\]

\[
\leq \frac{\lambda}{2(1-\lambda)} d(z_{n-2}, z_{n-1})
\]

From V, VI, VII, VIII we obtain

\[ d(z_n, z_{n+1}) \leq k d(z_{n-2}, z_{n-1}) \] where

\[ k = \max \{ \lambda, \lambda + \lambda^2, \frac{1}{1-\lambda}, \frac{2\lambda}{2(1-\lambda)} \} \]

Combining Cases 1, 2, 3 we get

\[ d(z_n, z_{n+1}) \leq k \omega_n \]

where \( \omega_n \in \{ d(z_{n-2}, z_{n-1}), d(z_{n-1}, z_n) \} \) and

\[ k = \max \{ \lambda, \lambda + \lambda^2, \frac{1}{1-\lambda}, \frac{2\lambda}{2(1-\lambda)} \} \]

Following the procedure of Assad and Kirk [3], it can be easily verify by induction that for \( n > 1 \)

\[ d(z_n, z_{n+1}) \leq k^{n+1} \omega_0 \]

where \( \omega_0 \in \{ d(z_0, z_1), d(z_1, z_2) \} \).

For \( n > m \) and using (2.5) and the triangle inequality we have

\[ d(z_n, z_m) \leq d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + \cdots + d(z_{m-1}, z_m) \]

\[
\leq (k^{n+1} + k^{n+2} + \cdots + k^{m+1}) \omega_0
\]

\[
\leq \left( k^{n+1} + k^{n+2} + \cdots + k^{m+1} \right) \omega_0 \rightarrow 0, \text{ as } m \to \infty.
\]

The sequence is Cauchy. Since \( z_n = x_{n+1} \in fK \cap K \) is complete, there is some \( z \in fK \cap K \) such that \( z_n \to z \). Let \( w \in K \) be such that \( Tw = z \). By the construction of \( \{ z_n \} \), there is a subsequence \( \{ z_{n_k} \} \) such that \( z_{n_k} = y_{n_k} = f_{n_k} \) and \( f_{n_k} \to z \). We show that \( f_w = z \).

\[
d(f_w, z) \leq d(f(w), f_{n_k}) + d(f_{n_k}, z) \leq \lambda \mu_{n_k} + d(f_{n_k}, z)
\]

where \( \mu_{n_k} \in \{ d(T_w, T_{n_k}), d(T_{n_k}, f_{n_k}), d(T_w, f_w), \frac{d(T_w, f_{n_k}) + d(T_{n_k}, f_{n_k})}{2} \}
\]

Taking \( z_{n_k} = y_{n_k} = f_{n_k} \to z \) as \( n \to \infty \) yields

\[
\mu_n \in \{ d(z, f_w), \frac{d(z, f_w)}{2} \}
\]

Thus, we have

i) \( d(f_w, z) \leq \lambda d(z, f_w) + d(f_{n_k}, z) \leq \lambda d(z, f_w) \)

Since \( \lambda < 1 \) then \( d(f_w, z) = 0 \). This implies \( z = f_w \)

ii) \( d(f_w, z) \leq \frac{\lambda}{2} d(f_w, z) \)

Since \( \lambda < 1 \) then \( d(f_w, z) = 0 \). Hence \( z = f_w \). In all cases we have \( z = f_w \).

Suppose that \( T \) and \( f \) are weakly compatible, then we have \( z = f_w = T_w = fT_w = fTw = Tz \).

Next we prove that \( z = fz = Tz \). Suppose \( z \neq fz \) then using 2.1 we obtain

\[
d(fz, z) = d(fz, f_w) \leq \lambda \mu
\]

where

\[
\mu \in \{ d(Tz, Tw), d(Tz, fz), d(Tw, f_w), \frac{d(Tz, fz) + d(Tz, z)}{2} \}
\]

\[
\leq \{ d(z, fz), d(z, z), \frac{d(z, fz) + d(z, z)}{2} \}
\]

Case (i)

\[
d(fz, z) \leq \lambda d(fz, z)
\]

It is a contradiction. Hence \( z = fz \)

Case (ii)

\[
d(fz, z) \leq \frac{\lambda}{2} d(fz, z)
\]

It is also a contradiction. This implies that \( z = fz \). Therefore we obtain \( z = fz = Tz \). Thus \( T \) and \( f \) have a common fixed point. The uniqueness of the common fixed point follows easily from (2.1).

**Remark 2.4:** Theorem 2.2 is an extension of the result of jankovic [7].

Setting \( T = I_k \), the identity mapping of \( X \) in Theorem 2.2, we obtain the following result.

**Corollary 2.5:** Let \( (X, d) \) be metric space of hyperbolic type, \( K \) a non-empty closed subset of \( X \) and \( \delta K \) the boundary of \( K \). Let \( \delta K \) be nonempty such that \( f : K \to K \) satisfies the condition

\[
d(fx, fy) \leq \lambda \mu
\]

where

\[
\mu \in \{ d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2} \}
\]

for all \( x, y) \in K \), \( 0 \leq \lambda < 1 \) and \( f \) has the additional property that for each \( x \in \delta K \) and \( fx \in K \). Then \( f \) has a unique fixed point.

**Corollary 2.6:** Let \( X \) be a metric space of hyperbolic type, \( K \) a non-empty closed subset of \( X \) and \( \delta K \) the boundary of \( K \). Let \( \delta K \) be nonempty and let \( T : K \to X \) and \( f : K \cap T(K) \to X \) be two non-self- mappings satisfying the following conditions:

\[
d(fx, fy) \leq \lambda (d(Tx, fx) + d(Ty, fy))
\]

for all \( x, y) \in C \), \( 0 < \lambda < \frac{1}{2} \), if
(i) $\delta K \subset TK$, $fK \cap K \subset TK$,
(ii) $Tx \in \delta K \Rightarrow fx \in K$,
(iii) $fK \cap K$ is complete.

Then $f$ and $T$ have a coincidence point $z$ in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $z$ is the unique common fixed point of $f$ and $T$.

Example 2.7: Let $X$ be the set of real numbers with the usual metric, $K = [0, +\infty)$ and let $T : K \to X$ and $f : K \cap T(K) \to X$ be two non-self mappings defined by $Tx = 4x$ and $fx = \frac{4x}{1+4x}$ for all $x \in K$.

Taking $x = \frac{1}{2}$ and $y = \frac{1}{4}$ we obtain $\lambda = \frac{1}{b}$. Thus $T$ and $f$ satisfied (2.1) and all the hypotheses in Theorem 2.2 are satisfied. $T$ and $f$ have a unique common fixed point $z = 0$.

3. Conclusion

In this section, we proved that in a metric space of hyperbolic type, two non-self mappings $f$ and $T$ satisfying certain contractive conditions have a coincidence point. Moreover, if the maps are weakly compatible then $f$ and $T$ have a unique common fixed point. We gave an example to validate our results.

References