On Response of Elastic Isotropic Damped Shear Highway Bridge Supported by Sub-grade to Uniform Partially Distributed Moving Vehicle

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Abstract
Plates and Plate-like structures resting on elastic foundation, such as Winkler or Pasternak foundation, have wide application in modern Engineering practices. Areas of application include vibration of railway and highway bridges, as well as effects of moving loads on highway pavements. In this paper we attempt to analyse the effect of shear deformation, damping and Winkler foundation on the vibration of a typical highway bridge that is elastic and rectangular, as a result of moving vehicle on it. The moving vehicle (load) is assumed to be uniform partially distributed, while the Bridge (plate) is isotropic. We used MATLAB, in conjunction with a computer program developed, to solve the resulting set of algebraic system of equations obtained from the conversion of the coupled partial differential governing equations via finite difference method. The desired solutions are obtained. The results are consistent with the ones in literature. In particular, deflection is lower with shear deformation when compared with results in literature where the effect of shear deformation was neglected. Also, there is a significant effect of the damping and foundation on deflection of the bridge, as these two factors reduce the vibration of the bridge under a moving vehicle.

Keywords: Damped Shear Highway Bridge, Sub-grade, Uniform Partially distributed moving vehicle

Introduction
The solution of problem of carrying out an analysis of plate resting on elastic foundation, which is often encountered in the analysis of bridges and other geotechnical structures, demands modelling of the mechanical behaviour of the bridge[3,11,15]. Also the mechanical behaviour of the soil as elastic subgrade, and the form of interaction between the bridge and the soil are very important in the analysis[6,11]. Several foundation models have been reported in the literature [7,16] and investigations on the static deflection, the dynamic response and the dynamic stability of plates on elastic foundations have been carried out [1,15,16]. Many researchers use the Winkler model for soil structure interaction in the static and dynamic analysis of plates resting on elastic foundations where the vertical surface displacement of the plate is assumed to be proportional at any point to the contact pressure at that point [2,4,5,6]. In Winkler model, it is assumed that the foundation soil consists of linear elastic springs which are closely spaced and independent of each other [1]. The importance of moving load problem manifested in several fields of applied mathematics, Engineering, applied physics and transportation.[1,3,8,11,12] Bridges, guideways, overhead cranes, cableways, rails, roadways, runways, tunnels and pipelines are examples of structural elements.[1,3,6] They support moving masses. Various kinds of problems associated with moving loads, especially the medium on which they move, have attracted the attention of researchers since 1897. Recently, developments and results can be found in state-of-the-art reviews [2,4,6,14] The dynamic behaviour[5,7,] of an elastic shear highway bridge resting on a Winkler foundation and traversed by uniform partially distributed moving load is considered in this paper. The elastic bridge is the shear’s rectangular bridge. The effect of rotatory inertia is neglected. The study of such problem is of practical importance because of its applications in Engineering and other fields.[13,16] For example, this study is relevant when considering the reliability, safety and performance of modern highway bridges over which loads, like vehicles, move. In this analysis, both damping and the gravitational effects of the moving load are taken into account. The effect of shear deformation on the response of the damped bridge resting on a Winkler foundation under the action of partially distributed moving vehicle is investigated. The bridge is modelled as a plate while the moving vehicle is modelled as a normal moving load [1,9,10]. Numerical discussions of the deflections of the bridge on Winkler foundation of various stiffness are also presented.

Formulation of Problem
A highway bridge, modelled as a rectangular plate, with a moving small vehicle (moving load) and different boundary conditions is considered. The load is relatively small, so its inertia can be neglected, and is moving along the mid-space on the surface of the bridge, supported by a Winkler foundation [1,2,6].

Assumption:
The following assumptions are made[2,16]:
• The plate is of constant cross-section
• The moving load moves with a constant speed
• The moving load is guided in such a way that it keeps contact with the plate throughout the motion
• The plate is continuously supported by a Winkler foundation
• The moving load is a partially distributed moving load
• The rectangular shear plate is elastic.
Initial Conditions
\[ W(x, y, o) = \frac{\partial w}{\partial t} (x, y, 0) \]

Boundary Conditions.
\[ W(x, y, t) = M_X(x, y, t) = U \big|_Y (x, y, t) = 0, \quad \text{for} \quad x = 0 \quad \text{and} \quad x = a \]
\[ W(x, y, t) = M_Y(x, y, t) = U \big|_X (x, y, t) = 0, \quad \text{for} \quad y = 0 \quad \text{and} \quad y = b \]
Where \( M_X \) and \( M_Y \) are bending moments in the \( x \)- and \( y \)- directions respectively. \( U \big|_X (x, y, t) \) and \( U \big|_Y (x, y, t) \) are local rotations in the \( x \)- and \( y \)- directions respectively. \( W(x, y, t) \) is the traverse displacement of the plate at time \( t \).

Problem Solution
The set of dynamic equilibrium equations which govern the behaviour of Mindlin’s plate (with the effects both rotatory inertia and shear deformation) supported by Pasternak foundation and traversed by a partially distributed moving load may be written as [2,14,16]:

\[ Q_x - \frac{\partial M_{xy}}{\partial y} - \frac{\partial M_y}{\partial x} = \frac{\partial^2 \phi_x}{\partial t^2} + \frac{\partial \phi_x}{\partial t} + \frac{\partial^2 \phi}{\partial x \partial t} + \frac{\partial \phi}{\partial x} + \frac{\partial^2 \phi_y}{\partial x \partial y} + \frac{\partial \phi_y}{\partial x}, \quad \text{for} \quad y \in (0, h) \]

\[ Q_y - \frac{\partial M_{xy}}{\partial x} - \frac{\partial M_x}{\partial y} = \frac{\partial^2 \phi_y}{\partial t^2} + \frac{\partial \phi_y}{\partial t} + \frac{\partial^2 \phi}{\partial y \partial t} + \frac{\partial \phi}{\partial y} + \frac{\partial^2 \phi_x}{\partial y \partial x} + \frac{\partial \phi_x}{\partial y}, \quad \text{for} \quad x \in (0, b) \]

where \( \phi_x \) and \( \phi_y \) are local rotations in the \( x \)- and \( y \)- directions respectively. \( M_x \) and \( M_y \) bending moments in the \( x \) and \( y \)- directions respectively, \( M_{xy} \) is the twisting moments, \( Q_x \) and \( Q_y \) are the traversed Shearing forces in the \( x \)- and \( y \)- directions respectively, \( h \) and \( h_i \) are thickness of the plate and load respectively, \( s \) is the viscous damping coefficient, \( \rho \) and \( \rho_1 \) are the densities of the plate and the load per unit volume respectively. \( W(x, y, T) \) is the traverse displacement of the plate at time \( T \), \( P(x, y, T) \) is the applied dynamic load (force) and the last terms in equation (1) and (2) account for inertia effects of the load in \( x \)- and \( y \)- directions respectively. It is the velocity of a load \( (M_L) \) of rectangular dimensions \( E \) by \( U \) with one of its lines of symmetry moving along \( Y = Y_1 \).

The plate is \( L_X \) and \( L_Y \) in dimensions and \( \xi = \frac{U}{E} + \frac{E}{F} \) as shown in figure 1, also \( B = B_X B_Y \), where \( B_x = \)

\[ H(1 - H(x - \xi - \frac{\xi}{2})) \quad \text{for} \quad 0 < \frac{\xi}{2} \]

\[ H(1 + \frac{\xi}{2}) \quad \text{for} \quad \frac{\xi}{2} < T \quad \text{or} \quad (L_x + \epsilon)/U \]

\[ B_y = H \left( y - y_1 + \frac{\xi}{2} \right) - H \left( y - y_1 - \frac{\xi}{2} \right) \]

\( H(x) \) is the Heaviside function defined as

\[ H(x) = \begin{cases} 
1 & \text{for} \quad x > 0 \\
0 & \text{for} \quad x < 0 
\end{cases} \]

Where \( G_i \) is the modulus of rigidity of the plate, \( D \) is the flexural rigidity of the plate defined by

\[ D = \frac{1}{12} Eh^3 \left(1 - \nu^2 \right) \]
G_{1}h^{1/6}(1-\nu) is the Shear correction factor and \nu is the Poisson’s ratio of the plate. Also, in equation (1), (2) and (3);

\[
D_{i} = \frac{\partial w}{\partial t}
\]

The set of partial differential equations (7) – (15), are the partial differential equations to be solved for the following eleven dependent variables: Qx, Qy, Mx, My, Mxy, \psi_{x}, \psi_{y}, D_{i} and W.

The Non – Dimensional Form

The following variables: Qx, Qy, Mx, My, Mxy, \psi_{x}, \psi_{y}, D_{i} and W, in the above equations are the unknowns. These variables and others can be written in a non – dimensional form as follows:

\[
q_{x} = \frac{q_{x}}{\alpha G h}
\]

(16)

\[
q_{y} = \frac{q_{y}}{\alpha G h}
\]

(17)

\[
m_{x} = \frac{m_{x}}{\alpha G h \ell_{x}}
\]

(18)

\[
m_{y} = \frac{m_{y}}{\alpha G h \ell_{x}}
\]

(19)

\[
m_{xy} = \frac{m_{xy}}{\alpha G h \ell_{x}}
\]

(20)

\[
W = \frac{w}{h}
\]

(21)

\[
dt = \frac{L_{x} \partial t}{\chi c}
\]

(22)

\[
\varphi_{xt} = \frac{\ell_{x} \varphi_{xt}}{\chi c}
\]

(23)

\[
\varphi_{yt} = \frac{\ell_{y} \varphi_{yt}}{\chi c}
\]

(24)

For the parameters: a = \frac{\chi L_{x}}{\nu c \ell_{x}} m = \frac{M_{x}}{\alpha G h \ell_{x}} g_{n} = \frac{L_{x}}{\nu c v} x = \frac{\chi L_{x}}{\nu c \ell_{x}} y = \frac{\chi L_{x}}{\nu c \ell_{x}} t = \frac{T}{\ell_{x} \ell_{x}} s = \frac{\ell_{x}}{\nu c}

The dimensionless form of equations (7) – (15) using equations (16) – (24) are:

\[
\alpha G h q_{x} - \alpha G h \frac{\partial m_{x}}{\partial x} - \alpha G h \frac{\partial m_{xy}}{\partial y} = \frac{\rho L_{x}^{2} h^{3} \varphi_{xt}}{a} + \frac{\partial^{2} c^{2} u \partial x^{2}}{L_{x}^{2} \partial t} + \frac{y c u}{d(y^{2} - 1)} \left\{ a G h y^{2} c \frac{\partial m_{x}}{\partial t} + a G h y^{2} c u a c u h \frac{\partial m_{y}}{\partial x} \right\} B_{n} + \frac{y c u}{d(y^{2} - 1)} \left\{ a G h y^{2} c \frac{\partial m_{y}}{\partial t} + a G h y^{2} c u a c u h \frac{\partial m_{x}}{\partial x} \right\} \frac{B_{n}}{\alpha G h}
\]

(25)
H(x) is a defined by

\[
H(x) = \begin{cases} 
1 & x > 0 \\
0.5 & x = 0 \\
0 & x < 0 
\end{cases}
\]

Writing equations (25) – (33) in a more concise form, we have [7]

\[
\frac{\partial E}{\partial t} = \frac{\partial E}{\partial x} (49)
\]

\[
0 \leq x < 0
\]

Where

\[
N_1 = \frac{D}{\rho h^2 c}
\]

\[
N_2 = \frac{\rho c}{y_c}
\]

\[
N_3 = \frac{1}{(v^2 - 1)}
\]

\[
R = \frac{\rho^2 h^4}{\rho h^3}
\]

Equations (38) – (46) are the non-dimensional first order governing differential equations to be solved for the following nine dependent variables

\[
\partial_x, \partial_y, m_x, m_y, m_{xy}, w, dt, \varphi_x \text{ and } \varphi_y
\]

The non-dimensional boundary conditions are:

\[
dt = m_x = \varphi_y = 0 \text{ (at } x = 0 \text{ and } x = 1) \]

\[
dt = m_y = \varphi_x = 0 \text{ (at } y = 0 \text{ and } y = b/2) \]

**Finite Difference Algorithm for the Shear plate**

Equations (38) - (46) is solved using a numerical method based on the finite difference algorithm. These equations are to be transferred into their equivalent algebraic form. The finite difference definition of first order partial derivative of a function E(x,y,t) with respect to x,y and t respectively are as follows:[5,7]

\[
\frac{\partial E}{\partial x} = N_1 \left( \frac{\partial E}{\partial x} + \frac{\partial E}{\partial y} \right)
\]

\[
\frac{\partial E}{\partial y} = N_1 \left( \frac{\partial E}{\partial x} + \frac{\partial E}{\partial y} \right)
\]

\[
\frac{\partial E}{\partial t} = N_1 \left( \frac{\partial E}{\partial x} + \frac{\partial E}{\partial y} \right)
\]

Where E is the function value of the centre of a grid, which is well approximated by the average of its values at the grid nodes [5]

\[
E \left( x + \frac{h}{2}, y + \frac{k}{2}, t + \frac{r}{2} \right) = \frac{1}{8} \left( E_{i,j} + E_{i+1,j} + E_{i,k} + E_{i+1,k} + E_{i+1,j+1} + E_{i+1,j-1} + E_{i,k} + E_{i+1,k} \right)
\]

Hence,
\[
\begin{align*}
\frac{\partial q_y}{\partial x} &= \frac{1}{4h} \left( q_{x1_i,j}^{k+1} + q_{x1_i,j}^{k} - q_{x1_i,j}^{k-1} - q_{x1_i,j}^{k+2} + q_{x1_i,j}^{k+3} - q_{x1_i,j}^{k+4} \right) \\
\frac{\partial q_y}{\partial y} &= \frac{1}{4k} \left( d_{x1_i,j}^{k+1} + d_{x1_i,j}^{k} - d_{x1_i,j}^{k-1} - d_{x1_i,j}^{k+2} + d_{x1_i,j}^{k+3} - d_{x1_i,j}^{k+4} \right) \\
\frac{\partial t}{\partial x} &= \frac{1}{4r} \left( d_{x1_i,j}^{k+1} + d_{x1_i,j}^{k} - d_{x1_i,j}^{k-1} - d_{x1_i,j}^{k+2} + d_{x1_i,j}^{k+3} - d_{x1_i,j}^{k+4} \right) \\
\frac{\partial m_y}{\partial x} &= \frac{1}{4h} \left( m_{y1_i,j}^{k+1} + m_{y1_i,j}^{k} - m_{y1_i,j}^{k-1} - m_{y1_i,j}^{k+2} + m_{y1_i,j}^{k+3} - m_{y1_i,j}^{k+4} \right) \\
\frac{\partial m_y}{\partial y} &= \frac{1}{4k} \left( m_{y1_i,j}^{k+1} + m_{y1_i,j}^{k} - m_{y1_i,j}^{k-1} - m_{y1_i,j}^{k+2} + m_{y1_i,j}^{k+3} - m_{y1_i,j}^{k+4} \right) \\
\frac{\partial q_y}{\partial t} &= \frac{1}{4r} \left( q_{y1_i,j}^{k+1} + q_{y1_i,j}^{k} - q_{y1_i,j}^{k-1} - q_{y1_i,j}^{k+2} + q_{y1_i,j}^{k+3} - q_{y1_i,j}^{k+4} \right) \\
\frac{\partial m_y}{\partial t} &= \frac{1}{4k} \left( m_{y1_i,j}^{k+1} + m_{y1_i,j}^{k} - m_{y1_i,j}^{k-1} - m_{y1_i,j}^{k+2} + m_{y1_i,j}^{k+3} - m_{y1_i,j}^{k+4} \right) \\
\frac{\partial q_y}{\partial x} &= \frac{1}{4h} \left( q_{x1_i,j}^{k+1} + q_{x1_i,j}^{k} - q_{x1_i,j}^{k-1} - q_{x1_i,j}^{k+2} + q_{x1_i,j}^{k+3} - q_{x1_i,j}^{k+4} \right) \\
\frac{\partial m_y}{\partial y} &= \frac{1}{4k} \left( m_{y1_i,j}^{k+1} + m_{y1_i,j}^{k} - m_{y1_i,j}^{k-1} - m_{y1_i,j}^{k+2} + m_{y1_i,j}^{k+3} - m_{y1_i,j}^{k+4} \right)
\end{align*}
\]
\[ \frac{\partial \phi_y}{\partial x} = \frac{1}{4k} \left( \phi_{u_{i+1,j}}^{k+1} + \phi_{u_{i,j}}^{k+1} - \phi_{u_{i-1,j}}^{k+1} - \phi_{u_{i,j}}^k + \phi_{u_{i+1,j}}^k - \phi_{u_{i,j}}^k \right) \]  

(74)

\[ \frac{\partial \phi_y}{\partial y} = \frac{1}{4h} \left( \phi_{v_{i,j+1}}^{k+1} + \phi_{v_{i,j}}^{k+1} - \phi_{v_{i,j-1}}^{k+1} - \phi_{v_{i,j}}^k + \phi_{v_{i,j+1}}^k - \phi_{v_{i,j}}^k \right) \]  

(75)

\[ \frac{\partial W}{\partial t} = \frac{1}{4r} \left( w_{i+1,j}^{k+1} + w_{i,j}^{k+1} + w_{i-1,j}^{k+1} - w_{i,j}^k - w_{i+1,j}^k - w_{i,j}^k \right) \]  

(76)

\[ \phi_{zt} = \frac{1}{8} \left( \phi_{u_{i+1,j}}^{k+1} + \phi_{u_{i,j}}^{k+1} + \phi_{u_{i-1,j}}^{k+1} + \phi_{v_{i,j}}^k + \phi_{v_{i+1,j}}^k + \phi_{v_{i,j}}^k \right) \]  

(77)

\[ \phi_{yt} = \frac{1}{8} \left( \phi_{v_{i+1,j}}^{k+1} + \phi_{v_{i,j}}^{k+1} + \phi_{v_{i-1,j}}^{k+1} + \phi_{v_{i,j}}^k + \phi_{v_{i+1,j}}^k + \phi_{v_{i,j}}^k \right) \]  

(78)

\[ w = \frac{1}{8} \left( w_{i+1,j}^{k+1} + w_{i,j}^{k+1} + w_{i-1,j}^{k+1} + w_{i,j}^k + w_{i+1,j}^k + w_{i,j}^k \right) \]  

(79)

\[ m_z = \frac{1}{8} \left( m_{z,i,j}^{k+1} + m_{z,i,j}^{k+1} + m_{z,i,j}^{k+1} + m_{z,i,j}^k + m_{z,i,j}^k + m_{z,i,j}^k + m_{z,i,j}^k \right) \]  

(80)

\[ m_y = \frac{1}{8} \left( m_{y,i,j}^{k+1} + m_{y,i,j}^{k+1} + m_{y,i,j}^{k+1} + m_{y,i,j}^k + m_{y,i,j}^k + m_{y,i,j}^k + m_{y,i,j}^k \right) \]  

(81)

\[ q_x = \frac{1}{8} \left( q_{x,i,j}^{k+1} + q_{x,i,j}^{k+1} + q_{x,i,j}^{k+1} + q_{x,i,j}^k + q_{x,i,j}^k + q_{x,i,j}^k + q_{x,i,j}^k \right) \]  

(82)

3.3 The algebraic form of the non-dimensional first order governing differential equations

The substitution of the definition of the above finite difference scheme into equations (38) – (46) and multiplying both sides of the resulting equations by 4h, setting \( h/r = u \) and \( h/k = H \), yields respectively[1,2]: 
\[
\frac{N_i N_j}{\rho y c} \left( q_i^{k+1} + q_j^{k+1} - q_{ij}^{k+1} - q_{ji}^{k+1} \right)
\]

\[
+ \frac{N_i N_j}{\rho y c} H \left( q_{ij}^{k+1} + q_{ji}^{k+1} - q_{ij}^{k+1} - q_{ji}^{k+1} \right)
\]

\[
+ \frac{N_i K L_i h}{\rho y^2 c^2} \left( w_i^{k+1} + w_{ij}^{k+1} - w_{ij}^{k+1} - w_{ij}^{k+1} \right)
\]

\[
+ \left( \frac{N_i}{L_x} + \frac{\gamma N_i m}{L_x a} B_n + \frac{N_i m y c u}{L_x a} B_n \right) u \left( d_{ij}^{k+1} + d_{ji}^{k+1} + d_{ij}^{k+1} + d_{ji}^{k+1} \right)
\]

\[
+ \frac{N_i u^2 m h}{L_x a} \left( \varphi_{ij}^{k+1} + \varphi_{ji}^{k+1} + \varphi_{ij}^{k+1} + \varphi_{ji}^{k+1} \right)
\]

\[
- \frac{N_i u^2 m h}{L_x^2 a} \left( m_{ij}^{k+1} + m_{ij}^{k+1} + m_{ij}^{k+1} + m_{ij}^{k+1} \right)
\]

\[
- \frac{N_i u^2 m h}{L_x a} \left( q_{ij}^{k+1} + q_{ij}^{k+1} + q_{ij}^{k+1} + q_{ij}^{k+1} \right)
\]

\[
+ \frac{N_i u^2 m h}{L_x a} \left( m_{ij}^{k+1} + m_{ij}^{k+1} + m_{ij}^{k+1} + m_{ij}^{k+1} \right)
\]

\[
- \frac{N_i u^2 m h}{L_x a} \left( q_{ij}^{k+1} + q_{ij}^{k+1} + q_{ij}^{k+1} + q_{ij}^{k+1} \right)
\]

\[
- \frac{N_i u^2 m h}{L_x a} \left( q_{ij}^{k+1} + q_{ij}^{k+1} + q_{ij}^{k+1} + q_{ij}^{k+1} \right)
\]

\[
- \frac{N_i u^2 m h}{L_x a} \left( q_{ij}^{k+1} - q_{ij}^{k+1} - q_{ij}^{k+1} \right) \right) B_n
\]

\[
= - \frac{N_i N_j}{\rho y c} \left( q_i^{k+1} + q_j^{k+1} - q_{ij}^{k+1} - q_{ji}^{k+1} \right)
\]

\[
+ \frac{N_i N_j}{\rho y c} H \left( q_{ij}^{k+1} + q_{ji}^{k+1} - q_{ij}^{k+1} - q_{ji}^{k+1} \right)
\]

\[
+ \frac{N_i K L_i h}{\rho y^2 c^2} \left( w_i^{k+1} + w_{ij}^{k+1} + w_{ij}^{k+1} + w_{ij}^{k+1} \right)
\]

\[
- \left( \frac{N_i}{L_x} + \frac{\gamma N_i m}{L_x a} B_n + \frac{N_i m y c u}{L_x a} B_n \right) u \left( -d_{ij}^{k+1} - d_{ji}^{k+1} - d_{ij}^{k+1} - d_{ji}^{k+1} \right)
\]

\[
- 4 h^2 m N_i g B_n \frac{N_i u^2 m h}{L_x^2 a} \left( \varphi_{ij}^{k+1} + \varphi_{ij}^{k+1} + \varphi_{ij}^{k+1} + \varphi_{ij}^{k+1} \right)
\]

\[
+ \frac{N_i u^2 m h}{L_x a} \left( m_{ij}^{k+1} + m_{ij}^{k+1} + m_{ij}^{k+1} + m_{ij}^{k+1} \right)
\]

\[
- \frac{N_i u^2 m h}{L_x a} \left( q_{ij}^{k+1} + q_{ij}^{k+1} + q_{ij}^{k+1} + q_{ij}^{k+1} \right)
\]

\[
- \frac{N_i u^2 m h}{L_x a} \left( m_{ij}^{k+1} + m_{ij}^{k+1} + m_{ij}^{k+1} + m_{ij}^{k+1} \right)
\]

\[
- \frac{N_i u^2 m h}{L_x a} \left( q_{ij}^{k+1} + q_{ij}^{k+1} + q_{ij}^{k+1} + q_{ij}^{k+1} \right)
\]

\[
- \frac{N_i u^2 m h}{L_x a} \left( q_{ij}^{k+1} + q_{ij}^{k+1} + q_{ij}^{k+1} \right) \right) B_n
\]

(83)
\[
\frac{N_i N_j h}{\rho c} \left( q_{i,j}^{k+1} + q_{i,j}^{k+1} + q_{i,j}^{k+1} + q_{i,j}^{k+1} \right)
\]
\[
+ \frac{N_i N_j}{\rho c} \left( m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} - m_{i,j}^{k+1} \right)
\]
\[
- \frac{N_i N_j}{\rho c} H \left( m_{x_{i,j}}^{k+1} + m_{x_{i,j}}^{k+1} - m_{x_{i,j}}^{k+1} - m_{x_{i,j}}^{k+1} \right)
\]
\[
+ \frac{\rho h^2}{12} \left( \frac{N_i}{L_i} \phi_{i,j}^{k+1} + \phi_{i,j}^{k+1} + \phi_{i,j}^{k+1} + \phi_{i,j}^{k+1} \right)
\]
\[
+ \frac{N_i c u^2}{L_i} \left( \phi_{i,j}^{k+1} + \phi_{i,j}^{k+1} - \phi_{i,j}^{k+1} - \phi_{i,j}^{k+1} \right)
\]
\[
+ \frac{N_i u^2}{L_i} \left( m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} \right)
\]
\[
+ \frac{N_i u^2}{L_i} \left( m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} \right)
\]
\[
+ \frac{N_i u^2}{L_i} \left( m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} \right)
\]
\[
+ \frac{N_i u^2}{L_i} \left( m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} \right)
\]
\[
+ \frac{N_i u^2}{L_i} \left( m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} \right)
\]
\[
- \frac{N_i u^2}{L_i} \left( m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} \right) \right] B_n
\]
\[
= - \frac{N_i N_j h}{\rho c} \left( q_{i,j}^{k+1} + q_{i,j}^{k+1} + q_{i,j}^{k+1} + q_{i,j}^{k+1} \right)
\]
\[
+ \frac{N_i N_j}{\rho c} \left( m_{i,j}^{k+1} + m_{i,j}^{k+1} + m_{i,j}^{k+1} - m_{i,j}^{k+1} \right)
\]
\[
+ \frac{N_i N_j}{\rho c} H \left( m_{x_{i,j}}^{k+1} + m_{x_{i,j}}^{k+1} + m_{x_{i,j}}^{k+1} + m_{x_{i,j}}^{k+1} \right)
\]
\[
+ \frac{\rho h^2}{12} \left( \frac{N_i}{L_i} \phi_{i,j}^{k+1} - \phi_{i,j}^{k+1} - \phi_{i,j}^{k+1} - \phi_{i,j}^{k+1} \right)
\]
\[
+ \frac{N_i c u^2}{L_i} \left( \phi_{i,j}^{k+1} - \phi_{i,j}^{k+1} - \phi_{i,j}^{k+1} - \phi_{i,j}^{k+1} \right)
\]
\[
+ \frac{N_i u^2}{L_i} \left( m_{i,j}^{k} - m_{i,j}^{k} - m_{i,j}^{k} - m_{i,j}^{k} \right)
\]
\[
+ \frac{N_i u^2}{L_i} \left( m_{i,j}^{k} - m_{i,j}^{k} - m_{i,j}^{k} - m_{i,j}^{k} \right)
\]
\[
+ \frac{N_i u^2}{L_i} \left( m_{i,j}^{k} - m_{i,j}^{k} - m_{i,j}^{k} - m_{i,j}^{k} \right)
\]
\[
+ \frac{N_i u^2}{L_i} \left( m_{i,j}^{k} - m_{i,j}^{k} - m_{i,j}^{k} - m_{i,j}^{k} \right)
\]
\[
- \frac{N_i u^2}{L_i} \left( m_{i,j}^{k} - m_{i,j}^{k} - m_{i,j}^{k} - m_{i,j}^{k} \right) \right] B_n
\]

(84)
\[
- \frac{N_1 N_2}{\rho \gamma c} \frac{h}{2} \left( q_{x_{1,j},i}^{k+1} + q_{x_{1,j},i}^{k} + q_{x_{1,j},i}^{k+1} + q_{x_{1,j},i}^{k+1} \right) \\
+ \frac{N_1 N_2}{\rho c} \left( m_{y_{1,j},i}^{k+1} + m_{y_{1,j},i}^{k} + m_{y_{1,j},i}^{k+1} + m_{y_{1,j},i}^{k+1} - m_{y_{1,j},i}^{k} \right) \\
+ \frac{N_1 N_2}{\rho c} H \left( m_{y_{1,j},i}^{k+1} + m_{y_{1,j},i}^{k} + m_{y_{1,j},i}^{k+1} - m_{y_{1,j},i}^{k+1} \right) \\
+ \frac{\rho_1 h^3}{12L} \frac{N_1 u}{L^2 \rho h} \left( \phi_{y_{1,j},i}^{k+1} + \phi_{y_{1,j},i}^{k} + \phi_{y_{1,j},i}^{k+1} + \phi_{y_{1,j},i}^{k+1} \right) \\
+ \frac{N_1 u}{L^2 \rho h} \left( \phi_{y_{1,j},i}^{k+1} + \phi_{y_{1,j},i}^{k} - \phi_{y_{1,j},i}^{k+1} - \phi_{y_{1,j},i}^{k+1} \right) \\
+ \frac{N_1 u^2}{L^2 \rho h} \left( m_{y_{1,j},i}^{k+1} + m_{y_{1,j},i}^{k} + m_{y_{1,j},i}^{k+1} + m_{y_{1,j},i}^{k+1} \right) \\
+ \frac{N_1 u^2}{L^2 \rho h} H \left( m_{y_{1,j},i}^{k+1} + m_{y_{1,j},i}^{k} + m_{y_{1,j},i}^{k+1} - m_{y_{1,j},i}^{k+1} \right) \\
- \frac{N_1 u^2}{L^2 \rho h} \left( m_{y_{1,j},i}^{k+1} + m_{y_{1,j},i}^{k} + m_{y_{1,j},i}^{k+1} - m_{y_{1,j},i}^{k+1} \right) \right]\nB_n \\
= \frac{N_1 N_2}{\rho \gamma c} \frac{h}{2} \left( q_{x_{1,j},i}^{k} + q_{x_{1,j},i}^{k} + q_{x_{1,j},i}^{k} + q_{x_{1,j},i}^{k} \right) \\
+ \frac{N_1 N_2}{\rho c} \left( m_{y_{1,j},i}^{k} + m_{y_{1,j},i}^{k} - m_{y_{1,j},i}^{k} - m_{y_{1,j},i}^{k} \right) \\
+ \frac{N_1 N_2}{\rho c} H \left( m_{y_{1,j},i}^{k} + m_{y_{1,j},i}^{k} - m_{y_{1,j},i}^{k} - m_{y_{1,j},i}^{k} \right) \\
+ \frac{\rho_1 h^3}{12L} \frac{N_1 u}{L^2 \rho h} \left( \phi_{y_{1,j},i}^{k} + \phi_{y_{1,j},i}^{k} - \phi_{y_{1,j},i}^{k} - \phi_{y_{1,j},i}^{k} \right) \\
+ \frac{N_1 u}{L^2 \rho h} \left( \phi_{y_{1,j},i}^{k} + \phi_{y_{1,j},i}^{k} - \phi_{y_{1,j},i}^{k} - \phi_{y_{1,j},i}^{k} \right) \\
+ \frac{N_1 u^2}{L^2 \rho h} \left( -m_{y_{1,j},i}^{k} - m_{y_{1,j},i}^{k} - m_{y_{1,j},i}^{k} - m_{y_{1,j},i}^{k} \right) \\
+ \frac{N_1 u^2}{L^2 \rho h} H \left( m_{y_{1,j},i}^{k} + m_{y_{1,j},i}^{k} - m_{y_{1,j},i}^{k} - m_{y_{1,j},i}^{k} \right) \\
- \frac{N_1 u^2}{L^2 \rho h} \left( -m_{y_{1,j},i}^{k} - m_{y_{1,j},i}^{k} - m_{y_{1,j},i}^{k} - m_{y_{1,j},i}^{k} \right) \right]\nB_n \tag{85}
\]
\[ u \left( m_{x_{i,j}}^{k+1} + m_{y_{i,j}}^{k+1} + m_{z_{i,j}}^{k+1} + m_{h_{i,j}}^{k+1} \right) \]
\[ + N_i \left( \phi_{x_{i,j}}^{k+1} + \phi_{y_{i,j}}^{k+1} - \phi_{x_{i,j}}^{k+1} - \phi_{y_{i,j}}^{k+1} \right) \]
\[ + vHN_i \left( \phi_{y_{i,j}}^{k+1} + \phi_{y_{i,j}}^{k+1} + \phi_{y_{i,j}}^{k+1} + \phi_{y_{i,j}}^{k+1} \right) \]
\[ = + u \left( m_{x_{i,j}}^{k} + m_{y_{i,j}}^{k} + m_{h_{i,j}}^{k} + m_{h_{i,j}}^{k} \right) \]
\[ + N_i \left( -\phi_{x_{i,j}}^{k} - \phi_{y_{i,j}}^{k} + \phi_{y_{i,j}}^{k} + \phi_{y_{i,j}}^{k} \right) \]
\[ + vHN_i \left( -\phi_{y_{i,j}}^{k} - \phi_{y_{i,j}}^{k} + \phi_{y_{i,j}}^{k} + \phi_{y_{i,j}}^{k} \right) \] (86)

\[ u \left( m_{y_{i,j}}^{k+1} + m_{y_{i,j}}^{k+1} + m_{y_{i,j}}^{k+1} + m_{y_{i,j}}^{k+1} \right) \]
\[ + HN_i \left( \phi_{y_{i,j}}^{k+1} + \phi_{y_{i,j}}^{k+1} - \phi_{x_{i,j}}^{k+1} - \phi_{x_{i,j}}^{k+1} \right) \]
\[ + vN_i \left( \phi_{y_{i,j}}^{k+1} + \phi_{y_{i,j}}^{k+1} - \phi_{x_{i,j}}^{k+1} - \phi_{x_{i,j}}^{k+1} \right) \]
\[ = + u \left( m_{x_{i,j}}^{k} + m_{y_{i,j}}^{k} + m_{h_{i,j}}^{k} + m_{h_{i,j}}^{k} \right) \]
\[ + HN_i \left( -\phi_{y_{i,j}}^{k} - \phi_{y_{i,j}}^{k} + \phi_{y_{i,j}}^{k} + \phi_{y_{i,j}}^{k} \right) \]
\[ + vN_i \left( -\phi_{y_{i,j}}^{k} - \phi_{y_{i,j}}^{k} + \phi_{y_{i,j}}^{k} + \phi_{y_{i,j}}^{k} \right) \] (87)

\[ u \left( m_{y_{i,j}}^{k+1} + m_{y_{i,j}}^{k+1} - m_{y_{i,j}}^{k+1} + m_{y_{i,j}}^{k+1} \right) \]
\[ + N_i \left( \frac{1-v}{2} \left( \phi_{y_{i,j}}^{k+1} - \phi_{y_{i,j}}^{k+1} - \phi_{y_{i,j}}^{k+1} \right) \right) \]
\[ - N_i H \left( \frac{1-v}{2} \left( \phi_{y_{i,j}}^{k+1} - \phi_{y_{i,j}}^{k+1} - \phi_{y_{i,j}}^{k+1} \right) \right) \]
\[ = u \left( m_{x_{i,j}}^{k} + m_{y_{i,j}}^{k} + m_{h_{i,j}}^{k} + m_{h_{i,j}}^{k} \right) \]
\[ + N_i \left( \frac{1-v}{2} \left( \phi_{y_{i,j}}^{k} - \phi_{y_{i,j}}^{k} + \phi_{y_{i,j}}^{k} \right) \right) \]
\[ + N_i H \left( \frac{1-v}{2} \left( \phi_{y_{i,j}}^{k} - \phi_{y_{i,j}}^{k} + \phi_{y_{i,j}}^{k} \right) \right) \] (88)
These set of algebraic equations to be solved may now be written in matrix form as follows [7].

\[
H_{i,j+1}S'_{i,j+1} + I_{i+1,j+i}S'_{i+1,j+1} = -G_{i,j}S'_{i,j} - J_{i+1,j}S'_{i+1,j} + K_{i+1,j}S'_{i+1,j+1} + L_{i+1,j+1}S'_{i+1,j+1} + P_i
\]  

(99)
i = 1, 2, 3 ..., N-1; j = 1, 2, 3, ... M-1

where N and M are the numbers of the nodal points along X and Y axes respectively.

Each term in equations (99) is a 9x9 matrix. Which is in the form:
\[
H_{i,j+1}S_{i,j+1} = \begin{pmatrix}
\nu N_1 & -vHN_1 & 0 & 0 \\
nu N_1 & -HN_1 & 0 & 0 \\
nu N_1 & -N_1e_1H & -N_1e_1 & 0 \\
0 & 0 & u & 0 \\
0 & 0 & 0 & u \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e_2 \\
(-e_2y) \\
0 \\
A \\
0 \\
B \\
e_\gamma \ \\
u \\
+e_8 \\
+e_8 cu \\
\end{pmatrix}
\begin{pmatrix}
m_{x_{i+1}} \\
m_{y_{i+1}} \\
m_{x_{i+1}} \\
q_{x_{i+1}} \\
q_{y_{i+1}} \\
\varphi_{x_{i+1}} \\
\varphi_{y_{i+1}} \\
w_{i+1} \\
d_{i+1}
\end{pmatrix}
\begin{pmatrix}
\gamma e_6 \\
-e_6u \\
+e_9 \\
e_6L_\gamma uu \\
+e_6vuL_\gamma H
\end{pmatrix}
\begin{pmatrix}
h \\
N_1L_x \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\begin{pmatrix}
e_6 \gamma \\
0 \\
(-e_6 \gamma) \\
0 \\
0 \\
0 \\
\end{pmatrix}
\begin{pmatrix}
e_6h \gamma \\
(-e_11) \\
(-e_10u - e_11) \\
0 \\
0 \\
\end{pmatrix}
\]

Where,
\[
A = \begin{pmatrix}
-e_6 \\
-e_6u \\
+e_9 \\
\end{pmatrix}
\]
\[
B = \begin{pmatrix}
e_6h^2 \gamma \\
2 \\
L_\gamma \\
\end{pmatrix}
\]
\[
C = \begin{pmatrix}
-e_10u \\
-e_11 \\
-\gamma e_8 H \\
\end{pmatrix}
\]
\[
D = \begin{pmatrix}
-e_6L_\gamma uu \\
-\gamma e_8 H \\
-e_6L_\gamma u \\
-e_6L_\gamma uH \\
\end{pmatrix}
\]

and
\[ e_1 = \frac{1 - v}{2} \]
\[ e_2 = \frac{m h u^2 N_a}{2aL^2} B_n \]
\[ e_4 = \frac{m h N}{2aL} B_n \]
\[ e_6 = \frac{N_i N_s}{\rho v c} \]
\[ e_8 = \frac{\gamma N_m B}{L a} \]
\[ e_9 = B_n \frac{N_i h u^3 \rho_L}{12 \rho h L^2} \]
\[ e_{10} = \frac{N_i \gamma^2}{12} \]
\[ e_{11} = B_n \frac{N_i h u^3 \rho_L}{12 L^2 \rho h} \]

**Result Discussion**

The numerical calculations were carried out for a simply supported rectangular isotropic damped highway bridge (plate) resting on a Winkler foundation and subjected to a moving railway vehicle (load.). Shear deformation effect was considered, while the effect of rotatory inertia was neglected. For a specific value of the parameters, deflection of the highway bridge was calculated and plotted as a function of time. The following results were obtained: The response amplitude of the highway bridge resting on a Winkler foundation decreases with an increase in the value of \( K \), the foundation’s constant, for various time \( t \), but fixed values of both the velocity and contact area of the moving partially distributed vehicle. (as we can see in figures 3.). The response maximum amplitude decreases with an increase in the value of damping coefficient, for fixed value of \( K \), \( A_r \) and \( U \). (as we can see in figure 5). The response maximum amplitude of the shear highway bridge supported continuously by an elastic Winkler subgrade decreases with an increase in contact area \( (A_r) \). (as we can see in figure 2). Also for the same time range and fixed values of \( K \) and \( A_r \), it is observed that the shear highway bridge has a larger value of the maximum amplitude than non- Mindlin, for all the values of velocity considered. (as we can see in figure 1).

**Figure 1:** Deflection of the Shear and non Mindlin highway bridges for fixed \( K \), and various values of time

**Figure 2:** Deflection of the highway bridge for various values of contact area \( (A_r) \) and time
Conclusion

The structure of interest was a railway bridge modelled as an isotropic shear rectangular elastic highway bridge, on Winkler foundation, under the influence of a uniform partially distributed moving vehicle. The problem was to determine the dynamic response of the whole system. Finite Difference technique was adopted in solving the resulting first order coupled partial differential equations obtained from the governing equations, for the simply supported highway bridge. The effects of the foundation, damping and shear deformation on the dynamic response of the isotropic highway bridge to the moving vehicle, give more realistic results for practical application.

References


