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ON ITERATIVE TECHNIQUES FOR NUMERICAL SOLUTIONS OF LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract: This paper presents Differential Transformation Method (DTM) and Picard's Iterative Method (PIM) as computational techniques in solving linear and nonlinear differential equations. For numerical analysis of the methods, three examples are considered. The results obtained are compared with their corresponding exact solutions. A link between successive terms of the solutions using the two methods is noted. The DTM is very effective and reliable in obtaining approximate solutions. The PIM requires the satisfaction of Lipschitz continuity condition; though, its results also converge rapidly to the exact solutions.

Keywords: differential transform; Picard's iteration; differential equation; Lipschitz constant.

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1. Introduction

Many analytical, semi-analytical or purely numerical methods are available for the solution of differential equations encountered in management sciences, pure and applied sciences. Most of these methods are computationally intensive because they are trial-error in nature, or need complicated symbolic computations [1].

Youssef used Picard iteration technique with Gauss-seidel technique for initial value problem [2], Rach used Adomian Decomposition method and Picard's method [3]. Bellomo

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and Sarafyan also compared Adomian Decomposition method and Picard iterative scheme [4].

The differential transformation is a numerical method for solving differential equations. The concept of differential transform was first introduced by Zhou (1986) while solving linear and non-linear initial value problems in electric circuit analysis [5]. Chen and Liu applied this method to solve two-boundary problems [6]. Jang et al, apply the two-dimensional differential method to solve partial differential equations [7]. Edeki et al [8] applied the differential transform method (DTM) as a semi-analytical method to a certain class of ODEs. The DTM has been applied to other areas among- difference equations [9], differential-difference equations [10], two-dimensional integral equations [11], optimization of the rectangular fins with variable thermal parameters [12] and integro-differential equations [13].

In this paper, linear and nonlinear ordinary differential equations are considered using the DTM and the PIM. The numerical results from the two methods are compared with their exact solutions. The main advantage of the DTM is that, it can be applied directly to linear and nonlinear ordinary differential equations without linearization, discretization or perturbation. Also, it is capable of greatly reducing the size of computational work while still maintaining accuracy, and providing the series solution with fast convergence rate. The PIM is also effective but requires the satisfaction of the Lipschitz continuity condition.

2. Analysis of the Basic Methods

In this section, the basic concepts and theorems for the Differential Transform Method (DTM) and the Picard's Iterative Method (PIM) are systematically introduced.

2.1 *The Fundamental of the Differential Transform Method*

Let $y = f(x)$ be an arbitrary function expressed in Taylor series about a point $x = 0$ as

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\frac{d^k f}{dx^k} \right]_{x=0} \tag{1}$$

Then, the differential transformation of $f(x)$ is defined as

$$F(k)=\frac{1}{k!}\left[\frac{d^k y}{dx^k}\right]_{x=0} \tag{2}$$

As a result, the inverse differential transform of $F(k)$ is:

$$f(x)=\sum_{k=0}^{\infty} x^k Y(k) \tag{3}$$

2.2 *Special theorems of the DTM*

The following theorems can be deduced from equations (1), (2) and (3):

Theorem 1: If $y(x)=y_1(x)\pm y_2(x)$, then $Y(k)=Y_1(k)\pm Y_2(k)$

Theorem 2: If $y(x)=cy_1(x)$, then $Y(k)=cY_1(k)$, where c is a constant.

Theorem 3: If $y(x)=\frac{d^n y_1(x)}{dx^n}$, then $Y(k)=\frac{(k+n)!}{k!}Y_1(k+n)$

Theorem 4: If $y(x)=y_1(x)y_2(x)$, then $Y(k)=\sum_{k_1=0}^{k} Y_1(k_1)Y_2(k-k_1)$

Theorem 5: If $y(x)=x^n$, then $Y(k)=\delta(k-n)$ where $\delta(k-n)=\begin{cases}1, & k=n\\ 0, & k\neq n\end{cases}$.

2.3 *Analysis of the Picard Iteration Method*

Consider the first order ordinary differential equation (IVP)

$$y'=g(t,y)\ , \qquad y(t_0)=y_0 \tag{4}$$

To guarantee the existence and uniqueness of the solution of (4), we assume that $g(t,y)$ is Lipschitz continuous in a ball, $B_b^*(y_0)$; centre y_0 and radius b. We define a complete normed space $\left(\mathrm{H},<\cdot,\cdot>,\|\cdot\|_g\right)$ for the function $g(t,y)$ equipped with the sup-norm:

$$\|g\|=\sup_{t\in[0,T]}\left|g\left(t,y(t)\right)\right| \tag{5}$$

where H is a Hilbert space, $<\cdot,\cdot>$ an inner product, and $\|\cdot\|_v$ a norm operator w.r.t v, such that:

$$g\in C\left[a,b\right]=\Lambda_a\left(t_0\right)\times B_b^*(y_0) \tag{6}$$

where

$$\Lambda_a\left(t_0\right)=\left[t_0-a,t_0+a\right] \text{ and } B_b^*(y_0)=\left[y_0-b,y_0+b\right] \tag{7}$$

Thus, for every pair of points $\left(y_\alpha, y_\beta\right)$ in Gr_g; the graph of g, there exists a constant $M>0$, such that :

$$\left|g(t,y_\alpha)-g(t,y_\beta)\right|\le M\left|y_\alpha-y_\beta\right| \tag{8}$$

where M is a Lipschitz constant.

Now by integrating both sides of (4) we get:

$$\int_{t_0}^{t} y'\left(\tau\right)d\tau=\int_{t_0}^{t} g(\tau,y(\tau))d\tau \tag{9}$$

Thus, by fundamental theorem of calculus, (5) becomes:

$$y\left(t\right)-y\left(t_0\right)=\int_{t_0}^{t} g(\tau,y(\tau))d\tau$$

$$\therefore \qquad y\left(t\right)=y\left(t_0\right)+\int_{t_0}^{t} g(\tau,y(\tau))d\tau \tag{10}$$

For an arbitrary t, it is obvious that $y\left(t\right)$ appears both on the LHS and in the integrand of (10). Therefore, we resort to iterative approach (Picard) by choosing an initial guess $y\left(t_0\right)=y_0$ and setting for $n\ge 1$, $n\in\mathbb{Z}^+$:

$$y_{n+1}\left(t\right)=y_0+\int_{t_0}^{t} g(\tau,y_n(\tau))d\tau \tag{11}$$

Thus, the approximate solution to (4) is $\phi_{n+1}^{PIM}(\mathrm{t})=y\left(t\right)$, provided the limits in (11) exist such that:

$$\phi_{n+1}^{PIM}(\mathrm{t})=\lim_{n\to\infty} y_{n+1}(t)=\lim_{n\to\infty} y_n(t)=y\left(t\right) \tag{12}$$

3. Applications and Numerical Results

In this subsection, we will consider some differential equations (IVP) and solve them using both methods- the differential transform method DTM and the PIM as discussed above.

Example1 *Consider the IVP:*

$$y' - y + 1 = 0, \quad y(0) = 2 \tag{13}$$

with an exact solution:

$$y_{ex}(t) = 1 + e^t \tag{14}$$

Solution (DTM):

We rewrite (13) in a standard form and take the differential transform (DT) as follows;

$$DT\left[y'(t) = y(t) - 1\right]$$

By using the basic ideas and theorems of the DTM as stated above, we obtain the following recurrence relation as follows:

$$(k+1)Y(k+1) = Y(k) - \delta(k),$$

So,

$$Y(k+1) = \frac{1}{k+1}[Y(k) - \delta(k)] \tag{15}$$

with the initial conditions $Y(0) = 2$,

Hence, for $k \geq 0$, we obtain values for $Y(1), Y(2), Y(3), \cdots$ as showed below:

for $k = 0,\ Y(1) = 1$; for $k = 1,\ Y(2) = \frac{1}{2!}$; for $k = 2,\ Y(3) = \frac{1}{4!}$; for $k = 3,\ Y(4) = \frac{1}{5!}, \cdots$

$$y(t) = 2 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + ... + \frac{t^n}{\text{n!}} \tag{16}$$

$$\phi_4^{DTM}(t) = 2 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!}. \tag{17}$$

Solution (PIM):

We re-express (13) in an integral form of (11) :

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} g(\tau, y_n(\tau))d\tau$$

$$y_{n+1}(t) = 2 + \int_0^t (-1 + y_n(\tau))d\tau,\ t_0 = 0,\ y_0 = 2 \tag{18}$$

Hence, the following successive approximations are obtained:

$$y_0 = 2,\ y_1 = 2+t,\ y_2 = 2+t+\frac{t^2}{2!},\ y_3 = 2+t+\frac{t^2}{2!}+\frac{t^3}{3!},\ y_4 = 2+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!},$$
$$\cdots y_n(t) = 2+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots+\frac{t^n}{\text{n!}}$$

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$$y_n(t) = 2+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+...+\frac{t^n}{\text{n!}} \tag{19}$$

$$\Rightarrow\ \phi_4^{PIM}(t) = 2+t+\frac{t^2}{2!}+\frac{t^3}{3!} = \sum_{n=0}^{4} y_n^{DTM}(t) \tag{20}$$

Example 2 *Consider the IVP:*

$$y' - 2y = 2,\ y(0) = 0 \tag{21}$$

with an exact solution:

$$y_{ex}(t) = -1+e^{2t} \tag{22}$$

Solution (DTM):

We rewrite (21) in a standard form and take the differential transform (DT) as follows;

$$DT\left[y'(t) = 2y(t)+2\right],\ Y\left(0\right) = 0$$

$$(k+1)Y(k+1) = 2\left[Y(k)+\delta(k)\right],$$

$$\therefore \qquad Y(k+1) = \frac{2}{k+1}\left[Y(k)+\delta(k)\right] \tag{23}$$

with the initial conditions $Y(0) = 2$,

Thus, for $k \geq 0.$, we obtain values for $Y(1), Y(2), Y(3), \cdots$ as showed below:

$$\text{for } k = 0,\ Y(1) = 2;\ \text{for } k = 1,\ Y(2) = \frac{2^2}{2!};\ \text{for } k = 2,\ Y(3) = \frac{2^3}{3!};\ \text{for } k = 3,\ Y(4) = \frac{2^4}{4!},\cdots$$

Hence,

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k = 2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}+\cdots \tag{24}$$

$$\phi_4^{DTM}(t) = 2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}+\frac{(2t)^5}{5!} \tag{25}$$

Solution (PIM):

We re-express (21) in an integral form of (11):

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} g(\tau, y_n(\tau))d\tau$$

$$y_{n+1}(t) = 2\int_{0}^{t} (1 + y_n(\tau))d\tau, \; t_0 = 0 \;, y_0 = 0 \tag{26}$$

Hence, the following successive approximations are obtained:

$$y_0 = 0, \; y_1 = 2t, \; y_2 = 2t + \frac{(2t)^2}{2!}, \; y_3 = 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!}, \; \cdots$$

Thus,

$$\phi_6^{PIM}(t) = 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} \tag{27}$$

First order non-linear differential equations

Example 3 *Consider the IVP*:

$$y'(t) - y^2(t) = 1, \; y(0) = 0 \tag{28}$$

with an exact solution:

$$y_{ex}(t) = \tan(t) \tag{29}$$

Solution (DTM):

We rewrite (23) in a standard form and take the differential transform (DT) as follows;

$$DT\left[y' = 1 + y^2 \right],$$

$$(k+1)Y(k+1) = \left[\delta(k) + \sum_{r=0}^{k} Y(r)Y(k-r) \right],$$

$$\therefore \qquad Y(k+1) = \frac{1}{k+1}\left[\delta(k) + \sum_{r=0}^{k} Y(r)Y(k-r) \right] \tag{30}$$

with the initial conditions $Y(0) = 0$,

Therefore, for $k \geq 0$, we obtain values for $Y(1), Y(2), Y(3), \cdots$ as showed below:

for $k = 0, \; Y(1) = 1;$ for $k = 2, \; Y(3) = \frac{1}{3};$ for $k = 4, \; Y(5) = \frac{2}{15};$ for $k = 6, \cdots$

where $Y(0) = Y(2) = Y(4) = \cdots = Y(2k-2) = 0,$ for $k \geq 1$

Hence,

$$y(t) = \sum_{k=0}^{\infty} Y(k) t^k = t + \frac{1}{3} t^3 + \frac{2}{15} t^5 + \cdots \tag{31}$$

$$\therefore \qquad \phi_6^{DTM}(t) = t + \frac{1}{3} t^3 + \frac{2}{15} t^5. \tag{32}$$

Solution (PIM):

We re-express (28) in an integral form of (11):

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} g(\tau, y_n(\tau)) d\tau$$

$$y_{n+1}(t) = y_0 + \int_{0}^{t} (1 + y^2{}_n(\tau)) d\tau, \ t_0 = 0, \ y_0 = 0 \tag{33}$$

Hence, the following successive approximations are obtained:

$$y_0 = 0, \ y_1 = t, \ y_2 = t + \frac{t^3}{3}, \ y_3 = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{t^7}{63}, \cdots$$

As such,

$$\phi_8^{PIM}(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{t^7}{63} \tag{34}$$

Remark 3.1: We observe a link between the solutions obtained using the DTM and the PIM. This is expressed as:

$$y_n^{PIM} = \sum_{k=0}^{n} Y(k)^{DTM} t^k \tag{35}$$

3.3 Numerical Comparison of the exact solution, the DTM solution, and the PIM solution

In the subsection, comparisons between the solutions for each example are displayed in the following tables with their graphs in figures 1-3 respectively.

Table 1: Numerical comparison for Example 1

t	**Exact solution**	$\phi_3^{DTM}(t)$	$\phi_4^{PIM}(t)$	**Absolute Error (DTM)**	**Absolute Error (PIM)**
0.0	2.000000	2.000	2.000000	0.000000	0.000000
0.1	2.105171	2.105	2.105167	0.000171	4.25E-06
0.2	2.221403	2.220	2.221333	0.001403	6.94E-05
0.3	2.349859	2.345	2.349500	0.004859	0.000359
0.4	2.491825	2.480	2.490667	0.011825	0.001158
0.5	2.648721	2.625	2.645833	0.023721	0.002888
0.6	2.822119	2.780	2.816000	0.042119	0.006119
0.7	3.013753	2.945	3.002167	0.068753	0.011586
0.8	3.225541	3.120	3.205333	0.105541	0.020208
0.9	3.459603	3.305	3.426500	0.154603	0.033103
1.0	3.718282	3.500	3.666667	0.218282	0.051615

Table 2: Numerical comparison for Example 2

t	**Exact solution**	$\phi_5^{DTM}(t)$	$\phi_6^{PIM}(t)$	**Absolute Error (DTM)**	**Absolute Error (PIM)**
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.221403	0.221400	0.221403	2.76E-06	9.15E-08
0.2	0.491825	0.491733	0.491819	9.14E-05	6.03E-06
0.3	0.822119	0.821400	0.822048	0.000719	7.08E-05
0.4	1.225541	1.222400	1.225131	0.003141	0.00041
0.5	1.718282	1.708333	1.716667	0.009948	0.001615
0.6	2.320117	2.294400	2.315136	0.025717	0.004981
0.7	3.055200	2.997400	3.042219	0.057800	0.012981
0.8	3.953032	3.835733	3.923115	0.117299	0.029918
0.9	5.049647	4.829400	4.986864	0.220247	0.062783
1.0	6.389056	6.000000	6.266667	0.389056	0.122389

Table 3: Numerical comparison for Example 3

t	Exact solution	$\phi_6^{DTM}(t)$	$\phi_8^{PIM}(t)$	Absolute Error (DTM)	Absolute Error (PIM)
0.0	0.000000	0.000000	0.000000	0.000000	0.00000
0.1	0.100335	0.100355	0.100355	2E-05	2E-05
0.2	0.202710	0.203349	0.203350	0.000639	0.00064
0.3	0.309336	0.314184	0.314187	0.004848	0.004851
0.4	0.422793	0.443179	0.443205	0.020385	0.020411
0.5	0.546302	0.608333	0.608457	0.062031	0.062155
0.6	0.684137	0.837888	0.838332	0.153751	0.154196
0.7	0.842288	1.172883	1.174190	0.330594	0.331901
0.8	1.029639	1.669717	1.673046	0.640079	0.643408
0.9	1.260158	2.402712	2.410304	1.142554	1.150146
1.0	1.557408	3.466667	3.482540	1.909259	1.925132

Remark 3.1: We show in Figure [1-3], the graphs representing the solutions of the solved examples. Series [1-3] indicate solutions for exact, DTM and PIM respectively.

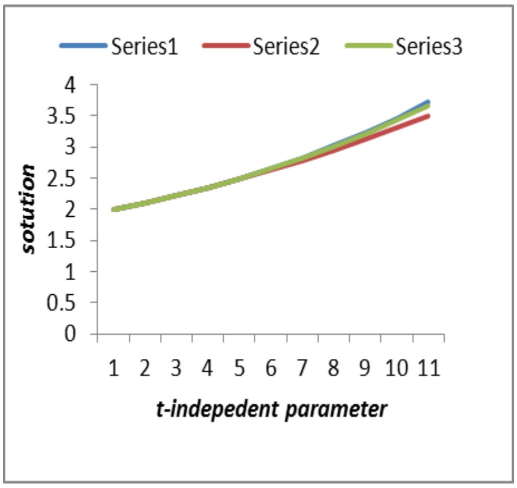

Fig 1: *Graph of example 1 Solutions*

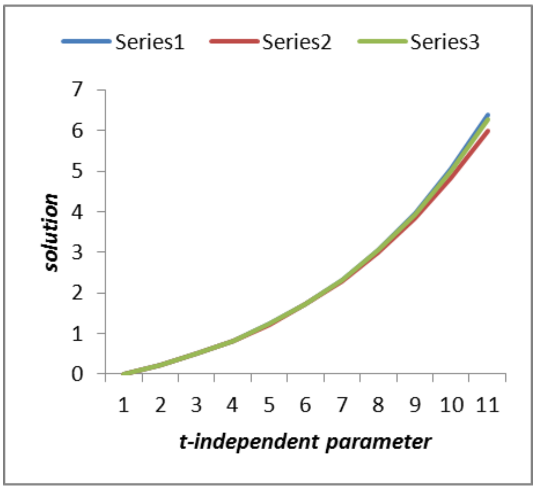

Fig 2: *Graph of example 2 Solutions*

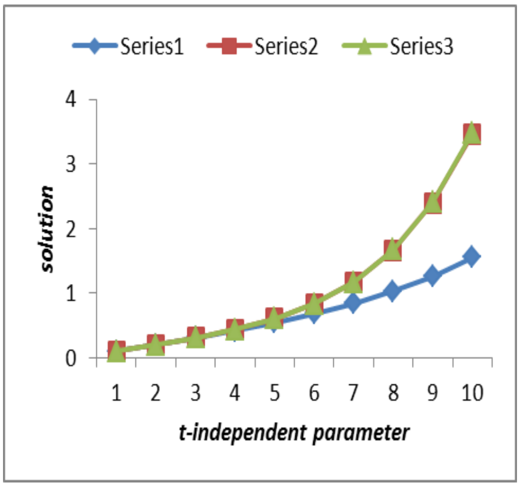

Fig. 3: Graph of example 3 solution

4.0 Discussion of Results and Concluding Remarks

In this paper, we have used the DTM and the PIM successfully in solving both linear and nonlinear differential equations (IVP), and the results obtained are compared with their corresponding exact solutions. It is observed and noted that all previous terms of the DTM are embedded in the corresponding stage of the PIM. More accuracy is recorded as the number of terms in the iterations is increased. Results from both methods converge faster to their exact solutions. The DTM transforms the differential equations to algebraic-recursive equations; hence, it is very effective and reduces the size of computational work without linearization, perturbation or discretization of the given problem while the PIM transforms a differential equation to its equivalent in integral form provided the Lipschitz continuity condition is satisfied.

Conflict of Interests

The authors declare that there is no conflict of interests.

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