The Effect of Stochastic Capital Reserve on Actuarial Risk Analysis via an Integro-differential Equation

Sunday Onos Edeki, Member, IAENG, Ini Adinya and Olabisi O. Ugbebor

Abstract—This work investigates the effect of stochastic capital reserve on actuarial risk analysis. The formulated mathematical problem is a risk-reserve process of an insurance company whose ruin and survival probabilities are analyzed via the solutions of a derived integro-differential equation (IDE). We further study the interplay between the parameters governing the ruin and the survival probabilities regarding the risk-reserve model; thereby establish a relationship between the probabilities and the initial risk reserve in terms of the other parameters.

Index Terms—Stochastic processes, Capital reserve, Risk theory and Integro-differential equations.

I. Introduction

Mathematical models are used in actuarial risk analysis and applied probability to describe the inability of an insurer to withstand ruin or insolvency. In these models; the probability of ruin, distribution of surplus immediately prior to ruin and deficit at the time of ruin are of paramount interest.

In 1903, the Swedish actuary, Filip Lunberg [1] introduced the theoretical foundation of ruin theory in modelling the ruin problem for an insurance company. The work of Lundberg was later reviewed and extended by Herald Cramer [2] and [3] in the 1930s, hence; the Cramer – Lunberg model (or classical compound-Poisson risk model or classical risk process or Poisson risk process) which is still the climax of insurance mathematics [4].

This model describes an insurance company faced with two opposing situations with respect to cash flows viz: the incoming cash premium and the outgoing claims.

The premium (from customers) arrives at constant rate $k > 0$ and the arrival of claims follows a Poisson process with intensity λ and are also independent and identically distributed non negative random variables with distribution F and mean μ (forming a compound Poisson process).

In extension of the classical model, Andersen [5] allows the claim interval times to have arbitrary distribution functions.

Gerber and Shiu [6] in their work, analyzed the behaviour of the insurer's surplus through the expected discounted penalty function – commonly referred to as Gerber-Shiu function and applied this function to the classical compound Poisson model while Powers [7] is of the opinion that insurers' surplus is better modeled by a family of diffusion processes.

Paulsen [8] proposed a general risk process with stochastic return on investments; taking account of three key factors – insurance risk, investment risk and inflation. In his work, he modeled those factors through semimartingales and obtained an integro–differential equation with an analytical expression for ruin probability under certain conditions.

Yuen and Wang [9] derived an integro–differential equation for the Gerber–Shiu expected discounted penalty function, and then obtained an exact solution to the equation and also obtained closed form expressions for the expected discounted penalty function in some special cases. They finally examined a lower bound for the ruin probability of the risk process.

Han and Yun [10] introduced the Optimal Homotopy Asymptotic Method (OHAM) for solving nonlinear integro-differential equations; with examples illustrating the reliability and efficiency of the proposed OHAM, directed towards obtaining approximate solutions of the nonlinear IDE.

In considering a perturbed market–modulated risk model with two sided jumps, Dong and Zhao [11] derived a system of differential equations for the Gerber–Shiu function and gave a numerical result based on Chebyshev polynomial approximation with illustrative examples. And Elghribi and Haouala [12] constructed, by Bochner subordination, a new model – an extension of the Sparre-Andersen model with investments that is perturbed by diffusion.

The paper is structured as follows: Section II deals with preliminaries on some basic concepts in actuarial risk analysis, section III is on integro-differential equation, application, and discussion of result while section IV is on conclusion.

II. Preliminaries

Some Basic Concepts in Actuarial Risk Analysis:

In a bid to calculate the risks and premium in an insurance company, a risky situation is encountered when the actuary wants to pay out a total claim amount. Hence, the following concepts:

Manuscript received July 3, 2013; revised March 17, 2014.

S. O. Edeki (Corresponding Author) is with the Department of Mathematics, College of Science & Technology, Covenant University, Canaan Land, Ota, Nigeria.
Corresponding Author e-mail : soedeki@yahoo.com
Ini Adinya is with the Department of Mathematics, University of Ibadan, Ibadan, Nigeria. E-mail Address: iniadinya@gmail.com.
Olabisi O. Ugbebor is with the Department of Mathematics, University of Ibadan, Ibadan, Nigeria. Email Address : ugbebor1@yahoo.com.

Definition 2.1 (Compound Poisson Process)

- the company begins its business at time, $t = 0$
- claims arrive at random time points:

$T_1, T_2, T_3, \cdots$ with respective claim sizes $H_1, H_2, H_3, \cdots$ giving rise to a marked point process $\{T_i, H_i\}_{i \geq 1}$ called risk process. Thus, the risk process has two components. Suppose $\{N(t), t \geq 0\}$ is a random counting process, then the total claim size is defined and denoted by:

$$C_t = \sum_{i=1}^{N_t} H_t \quad , \text{for } N_t \geq 1 \text{ or } 0 \text{ otherwise.} \qquad (2.1)$$

From (2.1), $\{C_t, t > 0\}$ is a compound Poisson process if $\{N_t, t > 0\}$ is a compound Poisson process.

2.2 Risk Reserve Process

In what follows, we denote a filtered probability space $(\mu, \beta, \Omega, F(\beta))$ upon which all the variables in this work are defined.

In order to minimize loss and to maximize profit, the insurance company imposes a certain amount called premium on the client. Henceforth, $K(t)$ is the corresponding total premium income of this insurance company in $(0,t]$ with C_t as defined in (2.1), therefore, the insurance company makes profit (resp. (loss)) if:

$$(K(t) - C(t)) > 0, \ (\text{resp. } K(t) - C(t)) < 0) \text{ on } (0,t].$$

Suppose the insurance company has an initial capital or initial reserve $U(0) = u \geq 0$ and makes a profit of $(K(t) - C(t))$ in $(0,t]$. Then it has a reserve known as risk reserve denoted by:

$$U_t = u + K(t) - C(t) \qquad (2.2)$$

The corresponding risk reserve process $\{U(t), t > 0\}$ is a stochastic process defined on the filtered probability space $(\mu, \beta, \Omega, F(\beta))$. This is a special type of stochastic process belonging to the class of Le'vy processes as considered in [13].

Classical Assumptions:

The following assumptions are made for suitability of the Cramer- Lundberg Model:

A_1 : $\{N(t), t \geq 0\}$ is a homogeneous Poisson process with parameter $\lambda = \frac{1}{\mu}$, such that

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n \geq 0$$

A_2 : the claim sizes $H_1, H_2, H_3, \cdots$ are independently and identically distributed as H.

A_3 : The premium income is a linear function in t i.e. $k_t = k(t) = kt, k > 0, t \geq 0.$

A_4 : The time horizon is finite.

Using (2.1) and A_3 in (2.2) gives:

$$U_t = u + k(t) - \sum_{i=1}^{N_t} H_t \qquad (2.3)$$

The major point using this model, is to investigate the probability that the insurer's surplus level finally falls below zero. As such, the probability of ruin is defined as:

$$\psi(u) = P^u\{\tau < \infty\} \qquad (2.4)$$

where the ruin time τ is such that $\tau = \inf\{U(t) < 0\}$ and $0 \leq \psi(u) < 1$ as a natural requirement for solvency otherwise $\psi(u) = 1$ implies a certainty of ruin; hence; the imposition:

$$E[U_t] > 0, \forall t > 0. \qquad (2.5)$$

where E is a mathematical expectation operator with respect to (the martingale measure) P.

Definition (a) Let $X = \sum_{i=1}^{n} a_i N_{A_i}$ be real-valued simple random variable defined on $(\mu, \beta, \Omega, F(\beta))$, then the mean value or expected value of X is defined as:

$$E(X) = \int_\Omega X dP := \sum_{i=1}^{n} a_i P(A_i) \qquad (2.6)$$

and the variance of X is :

$$V(X) = \int_\Omega |X - E(X)|^2 dP = E(|X|^2) - |E(X)|^2$$

where $|\cdot|$ denotes the Euclidean norm.

Definition (b) Let $g(x)$ defined on $[0, \infty)$ be the probability density function (pdf) of a non-negative random variable X, then the Laplace transform of $g(x)$ is defined and denoted by:

$$\mathcal{L}[g(x)] = \hat{g}(s) = E[e^{-sX}] = \int_0^\infty e^{-sx} g(x)dx \quad (2.7)$$

Definition (c) The n^{th} moment of X in definition (b) is defined as:

$$E[X^n] = (-1)^n \frac{d^n \hat{g}(s)}{ds^n}\bigg|_{s=1}, \text{ for } n \ge 0 \quad (2.8)$$

Lemma 2a Suppose $H_i, i \ge 1$ are stochastically independent and identically distributed as H and are independent of $\{T_t, t \ge 0\}$ then:

a) $\{C(t), t \ge 0\}$ has independent homogeneous increment

b) The Laplace transform of $C(t)$:

$$\mathcal{L}[C_t] = \hat{C}_t(s) = E[e^{\lambda t(\hat{H}(s)-1)}]$$

where $\hat{H}(s) = E(e^{-sH})$

Proof of a: This follows trivially from the properties of stochastic processes; see [13] and [14].

Proof of b: By (2.1) and (2.7), we have that:

$$\hat{C}_t(s) = E(e^{-sC(t)}) = E(e^{-s\sum_{i=1}^{n} H_i})$$

$$= E(e^{-s[H_1+H_2+H_3+H_4+\cdots+H_{N(t)}]})\cdot 1$$

$$= \sum_{n=0}^{\infty} E(e^{-s[H_1+H_2+H_3+H_4+\cdots+H_{N(t)}]})P(N_t = n)$$

$$= \sum_{n=0}^{\infty} E(e^{-sH})^n \left(\frac{(\lambda t)^n e^{-\lambda t}}{n!}\right)$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} E(e^{-sH})^n \left(\frac{(\lambda t)^n}{n!}\right)$$

$$= e^{-\lambda t} \left\{\sum_{n=0}^{\infty} E(\hat{H}(s))^n \left(\frac{(\lambda t)^n}{n!}\right)\right\}$$

$$= e^{-\lambda t}\left\{e^{\hat{H}(s)\lambda t}\right\}$$

$$= e^{\lambda t(\hat{H}(s)-1)} \quad \square$$

Corollary 2b. The mean value $E(C(t))$ and variance $Var(C(t))$ of $C(t)$ are easily computed via Lemma 2a, thus:

$E(C(t)) = \lambda t E(H)$ and $Var(C(t)) = \lambda t E(H^2)$.

Proof: From (2.8), we have:

$$E[C^n(t)] = (-1)^n \frac{d^n \hat{C}_t(s)}{ds^n}, \text{ for } n \ge 1$$

Therefore, $n = 1$ yields:

$$E\{C(t)\} = (-1)\frac{d\hat{C}_t(s)}{ds}$$

$$= (-1)\frac{d}{ds}\left(e^{\lambda t[\hat{H}(s)-1]}\right)$$

$$= (-1)\frac{d}{ds}\left(e^{\lambda t[E(e^{-sH})-1]}\right)$$

$$= (-1)\left[\lambda t E\{-He^{-sH}\}\left(e^{\lambda t[E(e^{-sH})-1]}\right)\right]$$

$$\therefore E(C(t))\big|_{s=0} = \lambda t E(H).$$

(which is the mean value).

Similarly, for $n = 2$, we have:

$$E\{C(t)\}^2 = (-1)^2 \frac{d^2 \hat{C}_t(s)}{ds^2}$$

$$= \left[\lambda t E\{H^2 e^{-sH}\}\left(e^{\lambda t[E(e^{-sH})-1]}\right)\right] + \left[\lambda t E\{-He^{-sH}\}\left(e^{\lambda t[E(e^{-sH})-1]}\right)\right]\lambda t E\{-He^{-sH}\}$$

Hence,

$$E(C(t))^2\big|_{s=0} = \lambda t E(H^2) + (\lambda t E(-H))^2$$

$$= \lambda t E(H^2) + (\lambda t E(H))^2$$

$$\therefore Var(C(t)) = E\{C(t)\}^2 - (E\{C(t)\})^2$$

$$= \lambda t E(H^2) + (\lambda t E(H))^2 - (\lambda t E(H))^2$$

$$= \lambda t E(H^2) \quad \square$$

Lemma 2c Suppose $= E(H) = v$ with $\lambda = \frac{1}{\mu}$ as in A_0 then, $\Lambda_1 = k\mu - v > 0$ where Λ_1 called safety loading guaranteed survival.

Proof: $U_t = u + kt - C_t$, by definition; see (2.3)

Thus, $E(U_t) = E(u + kt - C_t)$

$\Rightarrow$

$$\lim_{t\to\infty} \frac{\{E(U_t)\}}{t} = \lim_{t\to\infty}\frac{\{u\}}{t} + \lim_{t\to\infty} k - \lim_{t\to\infty}\frac{\{\lambda t E(H)\}}{t}$$

$$= k - \lambda v = \frac{k\mu - v}{\mu}$$

Hence, using the profit condition (2.5) gives:

$$\Lambda_1 = k\mu - v > 0 \qquad \lrcorner$$

III INTEGRO-DIFFERENTIIAL EQUATION (IDE)

This section deals with an Integro-differentiial equation satisfied by a survival probability as well as a corresponding ruin probability of an insurance company. For better understanding, we introduce the following concepts.

Definition 3. Let $\phi(u, z)$, (for $u > 0, z > 0$) both in monetary units) be the survival probability of an insurance company with u as initial capital, such that $0 < u < z$ and $u + k\Delta t < z$ with respect to the time interval $[0, \Delta t]$. As such $\phi(u, z) \to \psi(u)$ as $z \to \infty$, whence; $\psi(u)$ and $\bar{\psi}(u)$ are the survival and ruin probabilities respectively ,with a relation:

$$\psi(u) = 1 - \bar{\psi}(u) \tag{3.1}$$

3.1. Derivation of Integro-differentiial Equation Satisfied by $\psi(u)$ and $\bar{\psi}(u)$.

Here, the procedure for the derivation of a first order Integro-differentiial Equation (IDE) satisfied by the survival and ruin probabilities of an insurance company as defined above is carefully analyzed and studied.

Procedure:

Let the distribution function of the claim size H be $F(z)$ with probability density function (pdf) such that:

$$P(H \le z) = F(z) \text{ and } \frac{dF(z)}{dz} = f(z).$$

Now, for simplicity of the steps, the following points are noted and conditions are placed.

Considerations:

k_1 - Let the corresponding time interval be $[0, \Delta t]$

k_2 - Since the number of claims follow a Poisson process, claims can either be 0 or 1 during this length.

k_3 - Based on k_2, the probability of having more than one claim is trivial.

k_4 - Occurrence of a claim (no claim) happens with probability $\Delta t \; (1 - \lambda\Delta t)$.

Conditions:

c_1 - for no claim occurring in $[0, \Delta t]$ with negative risk reserve, the survival probability is zero (0)

c_2 - for two or more claims occurring in $[0, \Delta t]$, the survival probability is zero (0).

c_3 - for no claim occurring in $[0, \Delta t]$, the survival probability is: $\psi(u + k\Delta t)$.

c_4 - for one claim occurring in $[0, \Delta t]$ with non-negative risk reserve, the survival probability is:

$$\int_0^{u+k\Delta t} \psi(u + k\Delta t - z) f(z) dz$$

Therefore, running through the considerations and the conditions; using the law of total probability, we have:

$$\psi(u) = (1 - \lambda\Delta t + 0(\Delta t))\psi(u + k\Delta t) + (\lambda\Delta t + 0(\Delta t)) \int_0^{u+k\Delta t} \psi(u + k\Delta t - z) f(z) dz + 0(\Delta t) \tag{3.2}$$

$$\Rightarrow \psi(u) - \psi(u + k\Delta t) = (-\lambda\Delta t + 0(\Delta t))\psi(u + k\Delta t) + (\lambda\Delta t + 0(\Delta t)) \int_0^{u+k\Delta t} \psi(u + k\Delta t - z) f(z) dz + 0(\Delta t)$$

So,

$$\psi(u + k\Delta t) - \psi(u) = (\lambda\Delta t + 0(\Delta t))\psi(u + k\Delta t) - (\lambda\Delta t + 0(\Delta t)) \int_0^{u+k\Delta t} \psi(u + k\Delta t - z) f(z) dz + 0(\Delta t) \tag{3.3}$$

Setting $k\Delta t = h$ i.e. $\Delta t = \dfrac{h}{k}$ and assuming that $\psi(u)$ is differentiable, (3.3) becomes:

$$\psi(u+h)-\psi(u)=\left(\lambda\frac{h}{k}+O\left(\frac{h}{k}\right)\right)\psi(u+h)+O\left(\frac{h}{k}\right)$$

$$-\left(\lambda\frac{h}{k}+O\left(\frac{h}{k}\right)\right)\int_0^{u+h}(u+h-z)f(z)dz+O\left(\frac{h}{k}\right)$$

$\Rightarrow$

$$\frac{\psi(u+h)-\psi(u)}{h}=\frac{1}{h}\left\{\left(\lambda\frac{h}{k}+O\left(\frac{h}{k}\right)\right)\psi(u+h)\right.$$

$$\left.-\left(\lambda\frac{h}{k}+O\left(\frac{h}{k}\right)\right)\int_0^{u+h}(u+h-z)f(z)dz+O\left(\frac{h}{k}\right)\right\}$$

$$\therefore\frac{\psi(u+h)-\psi(u)}{h}=\frac{\lambda}{k}\{\psi(u+h)\}$$

$$-\frac{\lambda}{k}\int_0^{u+h}(u+h-z)f(z)dz$$

But $\dfrac{\psi(u+h)-\psi(u)}{h}\to\psi'(u)$ as $h\to 0$

Whence,

$$\psi'(u)=\frac{\lambda}{k}\left(\psi(u)-\int_0^u(u-z)f(z)dz\right)$$

$$=\frac{1}{\mu k}\left(\psi(u)-\int_0^u(u-z)f(z)dz\right)$$

where $\lambda=\dfrac{1}{\mu}$ following assumption A_0 □

3.2. The Laplacian Solution of the Integro-differentiial Equation (IDE)

In order to obtain the solution of the IDE, we apply the Laplace Transform, hence the following result:

Theorem 3.1: With regard to the above information, $\psi(u)$ and $\overline{\psi}(u)$ satisfy the IDE:

$$\psi'(u)=\frac{1}{\mu k}\left(\psi(u)-\int_0^u(u-z)f(z)dz\right).$$

Proof: From (2.7) in definition b, we recall that:

$$\mathcal{L}[g(x)]=\hat{g}(s)=E[e^{-sX}]=\int_0^\infty e^{-sx}g(s)ds$$

for a well-defined function $g(x)$ on $[0,\infty)$ with:

$$\mathcal{L}[g^n(x)]=s^n g(s)-s^{n-1}g(0)-s^{n-2}g'(0)$$

$$-s^{n-3}g''(0)-\cdots-g^{(n-1)}(0) \qquad (3.5)$$

and

$$\mathcal{L}[\int_0^t\rho(u)\eta(t-u)du]=\hat{\rho}(s)\hat{\eta}(s) \qquad (3.6)$$

where $g^n(x)$ denotes the nth derivative of $g(x)$.

$$\therefore\ \mathcal{L}[\psi'(u)]=\frac{1}{\mu k}\mathcal{L}\left\{\left(\psi(u)-\int_0^u(u-z)f(z)dz\right)\right\}$$

$$\Rightarrow\quad s\hat{\psi}(s)-\psi(0)=\frac{1}{\mu k}\left\{\hat{\psi}(s)-\hat{\psi}(s)\hat{f}(s)\right\}$$

$$\mu k\left(s\hat{\psi}(s)-\psi(0)\right)=\hat{\psi}(s)\left\{1-\hat{f}(s)\right\}$$

$$\mu ks\hat{\psi}(s)-\hat{\psi}(s)\left[1-\hat{f}(s)\right]=\mu k\psi(0)$$

$$\mu k\hat{\psi}(s)\left[s-\frac{1}{\mu k}\left(1-\hat{f}(s)\right)\right]=\mu k\psi(0)$$

$$\therefore\ \hat{\psi}(s)=\frac{\psi(0)}{s-\frac{1}{\mu k}\left(1-\hat{f}(s)\right)}=\frac{\psi(0)\mu k}{s\mu k-\left(1-\hat{f}(s)\right)} \qquad (3.7)$$

Theorem 3.2 Lundberg Inequality

For $u\geq 0$ and $L>0$ (with L referred to as Lundberg exponent), ruin (resp. survival) probability satisfies the Lundberg Inequality :

$$\overline{\psi}(u)\leq e^{-Lu}\quad\left(\text{resp. }\left(1-\psi(u)\right)\leq e^{-Lu}\right)$$

The Proof of Theorem 3.2 can be found in [4], [14], [15] ,[16] and [17].

3.3 Applications:

To explicitly illustrate the strength of the model, we consider the following results:

Case I

Corollary 3.1 Suppose the claim size H of an insurance company is exponentially distributed with parameter $\frac{1}{\phi}$, then the survival and the ruin probabilities are:

$\psi(u)=1-(1-\beta)e^{-\frac{\beta u}{\phi}}$ and $\bar{\psi}(u)=(1-\beta)e^{-\frac{\beta u}{\phi}}$

respectively, where $\beta\in(0,1)$.

Proof: Let the pdf of H be $f(z)$ such that $f(z)=\frac{1}{\phi}e^{-\frac{z}{\phi}}$, since H is exponentially distributed with parameter $\frac{1}{\phi}$ but from theorem 3.1,

$$\hat{\psi}(s)=\frac{\psi(0)\mu k}{s\mu k-\left(1-\hat{f}(s)\right)} \quad \text{see (2.7)}$$

$$\therefore\ \hat{f}(s)=\mathcal{L}[f(s)]=\mathcal{L}[e^{-sz}]=\int_0^\infty e^{-sz}f(z)dz$$

$$=\frac{1}{\phi}\int_0^\infty e^{-sz}e^{-\frac{z}{\phi}}dz=\frac{1}{\phi}\int_0^\infty e^{-\left(s+\frac{1}{\phi}\right)z}dz$$

$$=\frac{-1}{\phi\left(s+\frac{1}{\phi}\right)}e^{-\left(s+\frac{1}{\phi}\right)z}\Bigg|_0^\infty=\frac{1}{(\phi s+1)}$$

$$\therefore\ 1-\hat{f}(s)=1-\frac{1}{(\phi s+1)}=\frac{\phi s}{(\phi s+1)} \qquad (3.8)$$

Putting (3.8) in (3.7) yields:

$$\hat{\psi}(s)=\frac{\psi(0)}{s-\frac{1}{\mu k}\left(1-\hat{f}(s)\right)}=\frac{\psi(0)}{s-\frac{1}{\mu k}\left(\frac{\phi s}{\phi s+1}\right)}$$

$$=\left(\frac{(\phi s+1)\mu k\psi(0)}{s(\mu k\phi s+\mu k-\phi)}\right) \qquad (3.9)$$

Considering $\psi(s)$, $0<\beta<1$ and Lemma 2c, we set $\beta\mu k=\mu k-\phi=\alpha$, that is, $\mu k=\frac{\alpha}{\beta}$

Therefore, (3.9) becomes:

$$\hat{\psi}(s)=\left(\frac{(\phi s+1)}{s\left(\frac{\alpha}{\beta}\phi s+\alpha\right)}\frac{\alpha}{\beta}\psi(0)\right)$$

$$=\frac{(\phi s+1)}{\frac{s\alpha}{\beta}(\phi s+\beta)}\frac{\alpha}{\beta}\psi(0)=\frac{(\phi s+1)}{s(\phi s+\beta)}\psi(0)$$

But by partial fraction, $\frac{(\phi s+1)}{s(\phi s+\beta)}=\frac{A_1}{s}+\frac{A_2}{(\phi s+\beta)}$,

with $A_1=\frac{1}{\beta}$ and $A_2=\frac{\phi(\beta-1)}{\beta}$

$$\therefore\ \hat{\psi}(s)=\left[\frac{1}{\beta s}+\frac{\phi(\beta-1)}{\beta(\phi s+\beta)}\right]\psi(0)$$

$$=\left[\frac{1}{\beta s}+\frac{(\beta-1)}{\beta\left(s+\frac{\beta}{\phi}\right)}\right]\psi(0)$$

$$\Rightarrow\quad \hat{\psi}(s)=\left[\left(\frac{1}{\beta}\right)\left(\frac{1}{s}\right)+\left(\frac{\beta-1}{\beta}\right)\left(\frac{1}{s-\frac{\beta}{\phi}}\right)\right]\psi(0)$$

Thus, taking the Laplace transform of both sides gives:

$$\psi(s)=\mathcal{L}[\hat{\psi}(s)=\left[\left(\frac{1}{\beta}\right)\left(\frac{1}{s}\right)+\left(\frac{\beta-1}{\beta}\right)\left(\frac{1}{s-\frac{\beta}{\phi}}\right)\right]\psi(0)$$

$$=\left[\left(\frac{1}{\beta}\right)+\left(\frac{\beta-1}{\beta}\right)e^{\frac{-\beta s}{\phi}}\right]\psi(0)$$

Whence,

$$\psi(s)=\frac{1}{\beta}\left[1+(\beta-1)e^{\frac{-\beta s}{\phi}}\right]\psi(0) \qquad (3.10)$$

In order to compute $\psi(0)$, we restructure (3.10) and apply the consequence of Theorem 3.2 (Lundberg inequality) as follows:

$$\psi(0)=\frac{\beta\psi(u)}{1+(\beta-1)e^{\frac{-\beta z}{\phi}}} \text{ with } \lim_{u\to\infty}\psi(u)=1 \text{ such that}$$

$\psi(0)=\beta$ and $\bar{\psi}(u)=1-\beta$

Thus, (3.10) yields:

$$\psi(u)=1+(\beta-1)e^{\frac{-\beta z}{\phi}}$$

Equivalently, we denote:

$$\psi(u)=1+(\beta-1)e^{\frac{-\beta z}{\phi}} \qquad \text{(survival probability)}$$

$$\bar{\psi}(u)=(\beta-1)e^{\frac{-\beta z}{\phi}} \qquad \text{(ruin probability)}$$

Note a: In comparing these results, we see clearly that the ruin probability satisfies the Lundberg equation with the exponent $L=\frac{\beta}{\phi}$, upon the boundary conditions:

$\psi(0)=(1-\beta)$, $\bar{\psi}(\infty)=0$.

Case II

Result for Mixture Exponential Claim Sizes:

Other explicit results can be obtained for mixed exponential claim size distributions, thus the following:

Theorem 3.3

Suppose that the claim size H of an insurance firm is a mixture of n exponential distributions with the density function:

$$f(z)=\sum_{i=1}^{n}d_i\phi_i e^{-\phi_i z}\ ,\ z>0\ ,$$

then, the ruin probability is

$$\bar{\psi}(u)=\sum_{i=1}^{n}k_i e^{-r_i u}$$

for some constants k_i, provided that r_i are the distinct positive solutions of the Lundberg equation

$$\sum_{i=1}^{n}\left(A_i\frac{\phi_i}{s+\phi_i}\right)=\left(\frac{\lambda-ks}{\lambda}\right), \text{ where } A_i>0.$$

Proof:

We shall sketch the proof of theorem 3.3 by the application of Laplace Transform as follows,

Recall from theorem 3.1 that;

$$\hat{\psi}(s)=\frac{\psi(0)\mu k}{s\mu k-\left(1-\hat{f}(s)\right)}$$

But $f(z)=\sum_{i=1}^{n}d_i\phi_i e^{-\phi_i z}$ implies that

$$\hat{f}(s)=\sum_{i=1}^{n}\left(d_i\frac{\phi_i}{s+\phi_i}\right)$$ Hence, we have:

$$\hat{\psi}(s)=\frac{\psi(0)\mu k}{s\mu k-\left(1-\sum_{i=1}^{n}\left(d_i\frac{\phi_i}{s+\phi_i}\right)\right)}$$

For $\hat{\psi}(s)$ being a rational function, and the denominator equals zero whenever s is some r_i by definition, with the application of partial fraction theorem, we therefore write:

$$\hat{\psi}(s)=\frac{k_0}{s}+\sum_{i=1}^{n}\frac{k_i}{s+r_i}$$

Whereas all the terms under the summation can be transformed easily to exponentials by the application of Laplace Transform, while the first term becomes a constant function by Laplace for $z>0$, satisfying $\bar{\psi}(\infty)=0$ with $k_0=0$ ▫

Note b: An alternative and a detailed approach with regards to the solutions of ODE derived from the associated IDE is also applied by [18].

3.4 Discussion of Result

For the purpose of the discussion of result, we carefully studied the interplay between the associated parameters u, β, and ϕ through some numerical calculations; thus, we set $u>0$ say $(0, 200]$ and fixed the following:

$\beta=0\cdot1\in(0,1)$ and $\phi=3\cdot4$

Note: Fig 1and Fig 2 below, represent the graphs of the survival probability and the ruin probability respectively.

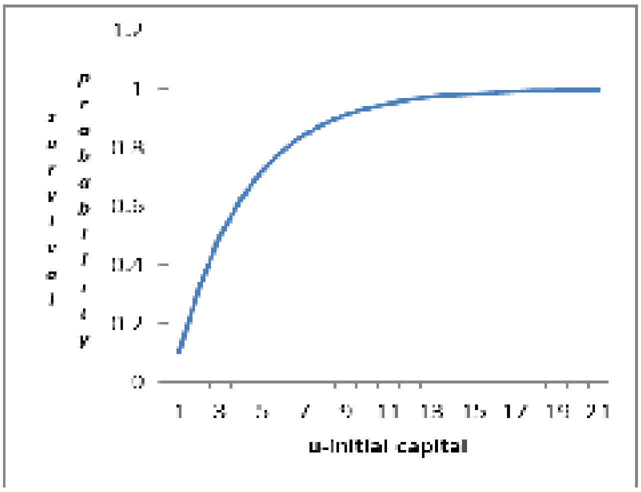

Fig 1 : A graphical representation of the survival model.

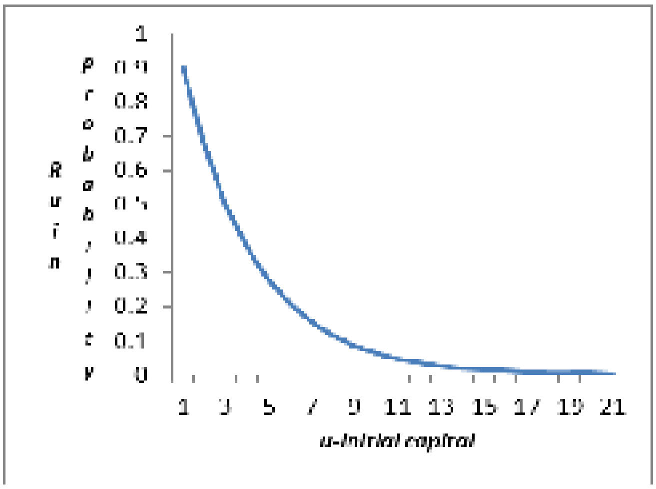

Fig 2 : A graphical representation of the ruin model.

IV. CONCLUSION

We therefore conclude that the insurance company has a better chance of survival when the initial reserve is reasonably high; since $\psi(u)$ increases as u increases. In other words, the ruin probability of the insurance company is minimal at such; since $\bar{\psi}(u)$ decreases as u increases. These are obviously shown in Fig 1 and Fig 2 above, respectively.

In the paper, two cases are considered; case I is when the claim size H of an insurance company is exponentially distributed with only one parameter while case II is when the claim size H is a mixture of n exponential distributions.

Remark : The validity of the result is ascertained for all values of $\beta \in (0,1)$, $u > 0$ and $\phi > 0$. Hence, the effectiveness of the results both in theoretical and computational views with regards to actuarial sciences.

ACKNOWLEDGEMENT

The authors are grateful to the anonymous reviewers and referees for their helpful suggestions and valuable comments towards the improvement of the paper.

REFERENCES

[1] F. Lundberg, ``Approximations of the Function/Reinsurance of collective Risks'' Ph.D Thesis, Uppsala, (1903).

[2] G. Blom, "Harald Cramer 1893-1985'' The Annals of Statistics 15(4), (1987), 1335-1350. doi:10.1214/aos/1176350596.

[3] H. Cramer, ``Historical review of Filip Lundberg's works on risk theory'' Scandinavian Actuarial Journal, Vol.1969, Supplement 3, (1969) 6-12.

[4] S.B. Volchan, ``The fundamental theorem of actuarial risk science'', 2012. www.appliedprobability.org

[5] E. Sparre Andersen, "On the collective theory of risk in case of contagion between claims''. Transaction of the XVth International congress of Actuaries 2(6), 1957.

[6] H.U Gerber and E.S.W. Shiu, "On the time value of ruin" North American Actuarial Journal 2(1),(1998),48-78. doi:10.1080/10920277.1998.10595671.

[7] M.R. Powers, "A Theory of risk, return and solvency''. Insurance, Mathematics and Economics 17 (2), (1995),101-118.

[8] J. Paulsen ``Risk theory in a stochastic environment '' Stochastic Processes and their Applications, 46, (1993),327-61.

[9] K. C. Yuen and G. Wang "Some Ruin Problems for a Risk Process with Stochastic interest". North American Actuarial Journal, 9(3),(2005),129-142.

[10] Y. D. Han and J.H. Yun, ``Optimal Homotopy Asymptotic Method for solving Integro-differential Equations'' *IAENG International Journal of Applied Mathematics*, 43 (3), (2013), 120-126.

[11] H. Dong and X. Zhao "Numerical Method for a Markov-Modulated Risk Model with Two-sided Jumps" Abstract and Applied Analysis. doi: 10.1155/2012/401562.

[12] M. Elghribi and E. Haouala, ``Laplace Transform of the Time of Ruin for a Perturbed Risk Process Driven by a Subordinator'' *IAENG International Journal of Applied Mathematics*, 39 (4), (2009), 221-230.

[13] O. O. Ugbebor and S.O. Edeki, On Duality Principle in Exponentially Lévy Market, Journal of Applied Mathematics & Bioinformatics, 3(2), (2013), 159-170.

[14] Frank Beichelt, ``Stochastic Processes in Science, Engineering and Finance'', Chapman and Hall/CRC 2006. Print ISBN: 978-1-58488-493-4.

[15] J. Grandell, ``Aspects of Ruin Theory'', Springer, New York (1991).

[16] M.C. Cardiu, ``Aspects of ruin probability in insurance'', U.P.B. Sci. Bull.,Series A, 68 (2), (2006).

[17] R. Kaas, M. Goovaerts, J. Dhaene and M. Denuit, ``Modern Actuarial Risk Theory'', Springer, U.S. (2001). Print ISBN: 978-0-7923-7636-1.

[18] C. Constantinescu and J. Lo, ``Ruin Theory Starter Kit # 1'' (2013), Institute and Faculty of Actuaries. www.actuaries.org.uk.