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ON APPROXIMATE AND CLOSED-FORM SOLUTION METHOD FOR INITIAL-VALUE WAVE-LIKE MODELS

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Abstract: This work presents a proposed Modified Differential Transform Method (MDTM) for obtaining both closed-form and approximate solutions of initial-value wave-like models with variable, and constant coefficients. Our results when compared with the exact solutions of the associated solved problems, show that the method is simple, effective and reliable. The results are very much in line with their exact forms. The method involves less computational work without neglecting accuracy. We recommend this simple proposed technique for solving both linear and nonlinear partial differential equations (PDEs) in other aspects of pure and applied sciences.

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Key Words: closed form solution, modified DTM, wave-like equations

1. Introduction

Wave equation is a second order Partial Differential Equation (PDE) used in the description of waves. It is of immense application in applied Mathematics, Engineering and Physics. Wave equations can be linear, or nonlinear initial-boundary value problems. A variety of numerical, analytical and semi-analytical methods have been developed and proposed to obtain approximate, and accurate analytical solutions of various forms of differential equations in literature. Some of these methods include: Homotopy Perturbation Method

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(HPM), Homotopy Analysis Method (HAM), Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Differential Transform Method (DTM) and so on [1]-[8].

DTM is an iterative process that is based on the expansion of Taylor series. It was first proposed by Zhou in 1986 when he used it to solve linear and non-linear initial value problems in the analysis of electric circuit [9]. DTM in most cases, provides analytical approximation, and exact solutions in rapidly convergent sequence form. Despites these advantages, many researchers have improved and modified the DTM for better results and applications [10-18].

The MDTM is useful in obtaining exact and approximate solutions of linear and non-linear differential systems. It has been used by several authors to solve different systems easily and accurately.

The main idea of this work is to use the modified DTM to solve some wave-like PDEs by considering both cases of constant and variable coefficients.

2. Notion and Basic Theorems of the MDTM, see [15], [17], [18]

Let $m(x,t)$ be an analytic function at (x_*, t_*) in a domain$D$, then in considering the Taylor series of$m(x,t)$, regard is given to some variables $s^{ov} = t$ instead of all the variables as in the classical DTM. Thus, the MDTM of $m(x,t)$ with respect to t at t_* is defined and denoted by:

$$M(x,h) = \frac{1}{h!}\left[\frac{\partial^h m(x,t)}{\partial t^h}\right]_{t=t_*}. \tag{1a}$$

Thus, we have:

$$m(x,h) = \sum_{h=0}^{\infty} M(x,h)(t-t_*)^h. \tag{1b}$$

The equation (**??**) is called the modified differential inverse transform of $M(x,h)$ with respect to$t$.

2.1. Basic Theorems and Properties of the MDTM

Theorem 1. *If $m(x,t) = \alpha f(x,t) \pm \beta g(x,t)$, then*

$$M(x,h) = \alpha F(x,h) \pm \beta G(x,h).$$

Theorem 2. *If* $m(x,t) = \frac{\alpha \partial^n m_*(x,t)}{\partial t^n}$, *then*

$$M(x,h) = \frac{\alpha (h+n)!}{h!} M_*(x, h+n).$$

Theorem 3. *If* $m(x,t) = \frac{p(x) \partial^n m_*(x,t)}{\partial x^n}$, *then*

$$M(x,h) = \frac{p(x) \partial^n M_*(x,h)}{\partial x^n}.$$

Theorem 4. *If* $m(x,t) = p(x) m_*^2(x,t)$, *then*

$$M(x,h) = p(x) \sum_{r=0}^{h} M_*(x,r) M_*(x,h-r).$$

Theorem 5. *If* $m(x,t) = t^n$, *then*

$$M_k(x) = \delta(k-n) = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

3. Illustrative and Numerical Examples

Here, we apply the proposed method to the following problems.

3.1. Cases 1 & 2 {Wave-Like Models with Variable, and Constant Coefficients}

Case Problem 1. Consider the wave-like model with variable coefficients:

$$\frac{\partial^2 u}{\partial t^2} = \frac{x^2}{2} \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

subject to the initial conditions:

$$u(x,0) = 1 \text{ and } u_t(x,0) = x^2. \tag{2}$$

Solution procedure to Case Problem 1. Taking the modified differential transform (MDT) of both sides of (1),we get

$$\frac{(k+2)!}{k!}U(x,k+2)=\frac{x^2}{2}\frac{\partial^2 U(x,k)}{\partial x^2}, \quad k\geq 0. \tag{3}$$

Corresponding to (3) is the recurrence formula (4) with initial conditions in (5):

$$U(x,k+2)=\frac{1}{(k+1)(k+2)}\frac{x^2}{2}\frac{\partial^2 U(x,k)}{\partial x^2}, \tag{4}$$

$$U(x,0)=1, \quad U(x,1)=x^2, \quad k\geq 0. \tag{5}$$

Using (5) in (4) gives the following components:

$$\begin{aligned} &U(x,0)=1, \quad U(x,1)=x^2, \quad U(x,2)=0, \\ &U(x,3)=\frac{x^2}{3!}, \quad U(x,4)=0, \quad U(x,5)=\frac{x^2}{5!}, \quad \cdots \end{aligned} \tag{6}$$

In general, we have:

$$U(x,2n)=0, \quad U(x,2n-1)=\frac{x^2}{(2n-1)!}, \quad n=1,2,3,\ldots \tag{7}$$

Substituting (6) and (7) into the solution series, we have:

$$\begin{aligned} u(x,t)&=\sum_{k=0}^{\infty}U(x,k)\,t^k \\ &=U(x,0)+U(x,1)\,t+U(x,2)\,t^2+U(x,3)\,t^3+U(x,4)\,t^4+U(x,5)\,t^5+\cdots \\ &=1+x^2\left\{t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}+\cdots\right\}=1+x^2\sum_{\eta=0}^{\infty}\frac{t^{2\eta+1}}{(2\eta+1)!}. \end{aligned} \tag{8}$$

Equation (8) is the closed-form solution of *case problem 1.*

Case Problem 2. Consider the wave-like model with constant coefficients:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}-3u, \tag{9}$$

subject to the initial conditions:

$$u(x,0)=0 \text{ and } u_t(x,0)=2\sin x. \tag{10}$$

Solution Procedure to Case Problem 2. Taking the modified differential transform of both sides of (9), we get:

$$\frac{(k+2)!}{k!}U\left(x,k+2\right)=\frac{\partial^2 U\left(x,k\right)}{\partial x^2}-3U\left(x,k\right),\quad k\geq 0. \tag{11}$$

Corresponding to (11) is the recurrence formula (12) with the initial conditions in (13):

$$U\left(x,k+2\right)=\frac{1}{\left(k+1\right)\left(k+2\right)}\left[\frac{\partial^2 U\left(x,k\right)}{\partial x^2}-3U\left(x,k\right)\right] \tag{12}$$

$$U\left(x,0\right)=0,\quad U\left(x,1\right)=2\sin x,\quad k\geq 0 \tag{13}$$

Using (13) in (12) gives the following:

$$\begin{gathered} U\left(x,0\right)=0,\quad U\left(x,1\right)=2\sin x,\quad U\left(x,2\right)=0,\\ U\left(x,3\right)=\frac{-2^3\sin x}{3!},\quad U\left(x,4\right)=0,U\left(x,5\right)=\frac{2^5\sin x}{5!},\cdots \end{gathered} \tag{14}$$

In general, we have:

$$U\left(x,2n\right)=0,\quad U\left(x,2n+1\right)=\frac{\left(-1\right)^n 2^{(2n+1)}\sin x}{\left(2n+1\right)!},\quad n=0,1,2,3,\ldots \tag{15}$$

Substituting these into the solution series, we have:

$$\begin{aligned} u\left(x,t\right)&=\sum_{k=0}^{\infty}U_k\left(x\right)t^k\\ &=U\left(x,0\right)+U\left(x,1\right)t+U\left(x,2\right)t^2+U\left(x,3\right)t^3+U\left(x,4\right)t^4+U\left(x,5\right)t^5+\cdots\\ &=2t\sin x-\frac{\left(2t\right)^3\sin x}{3!}+\frac{\left(2t\right)^5\sin x}{5!}-\frac{\left(2t\right)^7\sin x}{7!}+\cdots\\ &=\sin x\left[2t-\frac{\left(2t\right)^3}{3!}+\frac{\left(2t\right)^5}{5!}-\frac{\left(2t\right)^7}{7!}+\cdots\right]\\ &=\left(\sum_{j=0}^{\infty}\left(-1\right)^j\frac{\left(x\right)^{2j+1}}{\left(2j+1\right)!}\right)\left(\sum_{\eta=0}^{\infty}\left(-1\right)^\eta\frac{\left(2t\right)^{2\eta+1}}{\left(2\eta+1\right)!}\right). \end{aligned} \tag{16}$$

Equation (16) is the closed-form solution of *case problem 2.*

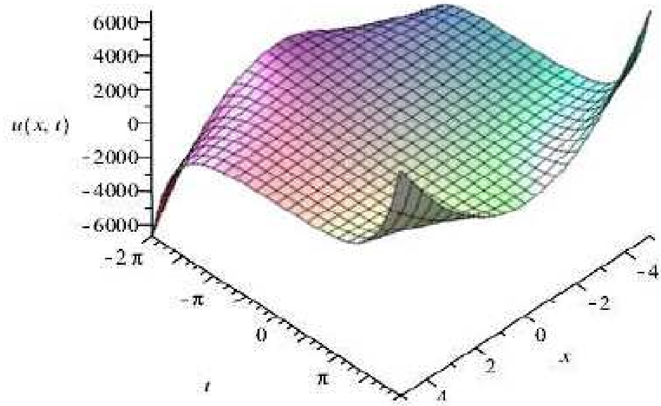

Figure 1: Exact solution of Case Problem 1.

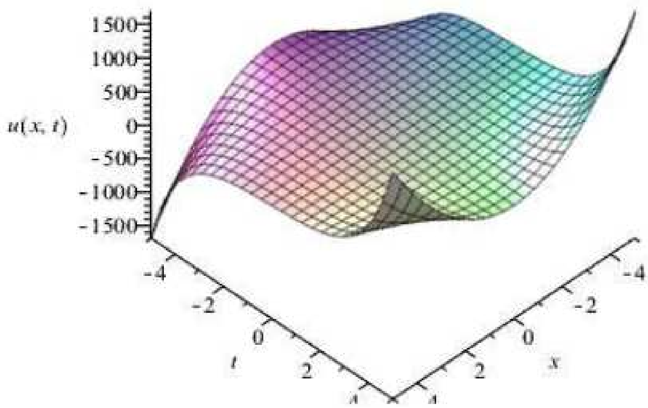

Figure 2: Approximate solution of Case Problem 1.

3.2. Discussion of Results

In this subsection, we will present graphs for the exact and the approximate solutions for discussion of results. The approximate solutions contain terms up to the power of seven (7).

Figure 1 and Figure 2 for exact and approximate solutions of *case problem 1* in that order.

4. Concluding Remarks

In this work, we solved initial-value wave-like models with variable, and constant coefficients for obtaining both closed-form and approximate solutions. For this, we used a proposed solution technique: a modified differential transform method (MDTM). Our results when compared with the exact solutions of the associated solved problems, showed that the method is simple, efficient, effective and reliable. The results are very much in line with their exact forms, even without neglecting accuracy. We therefore, recommend this solution technique for solving both linear and nonlinear partial differential equations (PDEs) in other aspects of pure and applied sciences.

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