Analytical Solutions of the Navier-Stokes Model by He's Polynomials

S. O. Edeki, *Member, IAENG*, G. O. Akinlabi, M.E. Adeosun

***Abstract*— Navier-Stokes models are of great usefulness in physics and applied sciences. In this paper, He's polynomials approach is implemented for obtaining approximate and exact solutions of the Navier-Stokes model. These solutions are calculated in the form of series with easily computable components. This technique is showed to be very effective, efficient and reliable because it gives the exact solution of the solved problems with less computational work, without neglecting the level of accuracy. We therefore, recommend the extension and application of this novel method for solving problems arising in other aspect of applied sciences. Numerical computations, and graphics done in this work, are through *Maple* 18.**

***Index Terms*— Analytical solutions; He's polynomials; Navier-Stokes model.**
***MSC*: 83C15, 65H20, 35Q30.**

I. Introduction

Navier-Stokes equations are basic models in physics used in describing the motion of viscous fluid substances. These models are very useful as they describe the physics of many phenomena relating mathematics, engineering, pure and applied sciences. In computational fluid dynamics, Navier-Stokes equations are the main equations, relating pressure and external forces acting on fluid to the response of the fluid flow [1].

In general form, the Navier-Stokes and continuity equations are given by:

$$\frac{\partial u}{\partial t}+\left(\underline{u}\cdot\nabla\right)\underline{u}=-\frac{1}{\rho}\nabla P+v\nabla^{2}\underline{u} \qquad (1.1)$$

$$\nabla\cdot\underline{u}=0 \qquad (1.2)$$

where u is the flow velocity, $\underline{u}$ is the velocity, v is the kinematics viscosity, P is the pressure, t is the time, ρ is the density, and ∇ is a del operator.

In considering unsteady, one dimensional motion of a viscous fluid in a tube; the equations of motion governing the flow field in the tube are Navier-Stokes equations in cylindrical coordinates [1]. These are denoted by:

$$\frac{\partial u}{\partial t}=P+v\left(\frac{\partial^{2}u}{\partial r^{2}}+\frac{1}{r}\frac{\partial u}{\partial r}\right) \qquad (1.3)$$

subject to:

$$u\left(r,0\right)=g\left(r\right) \qquad (1.4)$$

where $P=-\frac{\partial P}{\rho\partial z}$.

In a bid to providing numerical and/or exact solutions to linear and nonlinear differential equations; many researchers have considered and developed a lot of semi-analytical methods. These include Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Differential Transform Method (DTM), Laplace Transform Method (LTM), and their modified forms [3-7].

Recently, He [8] developed the Homotopy Perturbation Method (HPM) for solving differential equations. Basically, the merit of the HPM is to overcome the difficulties involved in calculating the nonlinear terms in the concerned problem. The HPM has wider applications when handling different classes of differential equations, integral equations, integro-differential equations and so on [9-15]. As a modification of the HPM, Ghorbani *et al.* [16,17] introduced the He's polynomials where nonlinear terms were split into series of polynomials which are calculated from HPM. It is remarked that He's polynomials are attuned with Adomian's polynomials, yet it is showed that the He's polynomials are easier to compute, and are very much user friendly.

In considering the solutions of the Navier-Stokes equation (of integer, and non-integer order), some of the semi-analytical methods have been applied [1-3,18-21].

In this work, it is therefore, our intention to provide analytical solutions to the Navier–Stokes model of the forms in (1.1)-(1.3) using the He's polynomials method. It is worth mentioning that the He's polynomials method is an alternative semi-analytical method, even without giving up accuracy.

II. The Overview of the He's Polynomial Method [16, 17, 22]

Suppose $\Im$ is an integral or a differential operator, then we consider the equation of the form:

$$\Im\left(\varpi\right)=0 \qquad (2.1)$$

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S.O. Edeki is with the Department of Mathematics, Covenant University, Canaanland, Otta, Nigeria: soedeki@yahoo.com (email of the c*orresponding author*)
G.O. Akinlabii is with the Department of Mathematics, Covenant University, Canaanland, Otta, Nigeria.
M.E. Adeosun is with the Department of Mathematics & Statistics, Osun State College of Technology,Esa-Oke, Nigeria

Let $H(\varpi, p)$ be a convex homotopy defined by:

$$H(\varpi, p) = p\Im(\varpi) + (1-p)G(\varpi) \qquad (2.2)$$

where $G(\varpi)$ is a functional operator with ϖ_0 as a known solution. Thus, we have:

$$H(\varpi,0) = G(\varpi) \text{ and } H(\varpi,1) = \Im(\varpi) \qquad (2.3)$$

whenever $H(\varpi, p) = 0$ is satisfied, and $p \in (0,1]$ is an embedded parameter. In HPM, p is used as an expanding parameter to obtain:

$$\varpi = \sum_{j=0}^{\infty} p^j \varpi_j = \varpi_0 + p\varpi_1 + p^2\varpi_2 + \cdots \qquad (2.4)$$

From (2.4), the solution is obtained as $p \to 1$. The convergence of (2.4) as $p \to 1$ has been considered in [23].

The method considers $N(\varpi)$ (the nonlinear term) as:

$$N(\varpi) = \sum_{j=0}^{\infty} p^j H_j = H_o + p^1 H_1 + p^2 H_2 + \cdots \qquad (2.5)$$

where H_k's are the so-called He's polynomials, which can be computed using:

$$H_k(\varpi) = \frac{1}{k!}\frac{\partial^k}{\partial p^k}\left(N\left(\sum_{j=0}^{k} p^j \varpi_j\right)\right)_{p=0}, \; n \geq 0 \qquad (2.6)$$

where $H_k(\varpi) = H_k(\varpi_0, \varpi_1, \varpi_2, \varpi_3, \cdots, \varpi_k)$.

III. The He's Polynomials and the Navier-Stokes Model

In this subsection, the He's Polynomials approach will be applied to the following Navier-Stokes model as follows:

A. *Problem 1: Consider the following Navier-Stokes model:*

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \qquad (3.1)$$

subject to:

$$w(r,0) = r \qquad (3.2)$$

Procedure w.r.t Problem 1:

We re-write (3.1) in an integral form as in (3.3) below:

$$w(r,t) = w(r,0) + \int_0^t \left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right\} dt \qquad (3.3)$$

Applying the convex homotopy method to (3.3) gives:

$$\sum_{n=0}^{\infty} p^n w_n = w(r,0) + p\int_0^t \left\{\sum_{n=0}^{\infty} p^n \frac{\partial^2 w_n}{\partial r^2} + \frac{1}{r}\sum_{n=0}^{\infty} p^n \frac{\partial w_n}{\partial r}\right\} dt \qquad (3.4)$$

Further simplification of (3.4) gives:

$$w_0 + pw_1 + p^2 w_2 + \cdots = r + p\int_0^t \left\{\left(\frac{\partial^2 w_0}{\partial r^2} + p\frac{\partial^2 w_1}{\partial r^2} + p^2\frac{\partial^2 w_2}{\partial r^2} + \cdots\right) + \frac{1}{r}\left(\frac{\partial w_0}{\partial r} + p\frac{\partial w_1}{\partial r} + p^2\frac{\partial w_2}{\partial r} + \cdots\right)\right\} dt \qquad (3.5)$$

From (3.5), comparing the coefficients of the equal powers of p in the following ways gives:

$$p^{(0)}: w_0 = r$$

$$p^{(1)}: w_1 = \int_0^t \left(\frac{\partial^2 w_0}{\partial r^2} + \frac{1}{r}\frac{\partial w_0}{\partial r}\right) dt$$

$$p^{(2)}: w_2 = \int_0^t \left(\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r}\right) dt$$

$$p^{(3)}: w_3 = \int_0^t \left(\frac{\partial^2 w_2}{\partial r^2} + \frac{1}{r}\frac{\partial w_2}{\partial r}\right) dt$$

$$\vdots$$

$$p^{(k)}: w_k = \int_0^t \left(\frac{\partial^2 w_{k-1}}{\partial r^2} + \frac{1}{r}\frac{\partial w_{k-1}}{\partial r}\right) dt, \; k \geq 1$$

Therefore, to obtain $w_1, w_2, w_3, \cdots$, we will apply $w_0 = r$ for the simplification of $p^{(1)}, p^{(2)}, p^{(3)}$, and so on, thus:

$$w_0 = r, w_1 = \frac{t}{r}, w_2 = \frac{1}{2}\frac{t^2}{r^3}, w_3 = \frac{3}{2}\frac{t^3}{r^5}, w_4 = \frac{75}{8}\frac{t^4}{r^7},$$

$$w_5 = \frac{735}{8}\frac{t^5}{r^9}, w_6 = \frac{19845}{16}\frac{t^6}{r^{11}}, w_7 = \frac{343035}{16}\frac{t^7}{r^{13}}\cdots$$

$$\therefore \quad w(x,t) = w_0 + w_1 + w_2 + w_3 + w_4 + w_5 + \cdots$$

$$= r + \frac{t}{r} + \frac{1}{2}\frac{t^2}{r^3} + \frac{3}{2}\frac{t^3}{r^5} + \frac{75}{8}\frac{t^4}{r^7} + \frac{735}{8}\frac{t^5}{r^9} + \frac{19845}{16}\frac{t^6}{r^{11}} + \frac{343035}{16}\frac{t^7}{r^{13}} + \cdots$$

$$= r + \sum_{j=1}^{\infty} \frac{1^1 \times 3^2 \times 5^2 \times \cdots \times (2j-3)^2}{r^{2j-1}} \frac{t^j}{j!} \qquad 3.6)$$

Remark: For graphical consideration of the approximate solution, we use $r \in [0, 0.5]$ and $t \in [0, 0.2]$. Figure 1 and Figure 2 below represent the 3D plots of the solution *(of problem 1)* for terms up to power seven and power five (in terms of the time variable t) respectively.

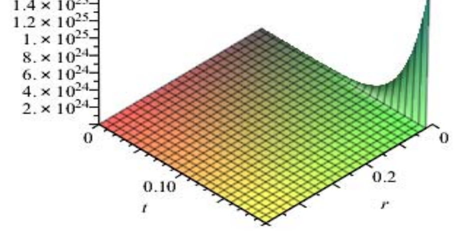

Figure1: He's polynomial solution up to t^7.

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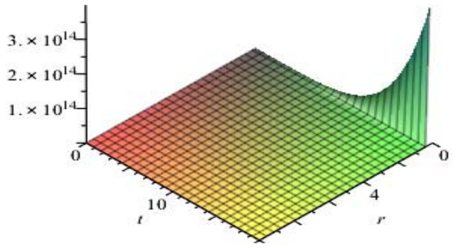

Figure2: He's polynomial solution up to t^5.

B. ***Problem 2:*** *Consider the following Navier-Stokes model:*

$$\frac{\partial w}{\partial t} = p + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \tag{3.7}$$

subject to:

$$w(r,0) = 1 - r^2 \tag{3.8}$$

Procedure w.r.t Problem 2:

We re-write (3.7) in an integral form as in (3.9) below:

$$w(r,t) = w(r,0) + \int_0^t \left\{ p + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \right\} dt \tag{3.9}$$

Applying the convex homotopy method to (3.9) gives:

$$\sum_{n=0}^{\infty} h^n w_n = w(r,0)$$
$$+ h\int_0^t \left\{ p + \sum_{n=0}^{\infty} h^n \frac{\partial^2 w_n}{\partial r^2} + \frac{1}{r}\sum_{n=0}^{\infty} h^n \frac{\partial w_n}{\partial r} \right\} dt \tag{3.10}$$

Further simplification of (3.10) gives:

$$w_0 + hw_1 + h^2 w_2 + \cdots = (1 - r^2)$$
$$+ h\int_0^t \left\{ p + \left(\frac{\partial^2 w_0}{\partial r^2} + h\frac{\partial^2 w_1}{\partial r^2} + h^2 \frac{\partial^2 w_2}{\partial r^2} + \cdots \right) \right.$$
$$\left. + \frac{1}{r}\left(\frac{\partial w_0}{\partial r} + h\frac{\partial w_1}{\partial r} + h^2 \frac{\partial w_2}{\partial r} + \cdots \right) \right\} dt \tag{3.11}$$

From (3.11), comparing the coefficients of the equal powers of h in the following ways gives:

$$h^{(0)} : w_0 = 1 - r^2$$

$$h^{(1)} : w_1 = \int_0^t \left(p + \frac{\partial^2 w_0}{\partial r^2} + \frac{1}{r}\frac{\partial w_0}{\partial r} \right) dt$$

$$h^{(2)} : w_2 = \int_0^t \left(\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} \right) dt$$

$$h^{(3)} : w_3 = \int_0^t \left(\frac{\partial^2 w_2}{\partial r^2} + \frac{1}{r}\frac{\partial w_2}{\partial r} \right) dt$$

$$\vdots$$

$$h^{(k)} : w_k = \int_0^t \left(\frac{\partial^2 w_{k-1}}{\partial r^2} + \frac{1}{r}\frac{\partial w_{k-1}}{\partial r} \right) dt, \; k \geq 2$$

Therefore, to obtain $w_1, w_2, w_3, \cdots$, we will apply $w_0 = 1 - r^2$ for the simplification of $p^{(1)}, p^{(2)}, p^{(3)}$, and so on, thus:

$$w_0 = 1 - r^2, w_1 = (p-4)t, w_\eta = 0 \text{ for } \eta \geq 2$$

$$\therefore \quad w(x,t) = w_0 + w_1 + w_2 + w_3 + w_4 + w_5 + \cdots$$
$$= (1 - r^2) + (p-4)t \tag{3.12}$$

Remark: for $p = 1$, the solution is:

$$w(r,t) = -r^2 - 3t + 1 \tag{3.13}$$

For graphical consideration of the obtained solution, we use $r \in [-1, 1.5]$ and $t \in [0,1]$. Figure 1 & Figure 2 below represent *3D plot* & *implicit plot* (respectively) of the solution *(of problem 2)* at $p = 1$.

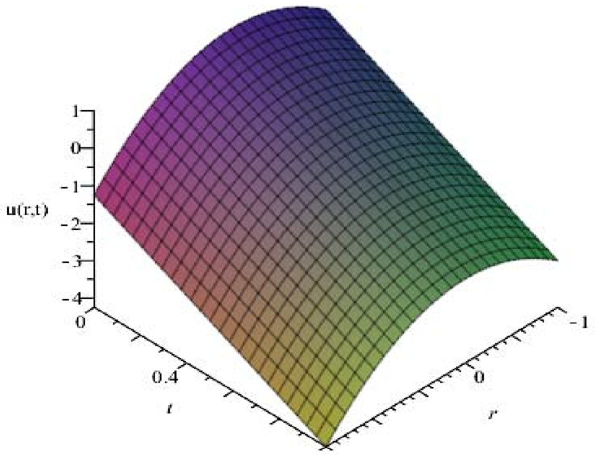

Figure1: He's polynomial solution (*3D plot*)

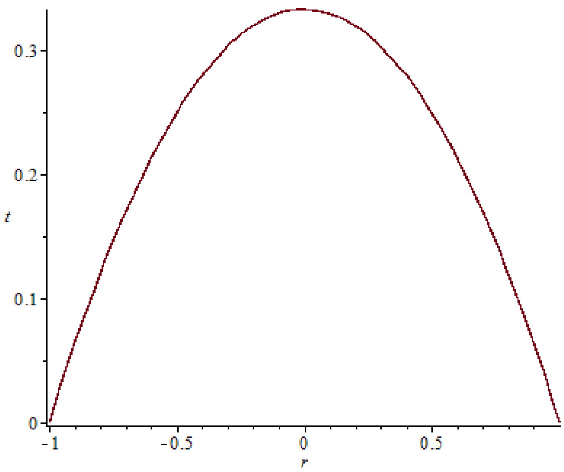

Figure2: He's polynomial solution (*implicit plot)*

IV. Concluding Remarks

In this paper, He's polynomials approach as a proposed solution technique has been applied successfully to the Navier-Stokes model for approximate and exact solutions. These solutions were calculated in the form of series with easily computable components. This technique is very much effective, efficient and reliable as it gives the exact solution of the solved problems with less computational work without neglecting the level of accuracy. As a way of comparison, our results are in very much in agreement with those obtained in [1, 2, 21] for a time-order one. We therefore, recommend an extension and application of this novel method for solving problems arising in other areas of applied sciences. Numerical computations, and graphics done in this work, are through *Maple* 18.

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The concerned authors: SOE, GOA and MEA contributed positively to this work, read and approved the final manuscript for publication.

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